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TWO-DIMENSIONAL PAULI EQUATION IN NONCOMMUTATIVE PHASE-SPACE

We study the Pauli equation in a two-dimensional noncommutative phase-space by considering a constant magnetic field perpendicular to the plane. The noncommutative problem is related to the equivalent commutative one through a set of two-dimensional Bopp-shift transformations. The energy spectrum and the wave function of the two-dimensional noncommutative Pauli equation are found, where the problem in question has been mapped to the Landau problem. In the classical limit, we have derived the noncommutative semiclassical partition function for one- and N-particle systems. The thermodynamic properties such as the Helmholtz free energy, mean energy, specific heat and entropy in noncommutative and commutative phase-spaces are determined. The impact of the phase-space noncommutativity on the Pauli system is successfully examined.

Keywords: noncommutative phase-space, Pauli equation, Bopp-shift, semiclassical partition function, thermodynamic properties.

1. Introduction

In a few last years, there has been a growing interest in the study of two-dimensional systems, which have become an active area of researches because of their implications in the nanofabrication technology. Such as in graphene [1, 2] and other materials like Weyl semimetals [3], semiconductor quantum wells, quantum Hall and fractional Hall effects [4, 5], as well the Dirac relativistic oscillator [6], *etc.* However, despite the experimental success, it is very important to understand these systems from a theoretical point of view in which quantum mechanics plays the central role. Motivated by the efforts to understand string theory [7] and black hole models and to describe the quantum gravitation [8–10] using a noncommutative geometry and by trying to have drawn a considerable attention to the phenomenological implications, we concentrate on studying the problem of a nonrelativistic spin-$1/2$ particle in the presence of an electromagnetic field within a two-dimensional noncommutative phase-space. We also mention several articles devoted to the noncommutative geometry, particularly in quantum field theory [11, 12] and quantum mechanics [13, 14].

We present the essential formulas of a noncommutative algebra we need in this work. At very tiny scales (the string scale), the position coordinates do not commute with one another, neither do the momenta.

In the two-dimensional noncommutative phase-space, the operators of coordinates x_j^{nc} and momenta p_j^{nc} satisfy the following Heisenberg-like commutation relations:

$$\begin{aligned} &\left[x_j^{nc}, x_k^{nc}\right] = [x_j, x_k]_\star = i\Theta\epsilon_{jk}, \\ &\left[p_j^{nc}, p_k^{nc}\right] = [p_j, p_k]_\star = i\eta\epsilon_{jk}, \quad (j,k=1,2). \\ &\left[x_j^{nc}, p_k^{nc}\right] = [x_j, p_k]_\star = i\tilde{\hbar}\delta_{jk}, \end{aligned} \tag{1}$$

The noncommutative phase-space can be obtained using the ordinary coordinates x_j and momenta p_j

operators and with replacing the ordinary product by the Moyal ⋆product, which can be used as follows [15]:

$$\mathcal{F}\left(x^{nc},p^{nc}\right)\mathcal{G}\left(x^{nc},p^{nc}\right)=\mathcal{F}\left(x,p\right)\star\mathcal{G}\left(x,p\right)= \\ =e^{\frac{i}{2}\left[\Theta_{ab}\partial_{x_a}\partial_{x_b}+\eta_{ab}\partial_{p_a}\partial_{p_b}\right]}\mathcal{F}\left(x_a,p_a\right)\mathcal{G}\left(x_b,p_b\right), \tag{2}$$

where $\mathcal{F}$, $\mathcal{G}$ are two functions that vary in terms of x, p and are assumed to be infinitely differentiable. The effective Planck constant (deformed Planck constant) is given by [16, 17]

$$\tilde{\hbar}=\hbar\left(1+\frac{\Theta\eta}{4\hbar^2}\right), \tag{3}$$

where $\frac{\Theta\eta}{4\hbar^2}\ll 1$ is the condition of consistency in the usual commutative spacetime quantum mechanics. It is expected to be generally satisfied, since the small parameters Θ and η are of the second order. δ_{ij} is the identity matrix, ϵ_{jk} is the Levi-Civita symbol, with $\epsilon_{12}=-\epsilon_{21}=1$, $\epsilon_{11}=\epsilon_{22}=0$. The quantities Θ, η are the real-valued noncommutative parameters with dimensions of length^2 and momentum^2, respectively, which are assumed to be extremely small. Note that the experimental and theoretical investigations of the noncommutative systems of noncommutativity constants led to obtaining the following upper bound on the value of the noncommutative parameters [17]

$$\Theta\preceq 4.10^{-40}\ \mathrm{m}^2;\quad \eta\preceq 1.76.10^{-61}\ \mathrm{Kg}^2\mathrm{m}^2\mathrm{s}^{-2}. \tag{4}$$

In addition, the recent studies [18–20] revealed that the noncommutative parameters associated with different particles are not the same in noncommutative quantum mechanics.

The set of operators x_i^{nc}, p_j^{nc} is related to the set x_i, p_j in usual quantum mechanics by a non-canonical linear transformation referred to as Bopp-shift as follows [21]:

$$\begin{aligned} x^{nc}&=x-\frac{1}{2\hbar}\Theta p_y; & p_x^{nc}&=p_x+\frac{1}{2\hbar}\eta y,\\ y^{nc}&=y+\frac{1}{2\hbar}\Theta p_x; & p_y^{nc}&=p_y-\frac{1}{2\hbar}\eta x. \end{aligned} \tag{5}$$

The quantum mechanical system will become merely a noncommutative one with the use of Eq. (5) or (2). Let $H\left(x,p\right)$ be the Hamiltonian operator of the usual quantum system, then the corresponding noncommutative Schrödinger equation is given by

$$H\left(x,p\right)\star\psi\left(x,p\right)= \\ =H\left(x_i-\frac{\Theta_{ij}}{2\hbar}p_j,\ p_i+\frac{\eta_{ij}}{2\hbar}x_j\right)\psi=E\psi. \tag{6}$$

Noting that the noncommutative term always can be treated as a perturbation in quantum mechanics.

In the ordinary two-dimensional commutative phase-space, the canonical variables x_j and p_i satisfy the following canonical commutation relations:

$$\begin{aligned} &[x_j,x_k]=0,\\ &[p_j,p_k]=0,\quad (j,k=1,2).\\ &[x_j,p_k]=i\hbar\delta_{jk}, \end{aligned} \tag{7}$$

The paper is organized as follows. The formulation of the two-dimensional noncommutative geometry is briefly outlined in Section I. The exact solution to the two-dimensional noncommutative Pauli equation is presented in Section II. Section III presents the thermodynamic properties of the problem in question, concluding with some remarks.

2. Two-Dimensional Noncommutative Pauli Equation

The time-independent Pauli equation is given by [22]

$$\frac{1}{2m_e}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2\psi+e\phi\psi+\mu_{\mathrm{B}}\boldsymbol{\sigma}\cdot\mathbf{B}\psi=E\psi, \tag{8}$$

where $\psi=\left(\psi_1\,\psi_2\right)^T$ is a two-component spinor, $\mathbf{p}=$ $=i\hbar\nabla$ is the momentum operator, m_e and e are the mass and charge of the electron, and c is the speed of light. As well, $\mu_{\mathrm{B}}=\frac{|e|\hbar}{2mc}$ is the Bohr magneton, $\mathbf{B}$ is the applied magnetic field vector, $\mathbf{A}\left(\mathbf{r},t\right)$ is the vector potential, $\phi\left(\mathbf{r},t\right)$ is the Coulomb potential, and $\boldsymbol{\sigma}=\left(\sigma_x,\sigma_y,\sigma_z\right)$ are the Pauli matrices.

The time-independent Pauli equation in the noncommutative phase-space is given by

$$\left\{\frac{1}{2m_e}\left(\mathbf{p}^{nc}-\frac{e}{c}\mathbf{A}^\star\right)^2+e\phi+\mu_{\mathrm{B}}\boldsymbol{\sigma}\cdot\mathbf{B}\right\}\bar{\psi}=E\bar{\psi}, \tag{9}$$

where $\bar{\psi}$ is the noncommutative spinor wave function. Let the magnetic field $\mathbf{B}$ be oriented along the axis (Oz), which is often referred to as the Landau system. Based on the proposal that noncommutative observables correspond to the commutative ones [23], we have the following deduced noncommutative symmetric gauge:

$$\mathbf{A}^\star=\left(A_x^\star,A_y^\star,A_z^\star\right)=\frac{B}{2}\left(-y^{nc},x^{nc},0\right),\ A_0^\star=e\phi^\star=0. \tag{10}$$

Here, the electron is unbound $\phi = 0$. Using Eq. (10), with $[p_i^{nc}, A_i^\star] = 0$, Eq. (9) becomes

$$\left\{\frac{(\mathbf{p}^{nc})^2}{2m_e} - \frac{e\mathbf{p}^{nc}.\mathbf{A}^\star}{cm_e} + \frac{e^2(\mathbf{A}^\star)^2}{2c^2m_e} + \mu_\mathrm{B}\sigma_z B\right\}\bar{\psi} = E\bar{\psi}, \tag{11}$$

where $\sigma_z = \pm 1$. It is easy to check that

$$(\mathbf{p}^{nc})^2 = p_x^2 + p_y^2 - \frac{\eta}{\hbar}L_z + \frac{\eta^2}{4\hbar^2}(x^2+y^2), \tag{12}$$

$$(\mathbf{A}^\star)^2 = \frac{B^2}{4}\left\{x^2 + y^2 + \frac{\Theta^2}{4\hbar^2}\left(p_x^2+p_y^2\right) - \frac{\Theta}{\hbar}L_z\right\}, \tag{13}$$

$$\mathbf{p}^{nc}\cdot\mathbf{A}^\star = \frac{-B}{2}\left\{\frac{\Theta}{2\hbar}\left(p_x^2+p_y^2\right) + \frac{\eta}{2\hbar}\left(y^2+x^2\right) - \left(1+\frac{\Theta\eta}{4\hbar^2}\right)L_z\right\}, \tag{14}$$

with

$$L_z = (\mathbf{x}\times\mathbf{p})_z = p_y x - p_x y. \tag{15}$$

Using the three-equations (12)–(14) above, the Pauli equation reads

$$\left\{\frac{(p_x^2+p_y^2)}{2\tilde{m}} - \tilde{\omega}L_z + \frac{\tilde{m}\tilde{\omega}^2}{2}(x^2+y^2) + \mu_\mathrm{B}\sigma_z B\right\}\bar{\psi} = E\bar{\psi}, \tag{16}$$

with

$$\tilde{m} = \frac{m_e}{\left(1+\frac{e\Theta B}{4c\hbar}\right)^2}, \quad \tilde{\omega} = \frac{eB\hbar + c\eta}{2c\hbar\tilde{m}\left(1+\frac{e\Theta B}{4c\hbar}\right)},$$
$$\frac{1}{2}\tilde{m}\tilde{\omega}^2 = \frac{1}{2m_e}\left(\frac{e\eta B}{2c\hbar} + \frac{\eta^2}{4\hbar^2} + \frac{e^2B^2}{c^2 4}\right). \tag{17}$$

We assume that $\tilde{\omega}$ is the deformed cyclotron frequency, where, in the $\Theta \to 0$, $\eta \to 0$ limits, $\tilde{\omega}$ is reduced to $\frac{\omega_c}{2} = \frac{eB}{2cm_e}$.

On the other hand, in the case of atomic hydrogen, the electron is bound to a proton by the Coulomb potential $A_0^\star$, which is given by

$$A_0^\star = \frac{e}{4\pi\epsilon_0}\frac{e}{\sqrt{x^2+y^2+\frac{\Theta^2}{4\hbar^2}(p_x^2+p_y^2) - \frac{\Theta}{\hbar}L_z}}. \tag{18}$$

Our system looks like a two-dimensional harmonic oscillator with an additional interaction $(-\tilde{\omega}L_z + \mu_\mathrm{B}\sigma_z B)$. This system corresponds to the Landau level problem, it corresponds to the motion of a charged particle in the xy plane that is subjected to the action of a uniform magnetic field (in the symmetric gauge) oriented along the axis (Oz). This means that the particle is in interaction with its orbital and spin angular momenta. The Hamiltonian from Eq. (16) can be written as

$$H_\mathrm{Pauli}^{nc} = H_{nc}^{ho} - \tilde{\omega}L_z + \mu_\mathrm{B}\sigma_z B. \tag{19}$$

This problem will be solved simply by introducing the operators of creation and annihilation of a harmonic oscillator. Thus, we define

$$a = \frac{1}{2}\sqrt{\frac{\tilde{\omega}}{\hbar}}(x-iy) + \frac{i}{2}\sqrt{\frac{1}{\hbar\tilde{\omega}}}(p_x - ip_y),$$
$$b = \frac{1}{2}\sqrt{\frac{\tilde{\omega}}{\hbar}}(x+iy) + \frac{i}{2}\sqrt{\frac{1}{\hbar\tilde{\omega}}}(p_x + ip_y), \tag{20}$$

with

$$a^\dagger = \frac{1}{2}\sqrt{\frac{\tilde{\omega}}{\hbar}}(x+iy) - \frac{i}{2}\sqrt{\frac{1}{\hbar\tilde{\omega}}}(p_x + ip_y),$$
$$b^\dagger = \frac{1}{2}\sqrt{\frac{\tilde{\omega}}{\hbar}}(x-iy) - \frac{i}{2}\sqrt{\frac{1}{\hbar\tilde{\omega}}}(p_x - ip_y). \tag{21}$$

The above equations satisfy the commutation relations

$$\left[a, a^\dagger\right] = \left[b, b^\dagger\right] = 1. \tag{22}$$

In terms of the ladder operators (20), (21) our Hamiltonian terms can be re-written as

$$L_z = \hbar\left(a^\dagger a - b^\dagger b\right), \tag{23}$$

$$H_{nc}^{ho} = \hbar\tilde{\omega}\left(a^\dagger a + b^\dagger b + 1\right) - \hbar\tilde{\omega}\left(a^\dagger a - b^\dagger b\right) = 2\hbar\tilde{\omega}\left(b^\dagger b + \frac{1}{2}\right). \tag{24}$$

Eigenstates of our Hamiltonian are labeled by the number j of excitation quanta of the oscillator a, and the number n of excitation quanta of the oscillator b,

$$a^\dagger a \mid n, j\rangle = j \mid n, j\rangle \text{ and } b^\dagger b \mid n, j\rangle = n \mid n, j\rangle, \tag{25}$$

where both n and j can take on any positive integer value. Therefore, our Pauli system becomes

$$\left\{\hbar\tilde{\omega}\left(3b^\dagger b - a^\dagger a + 1\right) + \mu_\mathrm{B}\sigma_z B\right\} \mid n, j\rangle = E \mid n, j\rangle, \tag{26}$$

where ± 1 are the eigenvalues of σ_z. Therefore, the energy spectrum of our system (discretely quantized) reads

$$E = \hbar\tilde{\omega}\,(3n - j + 1) \pm \mu_{\mathrm{B}} B. \tag{27}$$

The effect of the phase-space noncommutativity is reduced in $\tilde{\omega}$. Thus, by using Eq. (17), we have

$$E_{n,j}\,(\Theta,\eta) = \frac{eB\hbar + c\eta}{2cm_e}\left(1 + \frac{e\Theta B}{4c\hbar}\right)(3n - j + 1) \pm \mu_{\mathrm{B}} B. \tag{28}$$

The above spectrum is a bit different from that obtained in ref. [24] in the limit of $\eta \to 0$. However, the slight difference is, because the authors considered the magnetic field term proportional to $\frac{1}{2m_e}$. In the limits of $\Theta \to 0$ and $\eta \to 0$, the noncommutative energy spectrum becomes a commutative one, i.e., that of a commutative Landau system [25].

After finding the energy spectrum, we now find the wave function. The time-independent Pauli equation (16) reads

$$\left\{\frac{-\hbar^2}{2\tilde{m}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \tilde{\omega} L_z + {} + \frac{\tilde{\omega}^2\tilde{m}}{2} r^2 + \mu_{\mathrm{B}}\sigma_z B\right\}\bar{\psi} = E\bar{\psi}, \tag{29}$$

with $r = \sqrt{x^2 + y^2}$. Now, we use cylindrical coordinates (r, Φ) to solve the corresponding Pauli equation. The wave function $\bar{\psi}\,(r, \Phi)$ can be given by

$$\bar{\psi}\,(r, \Phi) = R\,(r)\, e^{im\Phi}, \tag{30}$$

knowing that $m = 0, \pm 1, \pm 2, \pm 3, ...$ are the eigenvalues of the orbital angular momentum operator L_z. By replacing Eq. (30) in Eq. (29), we obtain

$$\left\{\frac{-\hbar^2}{2\tilde{m}}\left(\frac{1}{r}\frac{d}{dr} + \frac{d^2}{dr} - \frac{m^2}{r^2}\right) - \tilde{\omega} m + {} + \frac{\tilde{\omega}^2\tilde{m}}{2} r^2 + \mu_{\mathrm{B}}\sigma_z B\right\} R = ER. \tag{31}$$

We can now solve the two-dimensional Pauli equation by assuming a new functional form for $R\,(r)$ which is

$$R\,(r) = \frac{\mathcal{Y}\,(r)}{\sqrt{r}}, \tag{32}$$

and choosing the lower eigenvalue of σ_z. Thus, the resulting equation for $\mathcal{Y}\,(r)$ is

$$\left\{-\frac{1}{2\tilde{m}}\frac{d^2}{dr^2} + \frac{m^2 - \frac{1}{4}}{2\tilde{m}r^2} - m\tilde{\omega} + {} + \frac{\tilde{m}\tilde{\omega}^2 r^2}{2} - \mu_{\mathrm{B}} B\right\}\mathcal{Y} = E\mathcal{Y}. \tag{33}$$

We have used natural units with $\hbar = c = e = 1$ to simplify this part only. Let us find the corresponding wave functions. We can re-write the left-hand side of Eq. (33) in the form $\mathcal{A}_1\mathcal{A}_2$, with

$$\begin{aligned} \mathcal{A}_1 &= \frac{d}{dr} - \left(\frac{|m| + \frac{1}{2}}{r}\right) + \tilde{m}\tilde{\omega} r, \\ \mathcal{A}_2 &= -\frac{d}{dr} - \left(\frac{|m| - \frac{1}{2}}{r}\right) + \tilde{m}\tilde{\omega} r, \end{aligned} \tag{34}$$

where the decomposition holds, if $|m| \leq 0$, which leads to $E_0 \geq 0$. The equality $E_0 = 0$ exists, if and only if the solution of the equation $\mathcal{A}\psi_0\,(r) = 0$. Thus,

$$\left(\frac{d}{dr} - \left(\frac{|m| + \frac{1}{2}}{r}\right) + \tilde{m}\tilde{\omega} r\right)\bar{\psi}_0\,(r) = 0. \tag{35}$$

Hence,

$$\frac{d\bar{\psi}_0\,(r)}{\bar{\psi}_0\,(r)} = \left(\frac{|m| + \frac{1}{2}}{r} - \tilde{m}\tilde{\omega} r\right) dr. \tag{36}$$

We solve the above equation to find

$$\begin{aligned} \bar{\psi}_0\,(r) &= k_0 r^{|m| + \frac{1}{2}} \exp\left[-\frac{1}{2}\tilde{m}\tilde{\omega} r^2\right] = \\ &= k_0 r^{|m| + \frac{1}{2}} \exp\left[-\frac{B + \eta}{4 + \Theta B} r^2\right], \end{aligned} \tag{37}$$

where k_0 is the normalization factor, $\bar{\psi}_0\,(r)$ is square integrable, as the polynomial factor is dominated by the exponential, and the overall integral is convergent. In the limits of $\Theta \to 0$ and $\eta \to 0$, the above result reduces to the commutative one, which corresponds to that of ref. [26] (with a little difference in "4" instead of "2". In our calculations $A \propto \frac{B}{2}$ (the symmetric gauge), whereas the author of the mentioned work considered $A \propto B$), and it is given by

$$\bar{\psi}_0\,(r) = k_0 r^{|m| + \frac{1}{2}} \exp\left[-\frac{B}{4} r^2\right]. \tag{38}$$

3. Semiclassical Partition Function and Thermodynamic Properties in Noncommutative Phase-Space

In the language of the classical treatment, we investigate the thermodynamic properties of the two-dimensional noncommutative Pauli equation using the semiclassical partition function. We initially focus on the calculation of the semiclassical partition function $\mathcal{Z}$. Our studied system is semiclassical, where the Hamiltonian is split as follows:

$$\mathcal{H}^{2D}_{\text{Pauli}} = H_{\text{classic}} + H_{ncl,\sigma}, \tag{39}$$

with $H_{ncl,\sigma} = \mu_{\text{B}}\sigma_z B$. Therefore, the noncommutative partition function is separable into two independent parts, as followed from our work Ref. [27]:

$$\mathcal{Z} = Z_{cl} Z_{ncl}, \tag{40}$$

where Z_{ncl} is the non-classical part of the partition function. To study our non-classical partition function, we assume that the passage between noncommutative classical mechanics and noncommutative quantum mechanics can be realized through the following generalized Dirac quantization condition [28, 29]:

$$\{f, g\} = \frac{1}{i\hbar}[F, G], \tag{41}$$

where F, G stand for the operators associated with classical observables f, g, and $\{,\}$ stands for the Poisson bracket. Using the condition above, we obtain from Eq. (1) that

$$\begin{aligned} &\left\{x^{nc}_j, x^{nc}_k\right\} = \Theta_{jk},\\ &\left\{p^{nc}_j, p^{nc}_k\right\} = \eta_{jk},\\ &\left\{x^{nc}_j, p^{nc}_k\right\} = \delta_{jk} + \tfrac{\Theta_{jl}\eta_{lk}}{4\hbar^2} = \delta_{jk}. \end{aligned} \tag{42}$$

It is important to mention that, in terms of the classical limit, $\frac{\Theta\eta}{4\hbar^2} \ll 1$ (check ref. [29]). Thus, $\left\{x^{nc}_j, p^{nc}_k\right\} = \delta_{jk}$. Now based on the proposal that the noncommutative observables F^{nc} corresponding to the commutative one $F(x,p)$ can be defined by [23, 30, 31]

$$F^{nc} = F(x^{nc}, p^{nc}). \tag{43}$$

For non-interacting particles, the classical partition function in the noncommutative phase-space for N particles is written as follows [27, 28]:

$$Z_{cl} = \frac{1}{N!(2\pi\tilde{\hbar})^{2N}} \int e^{-\beta H_{\text{classic}}} d^{2N}x^{nc} d^{2N}p^{nc}. \tag{44}$$

Let $\frac{1}{N!}$ be Gibbs's correction factor considered due to accounting for the indistinguishability, which means that there are $N!$ ways of arranging N particles at N sites; $\frac{1}{\tilde{\hbar}^2}$ is the appropriate factor that makes the volume of the noncommutative phase-space dimensionless; β is defined as $\frac{1}{K_{\text{B}}T}$, and K_{B} is the Boltzmann constant.

Using Eq. (40), we may derive the important thermodynamic quantities such as the Helmholtz free energy

$$F = -\frac{1}{\beta}\ln\mathcal{Z}, \tag{45}$$

and the average energy

$$U \equiv N\langle\varepsilon\rangle = -\frac{\partial}{\partial\beta}\ln\mathcal{Z}, \tag{46}$$

where ε is the mean energy, which is given by $-\frac{\partial}{\partial\beta}\ln\mathcal{Z}_1$. The specific heat (heat capacity) is

$$C_v = \frac{\partial}{\partial T}\langle\varepsilon\rangle, \tag{47}$$

and the entropy reads

$$S = -\frac{\partial F}{\partial T} = -\frac{K_{\text{B}}\ln\mathcal{Z}}{\beta^2} + \frac{1}{\beta}\frac{\partial}{\partial T}\ln\mathcal{Z}. \tag{48}$$

Now for a single particle, the noncommutative classical partition function is given by

$$Z_{cl,1} = \frac{1}{\tilde{\hbar}^2}\int e^{-\beta H_{\text{classic}}(x,p)} d^2x^{nc} d^2p^{nc}, \tag{49}$$

where d^2 is a shorthand notation serving as a reminder that the x and p are vectors in two-dimensional phase-space. The relation between Eqs. (44) and (49) is given by the formula

$$Z_{cl} = \frac{(Z_{cl,1})^N}{N!}. \tag{50}$$

From Eq. (5), we simply have

$$d^2x^{nc}d^2p^{nc} = \left(1 - \frac{\Theta\eta}{4\hbar^2}\right) d^2x d^2p. \tag{51}$$

We have also $\tilde{\hbar} \sim \triangle x^{nc} \triangle p^{nc}$, which is given by

$$\tilde{\hbar}^2 = \hbar^2\left(1 + \frac{\Theta\eta}{2\hbar^2}\right) + \mathcal{O}(\Theta^2\eta^2). \tag{52}$$

Following Eq. (49), we now present the single-particle noncommutative classical partition function as

$$Z_{cl,1}=\frac{1}{\tilde{\hbar}^2}\int e^{-\beta\left[\frac{p_x^2+p_y^2}{2\tilde{m}}-\tilde{\omega}L_z+\frac{\tilde{m}\tilde{\omega}^2}{2}\left(x^2+y^2\right)\right]}d^2x^{nc}d^2p^{nc}. \tag{53}$$

We should mention again (as we emphasized in our previous work [27]) that it is always possible within the classical limit to factorize our Hamiltonian into momentum and position terms. Thus, we have

$$Z_{cl,1}=\frac{1}{\tilde{\hbar}^2}\int e^{-\beta\frac{(p_x^2+p_y^2)}{2\tilde{m}}}\times$$
$$\times\, e^{-\beta\frac{\tilde{m}\tilde{\omega}^2}{2}(x^2+y^2)}e^{\beta\tilde{\omega}L_z}d^2p^{nc}d^2x^{nc}. \tag{54}$$

Using the same method used in our previous work [27], which depends on expanding exponentials containing $\tilde{\omega}$, and considering terms up to the second order in $\tilde{\omega}$, we find

$$Z_{cl,1}=\frac{1}{\tilde{\hbar}^2}\int e^{-\frac{\beta}{2}\left[\frac{p_x^2+p_y^2}{\tilde{m}}\right]}\left(1+\beta\tilde{\omega}L_z+\frac{1}{2}\beta^2\tilde{\omega}^2L_z^2\right)\times$$
$$\times\left(1-\beta\tilde{\omega}^2\frac{\tilde{m}}{2}(x^2+y^2)\right)d^2p^{nc}d^2x^{nc}. \tag{55}$$

Knowing that

$$\left(1-\frac{\Theta\eta}{4\hbar^2}\right)\left(1-\frac{\Theta\eta}{2\hbar^2}\right)=1-\frac{3\Theta\eta}{4\hbar^2}+\mathcal{O}(\Theta^2+\eta^2), \tag{56}$$

we have the convenient expression of $Z_{cl,1}$

$$Z_{cl,1}=\frac{1-\frac{3\Theta\eta}{4\hbar^2}}{h^2}\int e^{-\frac{\beta}{2}\left[\frac{p_x^2+p_y^2}{\tilde{m}}\right]}d^2pd^2x+$$
$$+\frac{\left(1-\frac{3\Theta\eta}{4\hbar^2}\right)}{h^2}\beta\tilde{\omega}\int e^{-\frac{\beta}{2}\left[\frac{p_x^2+p_y^2}{\tilde{m}}\right]}L_zd^2pd^2x+$$
$$+\frac{\left(1-\frac{3\Theta\eta}{4\hbar^2}\right)}{h^2}\beta^2\tilde{\omega}^2\int e^{-\frac{\beta}{2}\left[\frac{p_x^2+p_y^2}{\tilde{m}}\right]}L_z^2d^2pd^2x-$$
$$-\frac{\left(1-\frac{3\Theta\eta}{4\hbar^2}\right)}{h^2}\beta\tilde{\omega}^2\int e^{-\frac{\beta}{2}\left[\frac{p_x^2+p_y^2}{\tilde{m}}\right]}(x^2+y^2)d^2pd^2x. \tag{57}$$

On the right-hand side of the above equation, the second integral goes to zero, the third and fourth integrals cancel each other. Then, by using the known integral of the Gaussian function $\int e^{-ax^2}dx=\sqrt{\frac{\pi}{a}}$, we find

$$Z_{cl,1}=\frac{1-\frac{3\Theta\eta}{4\hbar^2}}{h^2}\int d^2xe^{-\frac{\beta}{2}\left[\frac{p_x^2+p_y^2}{\tilde{m}}\right]}d^2p=$$
$$=\frac{l^2\left(1-\frac{3\Theta\eta}{4\hbar^2}\right)}{\Lambda^2\left(1+\frac{e\Theta B}{4c\hbar}\right)^2}, \tag{58}$$

with $\int d^2x=l^2$, $\Lambda=h\left(2\pi m_eK_\mathrm{B}T\right)^{-\frac{1}{2}}$ are the area and the thermal de Broglie wavelength, respectively.

We also propose another method based on the substitution of variables with the Jacobian matrix to compute integral (53), explained in Appendix A, which gives the same results.

The non-classical partition function for N particles is given by

$$Z_{ncl}=Z_{ncl,1}^N=\left(\sum_{\sigma_z=\pm1}e^{\beta\mu_\mathrm{B}\sigma_zB}\right)^N=2^N\cosh^N\left(\beta\mu_BB\right). \tag{59}$$

It is worth to note that, for a canonical ensemble that is classical and discrete, the canonical partition function is defined using the summation, as in the case of $H_{ncl,\sigma}$. But, for a canonical ensemble that is classical and continuous, the canonical partition function is defined using the integration.

Finally, the Pauli partition function (40) for a system of N particles in the two-dimensional noncommutative phase-space is

$$\mathcal{Z}=\frac{2^Nl^{2N}}{\Lambda^{2N}N!}\frac{\left(1-\frac{3\Theta\eta}{4\hbar^2}\right)^N}{\left(1+\frac{eB\Theta}{8c\hbar}\right)^{2N}}\cosh^N\left(\beta\mu_\mathrm{B}B\right). \tag{60}$$

In the vanishing limit of the noncommutativity, i.e. $\Theta\to0$, $\eta\to0$, the expression for $\mathcal{Z}$ reduces to that of the usual commutative phase-space, which is

$$\mathcal{Z}=\frac{2^Nl^{2N}}{\Lambda^{2N}N!}\cosh^N\left(\beta\mu_\mathrm{B}B\right). \tag{61}$$

Following relations (45, 46, 47, 48), and (60), we can express the thermodynamic quantities in the noncommutative phase-space. Thus, we have

$$F^{nc}=-\frac{N}{\beta}\ln\left[\frac{2l^2}{\Lambda^2}\frac{\left(1-\frac{3\Theta\eta}{4\hbar^2}\right)}{\left(1+\frac{eB\Theta}{8c\hbar}\right)^2}\cosh\left(\beta\mu_\mathrm{B}B\right)\right]+\frac{1}{\beta}\ln N!, \tag{62}$$

where $\ln N! \approx N \ln N - N$,

$$S^{nc} = \frac{K_{\rm B} N}{\beta^2} \Bigg\{ 1 + \frac{\ln N!}{N} + \beta\mu_{\rm B} B \tanh(\beta\mu_{\rm B} B) -$$

$$- \ln\left[\frac{2l^2}{\Lambda^2} \frac{\left(1 - \frac{3\Theta\eta}{4\hbar^2}\right)}{\left(1 + \frac{eB\Theta}{8c\hbar}\right)^2} \cosh(\beta\mu_{\rm B} B)\right]\Bigg\}, \quad (63)$$

$$U^{nc} = N\left[\frac{1}{\beta} - \mu_{\rm B} B \tanh(\beta\mu_{\rm B} B)\right], \quad (64)$$

$$\langle\varepsilon^{nc}\rangle = \frac{1}{\beta} - \mu_{\rm B} B \tanh(\beta\mu_{\rm B} B), \quad (65)$$

$$C_v^{nc} = -K_{\rm B}\left[\frac{1}{\beta^2} + \frac{(\mu_{\rm B} B)^2}{\cosh^2(\beta\mu_{\rm B} B)}\right]. \quad (66)$$

In the vanishing limit of the noncommutativity, the result of this paper will be reduced to that of the commutative phase space. Namely,

$$F = -\frac{N}{\beta} \ln\left[\frac{2l^2}{\Lambda^2} \cosh(\beta\mu_{\rm B} B)\right] + \frac{1}{\beta} \ln N!, \quad (67)$$

as well

$$S = \frac{K_{\rm B} N}{\beta^2} \Bigg\{ 1 + \frac{\ln N!}{N} + \beta\mu_{\rm B} B \tanh(\beta\mu_{\rm B} B) -$$

$$- \ln\left[\frac{2l^2}{\Lambda^2} \cosh(\beta\mu_{\rm B} B)\right]\Bigg\}. \quad (68)$$

Through the higher derivatives, we can go deeper and calculate the rest of the thermodynamic properties, using the obtained partition function, such as the temperature T, pressure P, magnetization $\langle M\rangle$, and chemical potential μ.

4. Conclusion

In this work, we have discussed the problem of a charged particle with a spin that interacts with an electromagnetic field and moves in a two-dimensional noncommutative phase-space, by considering a constant magnetic field perpendicular to the plane. The approach that we have took to map the noncommutative problem to the equivalent commutative one is the Bopp-shift transformation. We found the energy spectrum, which is discretely quantized and the wave function of the two-dimensional noncommutative Pauli equation. Here, we can say that we successfully examined the influence of the noncommutativity on the problem in question. In addition, according to Eq. (17), we can see an emerge of a modified frequency $\tilde{\omega}$, which represents the effect of the noncommutativity on the cyclotron frequency. In the limits $\Theta \to 0$ and $\eta \to 0$, the noncommutative results reduce to those for the usual commutative phase-space.

Furthermore, within the classical treatment, some classical statistical quantities are determined in the two-dimensional noncommutative phase-space using a semiclassical partition function from the Pauli system of the one-particle and N-particle systems in two dimensions, all according to the canonical ensemble theory. It is shown that the Helmholtz free energy and entropy were significantly affected by the noncommutativity of the phase space. In contrast, the specific heat and average energy showed no dependence on the noncommutativity.

Note that result (58) is of the classical Maxwell–Boltzmann gas, as this happens in the classical calculation of the Landau problem. On the other hand, the quantum partition function for the Landau problem represents the de Haas–van Alphen effect.

The results of the present work can be used to expand the study onto a possible generalization to make consideration of *anyons*, i.e., particles with arbitrary non-integer spin, which can exist in a two-dimensional space.

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APPENDIX A.
Integration using the substitution of multiple variables with a Jacobian matrix

Here is a method based on the substitution of multiple variables with the determinant of the Jacobian matrix to compute integral (53).

The substitution of multiple variables is as follows:

$$\begin{cases} x = x, \\ y = y, \\ P_x = p_x + \tilde{m}\tilde{\omega} y, \\ P_y = p_y - \tilde{m}\tilde{\omega} x, \end{cases} \quad (A1)$$

where integral (53) is

$$\text{Int} = \frac{1}{\hbar^2} \int e^{-\beta\left[\frac{p_x^2+p_y^2}{2\tilde{m}} - \tilde{\omega} L_z + \frac{\tilde{m}\tilde{\omega}^2}{2}(x^2+y^2)\right]} d^2x^{nc} d^2p^{nc} =$$

$$= \frac{1 - \frac{3\Theta\eta}{4\hbar^2}}{h^2} \int e^{-\beta\left[\frac{p_x^2+p_y^2}{2\tilde{m}} - \tilde{\omega} L_z + \frac{\tilde{m}\tilde{\omega}^2}{2}(x^2+y^2)\right]} \times$$

$$\times\, dxdydp_xdp_y = \frac{1-\frac{3\Theta\eta}{4\hbar^2}}{h^2}\int e^{-\frac{\beta}{2\tilde{m}}\left[\left(P_x^2+P_y^2\right)\right]}\times$$
$$\times\left|\mathrm{Det\,J}\left(x,y,P_x,P_y\right)\right| dxdydP_xdP_y. \quad \text{(A2)}$$

The corresponding Jacobian matrix is

$$\mathrm{J}\left(x,y,P_x,P_y\right) = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial p_x} & \frac{\partial x}{\partial p_y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial p_x} & \frac{\partial y}{\partial p_y} \\ \frac{\partial P_x}{\partial x} & \frac{\partial P_x}{\partial y} & \frac{\partial P_x}{\partial p_x} & \frac{\partial P_x}{\partial p_y} \\ \frac{\partial P_y}{\partial x} & \frac{\partial P_y}{\partial y} & \frac{\partial P_y}{\partial p_x} & \frac{\partial P_y}{\partial p_y} \end{bmatrix} =$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \tilde{m}\tilde{\omega} & 1 & 0 \\ -\tilde{m}\tilde{\omega} & 0 & 0 & 1 \end{bmatrix}. \quad \text{(A3)}$$

The determinant of the Jacobian matrix J is

$$\mathrm{DetJ}\left(x,y,P_x,P_y\right) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \tilde{m}\tilde{\omega} & 1 & 0 \\ -\tilde{m}\tilde{\omega} & 0 & 0 & 1 \end{vmatrix} =$$
$$= 1\begin{vmatrix} 1 & 0 & 0 \\ \tilde{m}\tilde{\omega} & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \quad \text{(A4)}$$

Therefore, integral (A1) becomes

$$= \frac{1-\frac{3\Theta\eta}{4\hbar^2}}{h^2}\int e^{-\frac{\beta}{2\tilde{m}}\left[\left(P_x^2+P_Y^2\right)\right]}dP_xdP_y\int dxdy =$$
$$= \frac{1-\frac{3\Theta\eta}{4\hbar^2}}{h^2\left(1+\frac{e\Theta B}{4c\hbar}\right)^2}\frac{2\pi m_e}{\beta}\int dxdy. \quad \text{(A5)}$$

By using the known integral of the Gaussian function $\int e^{-a\left(x^2+y^2\right)}dx = \frac{\pi}{a}$, with $\int d^2x = l^2$, $\Lambda = h\left(2\pi m_e K_{\mathrm{B}}T\right)^{-\frac{1}{2}}$, we find

$$\mathrm{Int} = \frac{l^2\left(1-\frac{3\Theta\eta}{4\hbar^2}\right)}{\Lambda^2\left(1+\frac{e\Theta B}{4c\hbar}\right)^2}. \quad \text{(A6)}$$

1. A.D. Martino, L. Dell'Anna, R. Egger. Magnetic confinement of massless Dirac fermions in graphene. *Phys. Rev. Lett.* **98**, 066802 (2007).
2. K. Novoselov, A. Geim, S. Morozov *et al.* Two-dimensional gas of massless Dirac fermions in graphene. *Nature* **438**, 197 (2005).
3. V. Arjona, E.V. Castro, M.A.H. Vozmediano. Collapse of Landau levels in Weyl semimetals. *Phys. Rev. B* **96**, 081110 (R) (2017).
4. Y. Zhang, Y.W. Tan, H.L. Stormer, P. Kim. Experimental observation of the quantum Hall effect and Berry's phase in graphene. *Nature* **438**, 201 (2005).
5. K. Bolotin, F. Ghahari, M.D. Shulman, H.L. Stormer, P. Kim. Observation of the fractional quantum Hall effect in graphene. *Nature* **462**, 196 (2009).
6. F.M. Andrade, E.O. Silva, M.M. Ferreira, jr, E.C. Rodrigues. On the κ-Dirac oscillator revisited. *Phys. Lett. B* **731**, 327 (2014).
7. N. Seiberg, E. Witten. String theory and noncommutative geometry. *J. High. Energy. Phys.* **9**, 032 (1999).
8. D.M. Gingrich. Noncommutative geometry inspired black holes in higher dimensions at the LHC. *J. High. Energ. Phys.* **2010**, 022 (2010).
9. J.M. Gracia-Bondia. Notes on Quantum Gravity and Noncommutative Geometry. *Lect. Notes. Phys.* **807**, 3 (2010).
10. P. Nicolini. Noncommutative black holes, the final appeal to quantum gravity: A review. *Int. J. Mod. Phys. A* **24**, 1229 (2009).
11. I. Haouam. On the Fisk–Tait equation for spin-3/2 fermions interacting with an external magnetic field in noncommutative space-time. *J. Phys. Stud.* **24**, 1801 (2020).
12. R.J. Szabo. Quantum field theory on noncommutative spaces. *Phys. Rep.* **378**, 207 (2003).
13. I. Haouam. Analytical solution of $(2+1)$ dimensional Dirac equation in time-dependent noncommutative phase-space. *Acta Polytech.* **60**, 111 (2020).
14. I. Haouam. On the noncommutative geometry in quantum mechanics. *J. Phys. Stud.* **24**, 2002 (2020).
15. I. Haouam. The non-relativistic limit of the DKP equation in non-commutative phase-space. *Symmetry.* **11**, 223 (2019).
16. T. Harko, S. Liang. Energy-dependent noncommutative quantum mechanics. *Eur. Phys. J. C* **79**, 300 (2019).
17. O. Bertolami, J.G. Rosa, C.M.L. De Aragao, P. Castorina, D. Zappala. Noncommutative gravitational quantum well. *Phys. Rev. D* **72**, 025010 (2005).
18. P.M. Ho, H.C. Kao. Noncommutative quantum mechanics from noncommutative quantum field theory. *Phys. Rev. Lett.* **88,** 151602 (2002).
19. A. Stern. Noncommutative point sources. *Phys. Rev. Lett.* **100**, 061601 (2008).
20. A. Saha, S. Gangopadhyay, S. Saha. Noncommutative quantum mechanics of a harmonic oscillator under linearized gravitational waves. *Phys. Rev. D* **83**, 025004 (2011).
21. I. Haouam, L. Chetouani. The Foldy–Wouthuysen transformation of the Dirac equation in noncommutative phase-space. *J. Mod. Phys.* **9**, 2021 (2018).
22. W. Greiner. *Quantum Mechanics: An Introduction* (Springer, 2001) [ISBN: 978-3-540-67458-0].
23. M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu. Hydrogen atom spectrum and the Lamb shift in noncommutative QED. *Phys. Rev. Lett.* **86**, 2716 (2001).
24. A. Halder, S. Gangopadhyay. Pauli equation on noncommutative plane and the Seiberg–Witten map. *Mod. Phys. Lett. A* **31**, 1650087 (2016).
25. A. Khare, J. Maharana. Supersymmetric quantum mechanics in one, two and three dimensions. *Nucl. Phys. B* **244**, 409 (1984).
26. R.K. Singh. On the classical dynamics of charged particle in special class of spatially non-uniform magnetic field. Indian. *J. Phys.* **93**, 503 (2019).
27. I. Haouam. On the three-dimensional Pauli equation in noncommutative phase-space. *Acta Polytech.* **61**, 230 (2021).
28. M. Najafizadeh, S. Mehdi. Thermodynamics of classical systems on noncommutative phase space. *Chin. J. Phys.* **51**, 94 (2013).

29. W. Gao-Feng, L. Chao-Yun, L. Zheng-Wen, Q. Shui-Jie, F. Qiang. Classical mechanics in non-commutative phase space. *Chin. Phys. C* **32**, 338 (2008).
30. L. Kang, N. Chamoun. Hydrogen atom spectrum in non-commutative phase space. *Chin. Phys. Lett.* **23**, 1122 (2006).
31. S. Biswas. Bohr–van Leeuwen theorem in non-commutative space. *Phys. Lett. A* **381**, 3723 (2017).

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ДВОВИМІРНЕ РІВНЯННЯ ПАУЛІ
В НЕКОМУТАТИВНОМУ ФАЗОВОМУ ПРОСТОРІ

Розглянуто рівняння Паулі в двовимірному некомутативному фазовому просторі в присутності постійного магнітного поля, перпендикулярного площині. Некомутативну задачу зведено до еквівалентної комутативної шляхом двовимірних перетворень зі зсувом Боппа. Знайдено спектр енергії і хвильову функцію для двовимірного некомутативного рівняння Паулі в разі, коли задача може бути перетворена в задачу Ландау. У класичній границі знайдено некомутативні напівкласичні статистичні суми для одно- і N-частинкових систем. Розраховано такі термодинамічні величини, як вільна енергія Гельмгольца, середня енергія, теплоємність і ентропія в некомутативному і комутативному фазових просторах. Вивчено вплив некомутативності фазового простору на систему Паулі.

Ключові слова: некомутативний фазовий простір, рівняння Паулі, зсув Боппа, напівкласична статистична сума, термодинамічні властивості.