E.B. BELGHITAR,[1] M.T. MEFTAH,[2] Z. MALKI[3]

[1] Department of Physics, and LENREZA Laboratory, Kasdi Merbah University
*(Route de Ghardaia, Ouargla 30000, Algeria; e-mail: elhadj.belghitar@gmail.com)*

[2] Department of Physics, and LRPPS Laboratory, Kasdi Merbah University
*(Route de Ghardaia, Ouargla 30000, Algeria; e-mail: mewalid@yahoo.com)*

[3] LRPPS Laboratory, Kasdi Merbah University
*(Route de Ghardaia, Ouargla 30000, Algeria; e-mail: zahidamalki@gmail.com)*

UDC 539

# MODEL OF ANGULAR MOMENTUM TRANSPORT AT THE PROTOPLANETARY DISK EVOLUTION AND DISK SURFACE DENSITY

*We consider how the tidal effect of a protoplanetary disk interaction can be incorporated into calculations of its viscous evolution. The evolution of the disk occurs under the action of both internal viscous torques and external torques resulting from the presence of one or more embedded planets. The planets migrate under the effect of their tidal interaction with the disk (in the type-II migration regime). Torques on a planet are caused by its gravitational interaction with the density waves which occupy the Lindblad resonances in the disk. Our model simplifies the functional form of the rate of injection of the angular momentum $\Lambda(r)$ to construct and solve the evolution equation for a disk and an embedded protoplanet. The functional $\Lambda(r)$ depends on the tidal dissipation distribution in the disk which is concentrated in a vicinity of the protoplanet's orbit. We have found an analytic solution for the disk surface density.*

*Keywords*: protoplanetary disk, viscous torques, angular momentum, planet, accretion disk, surface density.

## 1. Introduction

We present the evolution of a protoplanetary disk under the action of both internal viscous torques and external torques from embedded planets [1, 2]. The orbital evolution of planets, due to their interaction with the ambient disk, has been covered in a few recent reviews [3–5]. It turns out that a change in the orbital elements of the protoplanet can be predicted as a consequence of the interaction with the disk [6]. The migration of the planet is the most important consequence observed in the overall evolution of the major semiaxis of the planet. Three regimes of migration, functions of the mass of planets, can be distinguished, with two limiting regimes of migration as the most important. For a mass less than about 50 Earth's masses, the planet cannot open a gap in the disk [3,4], so the evolution can be treated in the linear regime. This type of evolution is type I. The analytic formulation of the migration of this type has been presented in [7, 8]. For massive planets, the angular momentum deposition in the disk causes an annular gap in the disk at the orbit of the planet. The reduced mass available near the planet causes a slow-down of the migration speed from the linear rate. This is the type II migration. At equilibrium, the planet is locked in the middle of the gap to maintain the torque equilibrium [1, 9, 10]. Several numerical calculations fpr migrating massive planets were done in [11, 12]. In these works, a constant kinematic viscosity is considered. The more recent work [13] analyzed the influence of a planetary motion on the type I migration regime. The migration of massive planets in an ambiant disk has been studied in [14]. In this last model, the planets were locked in a disk with an initial Gaussian profile and a constant kinematic viscosity. The problem of type II migration is still relevant.

In this paper, we give the analytic solution of this problem and study the migration of planets in slim disks with an approximation of the external torque due to the planet. On the first step in the next section, we define the torque and, in Section 3, present our analytic solution.

We do not pretend to solve the problem without the resorting to a numerical method, but we will obtain an analytic solution for a simplified model which may be used to build migration models.

## 2. The Rate of Specific Angular Momentum Transfer

In the type I migration model, the planets are not massive enough to perturb the disk. We assume that the total torque is the sum of the contributions from the three different resonances: The first is the partial torque from the inner Lindblad resonances which drive the outward migration. The second is outer Lindblad resonances which drive the inward migration, and the third contribution is due to the corotation resonance. However, the partial torques from the inner and outer Lindblad resonances are of opposing signs, but of almost the same magnitude. To predict the direction of the migration in an analytic calculation is somewhat difficult, because a precise calculation of the torque is needed. Moreover, real disks can be turbulent with a domination of fluctuating torques that result from turbulent density fluctuations.

We adopt the cylindrical polar coordinates $(r, \Phi, z)$. The studied accretion disk rotates and has the axial symmetry around a star of mass $M_*$. The disk lies in the $z = 0$ plane, $r$ is the radial coordinate. The razor thin disk approximation implies that the disk scale height $H = C_s/\Omega$, where $C_s$ is the sound speed, and $\Omega$ is the angular velocity of rotation of the disk, is much smaller than the distance from the star $H/r \ll 1$. The disk is assumed to be axisymmetric, so that all variables are functions of only the radius $r$ and time $t$. We consider a geometrically thin stationary, rotating and axisymmetric accretion disk around a star of mass $M_*$, with a viscosity $\nu$ which can be written as a power-law in the radius [15]. The kinematic viscosity $\nu$ governs the transport of the angular momentum in the disk. We consider a more general case where $\nu = sr^n$, $n < 2$, and $s$ is a constant [16]. The rate of specific angular momentum transfer from the planet with mass $M_p = qM_*$ at the radius (major semiaxis) $r_p$ to the disk is given by $\Lambda(r)$ [17, 18]:

$$\Lambda(r) = \begin{cases} -\dfrac{q^2GM_*}{2r}\left(\dfrac{r}{\Delta_p}\right)^4; & r < r_p, \\ \dfrac{q^2GM_*}{2r}\left(\dfrac{r_p}{\Delta_p}\right)^4; & r > r_p, \end{cases} \tag{1}$$

where $\Delta_p = \max(H, |r - r_p|)$, and $H$ is the disk scale-height.

Protoplanetary disks evolve to cviscous transport of the angular momentum and the photo evaporation by the central star. Planets migrate due to the tidal interaction with the disk, and the disk is also subject to tidal torques from planets.

Treating the evolution equation with $\Lambda(r)$ given by (1) is analytically very complicated. To avoid these difficulties, we will modify the rate of specific angular momentum transfer $\Lambda(r)$ so that the resulting differential equation of the model can have an analytic solution which approximates the real evolution.

This modification allows us to solve only two equations and gives an acceptable result at a vicinity of the planet. In our model, the rate of specific angular momentum transfer is described by two equations, where $1 < n < 2$ and $0 < n' < 1$:

$$\Lambda'(r) = \begin{cases} -\dfrac{q^2GM_*}{2r}\left(\dfrac{r_p}{H}\right)^4\left(\dfrac{r}{r_p}\right)^{\frac{n'+1}{2}}; & r < r_p, \\ \dfrac{q^2GM_*}{2r}\left(\dfrac{r_p}{H}\right)^4\left(\dfrac{r}{r_p}\right)^{\frac{n'+1}{2}}; & r > r_p. \end{cases} \tag{2}$$

This choice of a model is based on two main facts:

The first is that the rate of specific angular momentum transfer $\Lambda(r)$ depends on the tidal dissipation, which is concentrated, in turn, when the viscosity $\nu$ is very large, at $r = r_p$ near the protoplanet orbit [19]. The viscosity expression proposed in [15, 20] leads us to choose this rate, as indicated by the formulas (2).

Another feature to note is that this model in a vicinity of the orbit admits an exact analytic treatment. In view of the importance of the viscosity near the planet orbit [19], we focus on the study near

$r_p$. We have mentioned this remark, when we compared the formulas (2) and (1), at $r \sim r_p$. Indeed, for all values of $n$ and $n'$. Near the planet orbit ($r \sim r_p$), we have $\Lambda'(r) = \Lambda(r)$.

Our model preserves the physical properties of the rate of specific angular momentum transfer. It is the fact that the moment is maximum close to the planet and decreases progressively, when the distance to the planet increases. The torque exerted on a planet can be obtained in the impulse approximation [19]. A loss of the angular momentum from the planet is due to the interaction between the external gas and the planet which is overtaken by the planet, while a net gain in the angular momentum is due to the gas in the interior part which is overtaken by the planet. The total torque will be the sum of all the torques and depends on the structure of the disk. We took values for $n$ and $n'$ in such way that the new function approximates the real function as much as possible.

The purpose of the present paper is to obtain an approximate solution of the evolution equation given by [1]

$$\frac{\partial \Sigma(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[3\sqrt{r}\frac{\partial}{\partial r}(\nu\Sigma\sqrt{r}) - 2\frac{\Lambda(r)r^{\frac{3}{2}}}{\sqrt{GM_*}}\Sigma\right]. \quad (3)$$

Here, $\Sigma = \Sigma(r,t)$ is the disk surface density in the cylindrical coordinate system, $t$ is the time, $\nu = sr^n$, $n < 2$, is the friction coefficient per unit density or kinematic viscosity. The torque $\Lambda(r)$ is given by (1).

To deal with the torque term $\Lambda(r)$, we shall use a polynomial-type approximation in order to obtain an approximate analytic solution of the evolution equation. Therefore, we use an approximation for the torque term by taking (2).

Such an approximation is good in the neighborhood of the planet orbit $r_p$, or for a thin disk $H \ll r$, if $1 < n < 2$ and $n' < 1$. We do not claim that our approximate analytic solution is accurate everywhere, but this approximation turns to become a good tool for studying the evolution of a planet orbit.

## 3. Model for Planetary Migration

We now give our simplified model with an evolution which is slightly different from the standard form used in the literature by the approximate function $\Lambda'(r)$ obtained in the last section. In our model, the evolution of the protoplanetary disk is characterized by the viscous transport of the angular momentum. The planets migrate under the influence of the tidal interaction with the disk which is subjected to tidal torques from planets. We assume that the shape of the disk has the cylindrical symmetry. So, all equations are expressed in a cylindrical coordinate system. The evolution equation of a protoplanetary disk and a planet is described by

$$\frac{\partial \Sigma(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[3\sqrt{r}\frac{\partial}{\partial r}(\nu\Sigma\sqrt{r}) - 2\frac{\Lambda'(r)r^{\frac{3}{2}}}{\sqrt{GM_*}}\Sigma(r,t)\right]. \quad (4)$$

Here, $\Sigma(r,t)$ is the disk surface density in the cylindrical coordinate system, $t$ is the time, $\nu = sr^n$, $n < 2$, is the friction coefficient per unit density or kinematic viscosity, and $M_*$ is the stellar mass. The ordinary viscous evolution of the disk is described by the first term on the right-hand side [15, 20], and the second term describes the effect of the planetary torque. We shall limit ourselves to the stationary regime

$$\frac{\partial \Sigma(r,t)}{\partial t} = -\lambda\Sigma(r,t). \quad (5)$$

In this case (the stationary regime), the viscosity $\nu$ is given by a radial power law. We assume a separable ansatz of the form

$$\Sigma(r,t) = \exp(-\lambda t)\phi(r), \quad (6)$$

where $\lambda$ is some constant, and $\phi$ is an arbitrary function of $r$. The latter expresses the amplitude of the matter density at any value of $r$ at a fixed time $t$ during the propagation of the matter. Substituting $\Lambda'(r)$ in (4) by its expression and setting $\alpha = \frac{q^2\sqrt{GM_*}}{3s}\left(\frac{r_p}{H}\right)^4$, $N = 1 - \frac{n}{2} > 0$ and $N' = 1 - \frac{n'}{2} > 0$, we get a homogeneous second-order differential equation, for $r < r_p$ and for $r > r_p$, respectively,

$$r^2\phi''(r) + \left[2n + \frac{3}{2} - \alpha r^N\right] r\phi'(r) + \left[n\left(n+\frac{1}{2}\right) - \alpha\left(1+\frac{n}{2}\right)r^N + \frac{\lambda}{3s}r^{2N}\right]\phi(r) = 0, \quad (7)$$

$$r^2\phi''(r) + \left[2n' + \frac{3}{2} + \alpha r^{N'}\right] r\phi'(r) + \left[n'\left(n'+\frac{1}{2}\right) + \alpha\left(1+\frac{n'}{2}\right)r^{N'} + \frac{\lambda}{3s}r^{2N'}\right]\phi(r) = 0. \quad (8)$$

However, this last equation cannot be immediately resolved. Thus, in the domain $r < r_p$, we define the new variable $Nx = \alpha r^N$. After some algebra, we have

$$(2-n)\,x\phi''(x) + [3\,(n+1) - (2-n)\,x]\,\phi'(x) + \\ + \left[\frac{\lambda}{3\alpha^2 s}\,(2-n)\,x + 2\frac{n\,(2n+1)}{(2-n)\,x} - (2+n)\right]\phi(x) = 0. \quad (9)$$

Substitute the function $\phi\,(x)$ by a new function $u\,(x)$ as follows

$$\phi(x) = x^b \exp\left(-\frac{x}{2}\right) u(x), \quad (10)$$

with

$$b = \frac{3}{2}\left(1 - \frac{3}{2-n}\right) < -\frac{3}{4}. \quad (11)$$

Inserting into (9), we get the equation

$$u''(x) + \left[-\beta + \frac{\gamma}{x} + \frac{\delta}{x^2}\right] u(x) = 0, \quad (12)$$

where

$$\beta = \frac{3}{4} + \frac{\lambda}{3\alpha^2 s} \quad (13)$$

$$\gamma = \frac{1}{2}\frac{n-1}{2-n} \quad (14)$$

$$\delta = \frac{(n-1)\,(n-3)}{4\,(n-2)^2}. \quad (15)$$

Introducing the new variable $z = 2x\sqrt{\beta}$, we obtain the Whittaker equation [21]

$$u''(z) + \left[-\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right] u(z) = 0, \quad (16)$$

where

$$\kappa = \frac{\gamma}{2\sqrt{\beta}} = \left(\frac{n-1}{2-n}\right)\frac{1}{2}\left(3 + \frac{4\lambda}{3\alpha^2 s}\right)^{-\frac{1}{2}} \quad (17)$$

and

$$2\mu = \sqrt{1 - 4\delta} = \frac{1}{2-n}. \quad (18)$$

The formal solution of this differential equation is of the form:

$$u(x) = C_1 M_{\kappa,\mu}(2x\sqrt{\beta}) + C_2 W_{\kappa,\mu}(2x\sqrt{\beta}), \quad (19)$$

where $M_{\kappa,\mu}(z)$, $W_{\kappa,\mu}(z)$ are the Whittaker functions, $C_1$ and $C_2$ are two constants depending on boundary conditions at $r = 0$ and $r = \infty$. In the original function $\phi(r)$ for $r < r_p$ and for $r > r_p$, we get

$$\Sigma(r,t) = \left(\frac{\alpha r^N}{N}\right)^b \exp(-\lambda t) \exp\left(-\frac{\alpha r^N}{2N}\right) \times \\ \times \left[C_1 M_{\kappa,\mu}\left(\frac{2\alpha}{N}\sqrt{\beta} r^N\right) + C_2 W_{\kappa,\mu}\left(\frac{2\alpha}{N}\sqrt{\beta} r^N\right)\right], \quad (20)$$

$$\Sigma(r,t) = \left(-\frac{\alpha r^{N'}}{N'}\right)^b \exp(-\lambda t) \exp\left(\frac{\alpha r^{N'}}{2N'}\right) \times \\ \times v\left[C'_1 M_{\kappa,\mu}\left(-\frac{2\alpha}{N'}\sqrt{\beta} r^{N'}\right) + C'_2 W_{\kappa,\mu}\left(-\frac{2\alpha}{N'}\sqrt{\beta} r^{N'}\right)\right], \quad (21)$$

where $C'_1$ and $C'_2$ are two constants depending on the boundary conditions at $r = 0$ and $r = \infty$.

## 4. Boundary Conditions

The dynamics of the accretion disk described by the differential equation (4) is an initial-value problem. Generally, the boundary condition has an influence on the global solution. So, the boundary conditions imposed on the accretion disk are of importance. The outer boundary $r \to \infty$ is a freely expanding surface.

We consider different boundary conditions: the zero stress or no accretion at the inner boundary $r = 0$ ($r = 0$ is an inner edge of the disk), and the zero mass inflow at the outer boundary $r \to \infty$.

First of all, we have the limiting cases as $z \to 0$ [21]

$$M_{\kappa,\mu}\,(z) \sim z^{\frac{1}{2}+\mu}, \quad (22)$$

$$W_{\kappa,\mu}(z) \sim \frac{\Gamma\,(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \kappa\right)} z^{\frac{1}{2}-\mu} + \frac{\Gamma\,(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - \kappa\right)} z^{\frac{1}{2}+\mu}. \quad (23)$$

As $z \to \infty$, we have

$$M_{\kappa,\mu}\,(z) \sim \frac{\Gamma\,(1+2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \kappa\right)} e^{\frac{1}{2}z} z^{-\kappa}, \quad (24)$$

$$W_{\kappa,\mu}(z) \sim e^{-\frac{1}{2}z} z^{\kappa}. \quad (25)$$

We note that

$$\frac{3}{4} < \frac{1}{2} + \mu < 1, \quad (26)$$

$$0 < \frac{1}{2} - \mu < \frac{1}{4}. \quad (27)$$

Because the Whittaker function $W_{\kappa,\mu}(z)$ has an exponential damping in the domain $r > r_p$, we put

$$\Sigma(r,t) = C'\left(-\frac{\alpha}{N'}\right)^{-\frac{3}{2}\frac{n'+1}{2-n'}} r^{-\frac{3}{4}(n'+1)} \exp(-\lambda t) \times$$
$$\times \exp\left(\frac{\alpha r^{N'}}{2N'}\right) W_{\kappa,\mu}\left(-\frac{2\alpha}{N'}\sqrt{\beta} r^{N'}\right) \quad (28)$$

for $r < r_p$. When $r \to 0$, we get

$$\Sigma(r,t) \simeq \left(\frac{\alpha}{N}\right)^b \exp(-\lambda t) \exp\left(-\frac{\alpha r^N}{2N}\right) [C_1 u_1 + C_2 u_2], \quad (29)$$

where

$$u_1 = \left(\frac{2\alpha}{N}\sqrt{\beta}\right)^{\frac{1}{2}+\mu} r^{N(b+\frac{1}{2}+\mu)}, \quad (30)$$

$$u_2 = \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)} \left(\frac{2\alpha}{N}\sqrt{\beta}\right)^{\frac{1}{2}-\mu} r^{N(b+\frac{1}{2}-\mu)} +$$
$$+ \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)} \left(\frac{2\alpha}{N}\sqrt{\beta}\right)^{\frac{1}{2}+\mu} r^{N(b+\frac{1}{2}+\mu)}. \quad (31)$$

As the term $r^{N(b+\frac{1}{2}+\mu)}$ tends to infinity at $r = 0$, we put $C_2 = 0$. Therefore, we have:

$$\Sigma(r,t) = C\left(\frac{\alpha}{N}\right)^{-\frac{3}{2}\frac{n+1}{2-n}} r^{-\frac{3}{4}(n+1)} \exp(-\lambda t) \times$$
$$\times \exp\left(-\frac{\alpha r^N}{2N}\right) M_{\kappa,\mu}\left(\frac{2\alpha}{N}\sqrt{\beta} r^N\right). \quad (32)$$

Finally, we obtain an approximate solution $\Sigma(r,t)$ of the evolution equation for $r < r_p$ and $r > r_p$, respectively, as

$$\Sigma(r,t) = C\left(\frac{\alpha}{N}\right)^{-\frac{3}{2}\frac{n+1}{2-n}} r^{-\frac{3}{4}(n+1)} \exp(-\lambda t) \times$$
$$\times \exp\left(-\frac{\alpha r^N}{2N}\right) M_{\kappa,\mu}\left(\frac{2\alpha}{N}\sqrt{\beta} r^N\right) \quad (33)$$

and

$$\Sigma(r,t) = C'\left(-\frac{\alpha}{N'}\right)^{-\frac{3}{2}\frac{n'+1}{2-n'}} r^{-\frac{3}{4}(n'+1)} \exp(-\lambda t) \times$$
$$\times \exp\left(\frac{\alpha r^{N'}}{2N'}\right) W_{\kappa,\mu}\left(-\frac{2\alpha}{N'}\sqrt{\beta} r^{N'}\right). \quad (34)$$

## 5. Results and Discussion

To present the behavior of the surface density near the planet, we take the approximation made in [16], where $t_\nu = \frac{2}{3}\frac{r^{2-n}}{s}$ is the local time scale at $r$ in the case of nonzero viscosity. For computational convenience, we introduce a set of dimensionless variables such that: $\overline{\alpha} = \alpha r_p^N$, $\rho = \frac{r}{r_p}$, $\tau = \frac{t}{t_\nu}$ and $K = \frac{\lambda}{3\alpha^2 s}$, $K$ is constant. We have $\beta = \frac{3}{4} + K$, $\kappa = \left(\frac{n-1}{2-n}\right)\frac{1}{2}(3+K)^{-\frac{1}{2}}$, $2\mu = \frac{1}{2-n}$, $N = 1 - \frac{n}{2}$, $N' = 1 - \frac{n'}{2}$, $1 < n < 2$ and $0 < n' < 1$ and $b = \frac{3}{2}\left(1 - \frac{3}{2-n}\right)$. We can then have the solution of the evolution equation as follows: for $\rho < 1$, ($r < r_p$)

$$\Sigma(\rho,\tau) = C\,(X_N)^b \exp\left(-2K(X_N)^2 N^2 \tau\right) \times$$
$$\times \exp\left(-\frac{X_N}{2}\right) M_{\kappa,\mu}(2X_N\sqrt{\beta}) \quad (35)$$

and, for $\rho > 1$, ($r > r_p$)

$$\Sigma(\rho,\tau) = C'(-X_{N'})^b \exp\left(-2K(X_{N'})^2 N'^2 \tau\right) \times$$
$$\times \exp\left(\frac{X_{N'}}{2}\right) W_{\kappa,\mu}(-2X_{N'}\sqrt{\beta}), \quad (36)$$

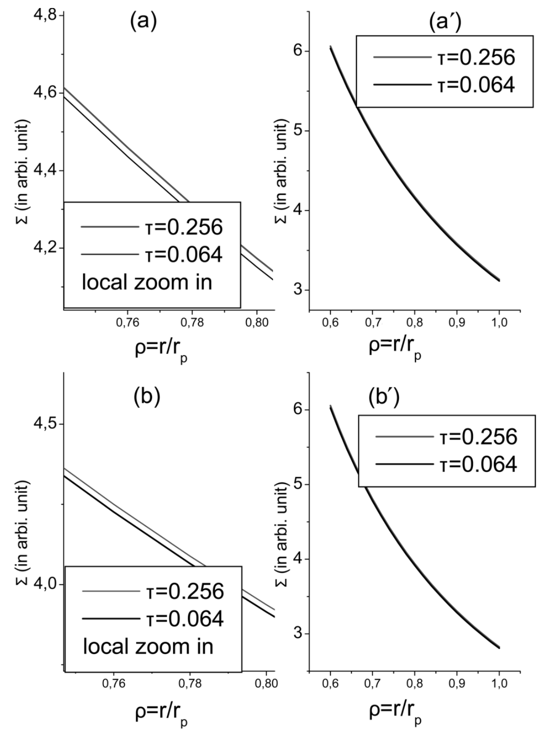


***Fig. 1.*** The behavior of the surface density as a function of the scaled time variable $\tau = 0.064$; $0.256$, for different values of $n$

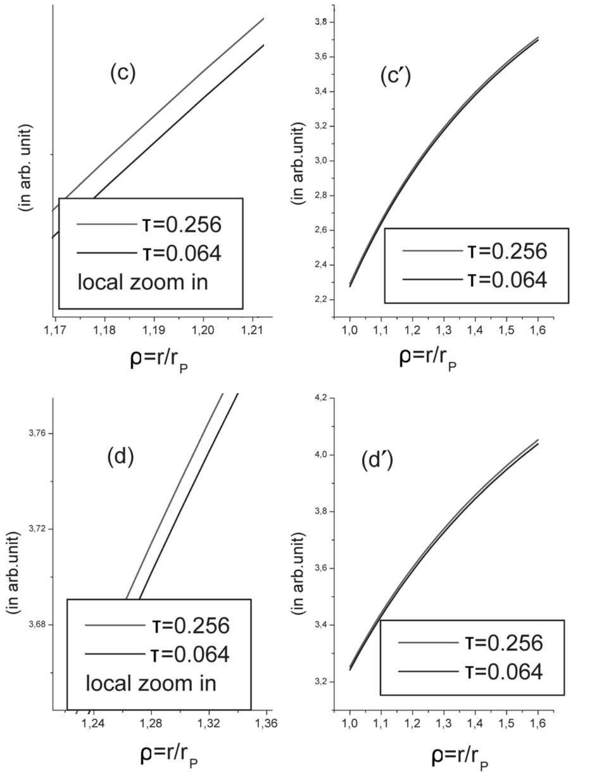


***Fig. 2.*** The behavior of the surface density as a function of the scaled time variable $\tau = 0.064$; 0.256, for different values of $n'$

where $X_N = \overline{\alpha}\rho^N/N$ and $X_{N'} = \overline{\alpha}\rho^{N'}/N'$. The resulting solutions suffice to illustrate the essential behavior implied by the evolution equation for the disk. With $\nu$ = constant, the solution to the evolution equation is possible [22].

In the Fig. 1, the curves show the behavior of the surface density as a function of $\rho$ for two scaled time values $\tau = 0.064$ and $\tau = 0.256$ and for two values of $n = 3/2$ and $n = 7/4$. Figure 2 shows the behavior of the surface density as a function of $\rho$ for two scaled time values $\tau = 0.064$ and $\tau = 0.256$ and for two values of $n' = 1/10$ and $n' = 1/2$.

The evolution of the surface density in the model of a protoplanetary disk is plotted: ($a$) for $n = 3/2$, ($b$) for $n = 7/4$, ($c$) for $n' = 1/10$, ($d$) for $n' = 1/2$. The curves show the behavior as a function of the scaled time variable $\tau$ for $\tau = 0.064$ (black line) and $\tau = 0.256$ (gray line). To highlight the difference between the different cases, we have done a zoom in of each figure (at the left of each).

The coupled evolution of a planet embedded within the evolving disk can be approximately modeled with straightforward generalizations of the one-dimensional evolution equation discussed in Section 3. As an illustration of such a calculation, Figs. 1 and 2 show the behavior of the surface density at a given time and distance, this expected behavior followed by the formation of a planet at ($r = r_p$) inside the disk. Through these curves, the density is very low near the planet ($r = r_p$), and it has a significant value away from the planet.

## 6. Concluding Remarks

In this work,, we have solved exactly the evolution equation for a disk with embedded protoplanets. In view of the importance of the second term on the right-hand side of the evolution equation, which describes the effect of the planetary torque, especially near the planet orbit, we have suggested an approximate rate $\Lambda'(r)$ of specific angular momentum transfer which have the same form as $\Lambda(r)$ near the planet orbit. The choice of $\Lambda'(r)$ is based on the idea of viscosity proposed earlier in [15].

We have presented the overview of a giant planet migration in evolving protoplanetary disks. Our disks evolve under condition of the viscous transport of angular momentum, while planets undergo the type II migration. In a one-dimensional model, we have performed the calculations within the approximation of the external torque due to the presence of a planet.

We have found analytic solutions for the surface density within a protoplanetary disk. Our result is similar to that in [1]. This solution is plotted in Figs. 1 and 2. Over the time, the surface density is very low near the planet, and it has a significant value away from the planet.

Finally, we have used the results of our exact solution of the evolution equation that allowed us to contribute to the study of the planetary migration.

1. D.N.C. Lin, J. Papaloizou. On the tidal interaction between protoplanets and the protoplanetary disc. III – Orbital migration of protoplanets. *Astrophysical J.* **309**, 846 (1986).
2. P. Goldreich, S. Tremaine. Disk-Satellite interactions. *Astrophysical J.* **241**, 425 (1980).
3. W. Kley, R.P. Nelson. Planet-disk interaction and orbital evolution. *ARA&A* **50**, 211(2012).
4. C. Baruteau, Q. Massey. In: Tidal effects in astronomy and astrophysics. *Lecture Notes in Physics* **201**, 861 (2013).

5. C. Baruteau, A. Crida, S.J. Paardekooper, *et al.* Planet-disc interactions and early evolution of planetary systems. arXiv: 1312.4293 (2013).
6. G. Laughlin, A. Steinacker, F.C. Adams. Type I planetary migration with MHD turbulence. *Astrophysical J.* **608**, 489 (2004).
7. S.J. Paardekooper, C. Baruteau, A. Crida, W. Kley. A torque formula for non-isothermal type I planetary migration – I. Unsaturated horseshoe drag. *MNRAS* **401**, 1950 (2010).
8. S. J. Paardekooper, C. Baruteau, W. Kley. A torque formula for non-isothermal type I planetary migration – II. Effects of diffusion. *MNRAS* **410**, 293 (2011).
9. W.R. Ward. Protoplanet migration by nebula tides. *ICARUS* **126**, 261 (1997).
10. W.R. Ward.Comments of the long-term stability of the Earth's obliquity. *ICARUS* **50**, 444 (1982).
11. R.G. Edgar. Giant planet migration in viscous power-law discs. *Astrophysical J.* **663**, 1325 (2007).
12. R.G. Edgar. Type II migration: Varying planet mass and disc viscosity. arXiv:0807.0625 (2008).
13. S. J. Paardekooper. Dynamical corotation torques on low-mass planets. *MNRAS* **444**, 2031 (2014).
14. A. Crida, A. Morbidelli. Cavity opening by a giant planet in a protoplanetary disc and effects on planetary migration. *MNRAS* **377**, 1324(2007).
15. D. Lynden-Bell, J. E. Pringle. The evolution of viscous discs and the origin of the nebular variables. *MNRAS* **168**, 603 (1974).
16. T. Tanaka. 2011. Exact time-dependent solutions for the thin accretion disc equation: Boundary conditions at finite radius. *MNRAS* **410**, 1007 (2011).
17. P.J. Armitage, M. Livio, S.H. Lubow, J.E. Pringle. Predictions for the frequency and orbital radii of massive extrasolar planets. *MNRAS* **334**, 248 (2002).
18. D.E. Trilling, W. Benz, T. Guillot, J.I. Lunine, W.B. Hubbard, A. Burrows. Orbital evolution and migration of giant planets: Modeling extrasolar planets. *Astrophysical J.* **500**, 428 (1998).
19. J. Papaloizou, D.N.C. Lin. On the tidal interaction between protoplanets and the primordial solar nebula. I – Linear calculation of the role of angular momentum exchange. *Astrophysical J.* **285**, 818 (1984).
20. J.E. Pringle. Accretion discs in astrophysics. *ARA&A* **19**, 137 (1981).
21. I.S. Gradshteyn, I.M. Ryzhik. *Table of Integrals, Series and Products* (Academic Press, 1980) [ISBN-13: 978-0-12-373637-6].
22. P.J. Armitage. *Astrophysics of Planet Formation* (Cambridge Univ. Press, 2010) [ISBN-13: 978-0-511-69136-2].
23. D. Lin, J. Papaloizou. On the tidal interaction between protoplanets and the protoplanetary disk. III. Orbital migration of protoplanets. *Astrophysical J.* **309**, 846 (1986).

Received 28.10.19

*Е.Б. Белджітар, М.Т. Мефтах, З. Малкі*

МОДЕЛЬ ТРАНСПОРТУ КУТОВОГО МОМЕНТУ ПРИ ЕВОЛЮЦІЇ ПРОТОПЛАНЕТНОГО ДИСКА ТА ПОВЕРХНЕВА ГУСТИНА ДИСКА

Ми розглядаємо, яким чином може бути включено припливний ефект від взаємодії протопланетного диска та планети в розрахунки його еволюції з урахуванням в'язкості. Еволюція диска триває під дією як моментів внутрішніх сил в'язкості, так і моментів зовнішніх сил, що виникають внаслідок присутності однієї або кількох планет. Планети мігрують під впливом припливної взаємодії з диском (у режимі міграції II типу). Моменти сил на планеті викликані її гравітаційною взаємодією з хвилями густини, які пов'язані з резонансами Ліндблада в диску. Наша модель спрощує функціональну форму швидкості інжекції кутового моменту $\Lambda(r)$ з метою побудови рівняння еволюції та його розв'язання для диска і протопланети. Функціонал $\Lambda(r)$ залежить від розподілу в диску припливної дисипації, що зосереджена поблизу орбіти протопланети. Ми знайшли аналітичний розв'язок для поверхневої густини диска.

*Ключові слова:* протопланетний диск, момент сил в'язкості, кутовий момент, планета, акреційний диск, поверхнева густина.