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UDC 539

ON THE ANALYTICAL SOLVING OF THE TER-MARTIROSIAN–SKORNYAKOV EQUATION FOR THREE PARTICLES AT NEGATIVE ENERGIES

Simple analytical expression for the solution of the Ter-Martirosian–Skornyakov equation for three particles at a negative energy has been obtained.

K e y w o r d s: Ter-Martirosian–Skornyakov equation, Mellin transformation.

1. Analytical Solution of the Equation

The analytical expression for the solution of the Ter-Martirosian–Skornyakov equation [1] has the form

$$\varphi(p) = \varphi_0(p) + \mu \int\limits_0^\infty \frac{1}{\alpha} - \sqrt{\lambda^2 + \frac{3}{4}q^2} \times$$

$$\times \ln \frac{p^2 + p\rho + q^2 + \lambda^2}{p^2 - p\rho + q^2 + \lambda^2}\, \varphi(\rho) d\rho. \quad (1)$$

Our main interest is in the case where $\lambda^2 \equiv \alpha^2 - \frac{3}{4}p_0^2$ contains large p_0^2 so that $\lambda^2 < 0$. Let us transform this equation into an equation with $\lambda^2 > 0$ and then perform the analytical continuation with negative λ^2. Now, we substitute

$$\rho = \lambda \frac{x^2 - 1}{x\sqrt{3}}, \quad q = \lambda \frac{y^2 - 1}{y\sqrt{3}}. \quad (2)$$

Then we obtain

$$\lambda^2 + \frac{3}{4}q^2 = \lambda^2 \frac{(y^2+1)^2}{4y^2}, \quad d\rho = \lambda \frac{1+\gamma^2}{\sqrt{3}y^2} dy, \quad (3)$$

$$(0,\infty) \to (1,\infty),$$

and the equation takes the form

$$\varphi(x) = \varphi_0(x) -$$

$$- \frac{2\mu}{\sqrt{3}} \int\limits_1^\infty \ln \frac{(x^2 + xy + y^2)(x^2y^2 - xy + 1)}{(x^2 - xy + y^2)(x^2y^2 + xy + 1)} \times$$

$$\times \frac{1}{1 - \frac{\alpha}{\lambda}\frac{y}{1+y^2}} \varphi(y) \frac{dy}{y}. \quad (4)$$

Consider the case where $\varphi_0(x)$ with $x < 1$ is continued analytically:

$$\varphi_0\left(\frac{1}{x}\right) = -\varphi_0(x). \quad (5)$$

Then the solution φ can be also continued. The equation becomes simplified:

$$\varphi(x) = \varphi_0(x) - \frac{2\mu}{3} \int\limits_0^\infty \ln \frac{x^2 + xy + y^2}{x^2 - xy + y^2} \times$$

$$\times \frac{1}{1 - \frac{\alpha}{\lambda}\frac{y}{1-y^2}} \varphi(y) \frac{dy}{y}. \quad (6)$$

Using now the Mellin transformation, we get

$$\Phi(\xi) = \int\limits_0^\infty \varphi(x) x^{i\xi - 1} dx,$$

$$\varphi(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\Phi(\xi)x^{-i\xi}d\xi. \quad (7)$$

The equation for $\varphi(\xi)$ takes the form

$$\Phi(\xi) = \Phi_0(\xi) + L(\xi)\int\limits_{-\infty}^{+\infty}M(\xi-\eta))\Phi(\eta)d\eta, \quad (8)$$

where

$$L(\xi) = -\frac{2\mu}{\sqrt{3}}\int\limits_0^{\infty}\ln\frac{1+t+t^2}{1-t+t^2}t^{-\xi-1}dt,$$

$$M(\xi) = \frac{1}{2\pi}\int\limits_0^{\infty}y^{i\xi}\frac{1}{1-\frac{\alpha}{\lambda}\frac{y}{1+y^2}}\frac{dy}{y}. \quad (9)$$

Both integrals can be calculated using the contour integration. In the first one, it is necessary to integrate by parts. It is worth to mention that

$$L(\xi) = -\frac{2\mu}{3}\frac{2\pi}{\xi}\frac{\sinh\frac{\pi\xi}{6}}{\cosh\pi\xi/2} \quad (10)$$

is known as the Danilov factor [2], and

$$M(\xi) = \delta(\xi_)+M_1(\xi), \quad (11)$$

where M_1 is a smooth function which depends on λ analytically on the plane with the cut from $-\alpha^2$ to $+\infty$ (see [4]). In this way, we reduce the equation to the form [3, 4]

$$[1-L(\xi)]\Phi(\xi) = \Phi_0(\xi) + $$
$$+ L(\xi)\int\limits_{-\infty}^{+\infty}M_1(\xi-\eta)\Phi(\eta)d\eta. \quad (12)$$

In the quartet case, $\mu = \frac{1}{\pi}$, and, in the doublet case, $\mu = -\frac{2}{\pi}$, and the factor $1-L(\xi)$ does not vanish. Dividing by it, we obtain the Fredholm equation with the smooth kernel. In this kernel, we must do the analytic continuation in the negative λ^2.

It should be noted that the Mellin transformation can be useful also for other models, in particular, for the Yamaguchi model.

It should be also noted that the integral for $M_1(\xi)$ is

$$M_1(\xi) = \frac{1}{1-\exp -2\pi\xi}\,\frac{\alpha}{\lambda}\,\frac{1}{2\sqrt{\frac{\alpha^2}{\lambda^2}-1}}\times$$
$$\times\left[\left(\frac{\alpha}{\lambda}+\sqrt{\frac{\alpha^2}{\lambda^2}-1}\right)^{i\xi}-\left(\frac{\alpha}{\lambda}-\sqrt{\frac{\alpha^2}{\lambda^2}-1}\right)^{i\xi}\right]. \quad (13)$$

Here, it is possible to pass to the negative λ^2. The corresponding $M_1(\xi)$ will be rapidly decreasing. Thus, the recipe for the energy above the threshold is as follows:

1. To solve Eq. (12) for $\Phi(\xi)$ supposing λ^2 to be negative.
2. To construct the answer using the formula

$$\varphi(p) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\Phi(\xi)\left(\frac{\rho\sqrt{\xi}}{2\lambda}+\sqrt{\frac{\xi}{4\lambda^2}p+1}\right)^{i\xi}d\xi. \quad (14)$$

If λ is negative, the integrand (the second factor) increases, if $|\xi| \to \infty$. However, apparently Eq. (12) implies in this case that the solution $\Phi(xi)$ decreases quickly. Therefore, the integrals must be convergent. This method is suitable for other kernels with logarithm.

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ЩОДО АНАЛІТИЧНОГО РОЗВ'ЯЗУВАННЯ
РІВНЯННЯ ТЕР-МАРТИРОСЯНА–СКОРНЯКОВА
ДЛЯ ТРЬОХ ЧАСТИНОК
ПРИ НЕГАТИВНИХ ЕНЕРГІЯХ

Отримано простий аналітичний вираз для розв'язку рівняння Тер-Мартиросяна–Скорнякова для трьох частинок при негативній енергії.

Ключові слова: рівняння Тер-Мартиросяна–Скорнякова, перетворення Мелліна.