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EFFECTS OF NONLINEARITIES IN PHYSICS AND DEMOGRAPHY

Nonlinearities appear in almost all systems. Earlier, we focused on those in plasmas, ionospheric scattering, and the world population. As turned out, the estimate of the population growth made in 1974 is in astonishing agreement with the United Nations estimates and agrees with our present data to within 2%. A particularly important role, both for the population evolution and wave interaction in plasmas, is played by non-Markovian effects (effects depending on the past time). For the population growth, this occurs due to a delay of one generation in the set of population limiting actions, while, for plasmas, it is caused by nonlinear frequency shifts.

Keywords: descriptive words, fundamental nonlinearities in nature, explosive instabilities, fusion research, demographic research.

1. Introduction

We are here focusing on nonlinear effects in physics, mainly plasmas, and demography, mainly populations. Such effects could be both of a periodic and explosive nature [1]. In both contexts, it has turned out that many authors leave out important nonlinear effects, while we have looked at both general basic equations fulfilling resonance conditions in both wave vectors $\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$ and frequency $\omega_1 = \omega_2 + \omega_3$ with unspecified coupling factors. We also derived coupling factors for special cases, mainly for plasmas. An important nonlinear effect here can be a nonlinear frequency shift, which may detune the resonance in frequency. In the case of nonlinear (explosive) instability, we also had to include cubic nonlinearities, giving nonlinear frequency shifts (energy conserving) or cubic nonlinear damping (dissipative). Of course, three-wave interactions would, in a more general case, couple further in $\mathbf{k}$-space, leading to a turbulent state with many excited waves. Initially, we focused on homogeneous plasmas, where, in general, the wave with the largest mode number (the pump wave) gave energy to two waves with smaller wave numbers. However, when our activity after 1976 became more directed toward the nuclear fusion [2–29], we started to look at inhomogeneous systems, where, for drift waves, the wave with an intermediate magnitude of mode number became the pump wave [8]. This introduces a problem for the wave dynamics, since we are looking at a system which is finite in space, and wave cascades toward larger wave lengths could cause a pileup of the wave energy at wavelengths of the system size. The transport, in the simplest case, could be described by the formula:

$$\chi = \gamma/k_x^2. \tag{1}$$

Here, γ is the growth rate of the wave, and k_x is the radial propagation vector. Thus, waves with long wavelengths give very large transport. To stop this pileup at very long wavelengths, which would give much larger transport than observed, we need a mechanism that absorbs the cascade toward longer wavelengths. As we have found, the strongest candidate for this is flowshear, which tears apart the largest eddies [17]. The generation of flowshear is a nonlinear effect. Our present concern is nonlinear effects in physics and demography. In both these fields, authors seem to forget important nonlinearities. These questions turn out to vary with the degree of detail we need. A very important observation was that of "profile consistency" or "profile resilience", first pointed out in Ref. 2. This observation was for density and temperature profiles in tokamaks and means that such profiles are often surprisingly insensitive to the exact location of sources.

For the population evolution, we used the equation:

$$\frac{\partial n}{\partial t} = \alpha n^2 + \beta n - \gamma \alpha n^3 (t - \tau), \tag{2}$$

where n is the number of people, t is time, and τ is the time delay.

Since β is ignorable at the levels of interest, we start by taking $\gamma = 0.1$ corresponding to the saturation level 10 billions (common estimate) in the absence of time delay. The numerical values $\alpha = 0.0048$ and $\beta = -0.0003$ were used to fit the evolution to the world population up to the year 1974. This is the model used in Ref. 3.

A way of interpreting the similarity between the wave interaction in plasmas and the population growth is in terms of profile resilience (or profile consistency) [2]. This is a phenomenon seen in plasmas [2] and shows that the density and temperature profiles are surprisingly insensitive to the exact location of sources. Both the transport equations in plasmas and the population growth are now described by first-order differential equations. In plasmas, this is mainly in space. But, for populations, it is rather in time. However, of course, we generally have both space and time variations in both cases. A discussion of this was given in Ref. 9. An important point to observe is that, in a typical fusion plasma, the particle velocity is around 10^8 m/s, while the fluid velocity is around 10^3 m/s. Thus, we need to include nonlinear frequency shifts in kinetic theory [5, 12–15], but we can safely use quasilinear theory to calculate the transport in fluid theory [22–25, 27, 29]. We note that particles and/or heat pinches can contribute to the profile resilience in a plasma [18], while the fact that the birth rates increase during wars may contribute to the profile resilience in human populations.

2. Basic Wave Interaction

Despite the fact that the population growth is described by a single equation [3], its variation is easily described in terms of three-wave interactions. One way to do this is to take a case where two waves are equal. However, the three-wave interaction can also be described as the dynamics of a nonlinear pendulum [7].

In our work on the transport due to drift waves, the basic equations for η_i modes are:

$$\frac{dn}{dt} + \mathbf{v_e} \cdot \nabla n + n \nabla \cdot \mathbf{v_e} + \nabla \cdot (n \mathbf{v}_*) = 0, \tag{3}$$

where $\mathbf{v_e}$ is the $\mathbf{E} \times \mathbf{B}$ drift, and $\mathbf{v}_*$ is the diamagnetic drift including the full pressure gradient and

$$\frac{3}{2} n \left(\frac{d}{dt} + \mathbf{v_e} \cdot \nabla \right) T_i + P_i \nabla \cdot \mathbf{v}_i = -\nabla \mathbf{q}_{*i}, \tag{4a}$$

$$\mathbf{q}_{*i} = \frac{5}{2} \frac{P_i}{m \Omega_{ci}} (\mathbf{e}_\parallel \times \nabla T_i). \tag{4b}$$

A new and fundamental feature of system (3) and (4) is that all curvature and magnetic drift effects are kept. Because of this, the linear density perturbation includes both adiabatic and isothermal limits. The only nonlinear effect kept is the convective nonlinearity due to the $\mathbf{E} \times \mathbf{B}$ drift. It occurs in both (4a) and (4b). There are several fundamental similarities between the developments in Figs. 1 and 2. In both cases, we have destabilizing quadratic nonlinearities and stabilizing cubic nonlinearities. The periodicity is in the case of population evolution due to a delay of action, while that for the wave interaction is due to the nonlinear phase dependence (nonlinear frequency shift) of the wave interaction. Both of these are non-Markovian effects. We also note that the delay of 25 years for the cubic nonlinear stabilization is instrumental in describing the actual population growth, which was not known when this work was done. The fact that the systems in Refs. 12 and 13 experience a nonlinear growth at small amplitudes can be seen by the fact that the growth rate in Ref. 12 gets larger, when the nonlinear effects are added. The system in Ref. 13 is actually the same as in Ref. 12, but with damping due to the turbulent diffusion added. In particular, we note the strong similarity between Fig. 2 in Ref. 13 and our Fig. 2.

3. Nonlinear Stabilization

There are many examples of similar nonlinear systems which we may discuss. An important point is the level of stabilization of a simple stable three-wave interaction with a linear growth. This has been discussed in several cases. The case that we are mainly interested in for the drift wave transport is that which occurs, when the instability is stabilized by the nonlinear $\mathbf{E} \times \mathbf{B}$ convection. The growth and damping can then be expressed as:

$$\frac{\partial}{\partial t} = \nu_{\mathbf{E}} \cdot \nabla. \qquad (5)$$

When we operate with Eq. (5) on the density or temperature, replacing $\frac{\partial}{\partial t}$ with the growth rate γ and expressing the $\mathbf{E} \times \mathbf{B}$ drift ($\nu_{\mathbf{E}}$) in the potential φ, we obtain the saturation level:

$$\frac{e\varphi}{T} = \frac{1}{k_x \rho_s} \frac{\gamma}{k_y c_s}. \qquad (6)$$

Here, e is the electronic charge, T is the temperature, k_y is the poloidal wave propagation vector, ρ_s is the Larmor radius, and c_s is the sound speed. It is important to note that, due to the similarity of the continuity and energy equations, Eq. (6) is valid both in Refs. 16–19 (from the energy equation) and in Ref. 23. The energy equation was first found in Ref. 25. But, in Ref. 23, it was called the "improved mixing-length level".

Using the saturation level in Eq. (6), we arrive at the improved mixing length transport:

$$D = \frac{\gamma^3/k_x^2}{\omega_r^2 + \gamma^2}. \qquad (7)$$

The result, Eq. (7), can also be obtained for the temperature diffusion, where D is replaced by χ. The similarity between the temperature diffusion and the particle diffusion is due to the similarity between the continuity and energy equations [24]. Equation (4) represents the "kernel" of the transport coefficient. In a quasilinear treatment of transport, we obtain an additional factor representing compressional effects like pinch effects for the temperature or density [25].

We note that the real frequency (non-Markovian) reduces the transport. Thus, we expect more transport from low-frequency modes than from high-frequency modes.

Of course, when nonlinear effects have a destabilizing influence at low amplitudes, such as in Ref. 21 for toroidal modes and in Refs. 12, 13 for slab modes, we have to consider the total effect of linear, quadratic, and cubic nonlinearities. However, since nonlinear frequency shifts act as a part of the resonance condition for frequencies, it will change the sign of the quadratic nonlinearities, just like it changes the sign of the wave energy [1]. This means that, at higher amplitudes, the quadratic nonlinear terms will

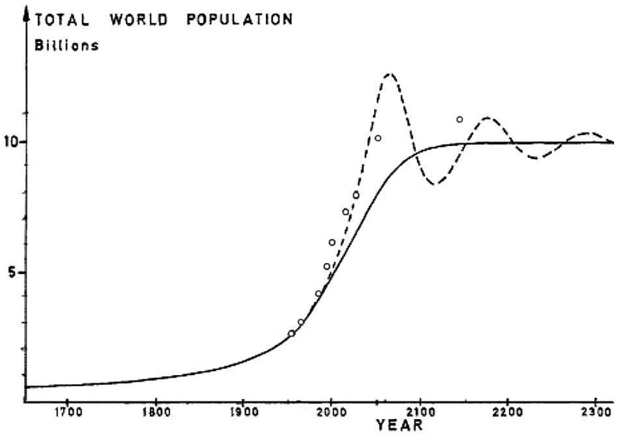

Fig. 1. The evolution of the world population as predicted in Ref. 3 in 1974 and the population according to the United Nations up to now and the extrapolation to 2150 (circles). In the calculated prediction, the full line corresponds to stabilizing terms calculated at present time (Markovian), while the dotted line includes a delay of one generation (25 years) before cubic effects start (non-Markovian)

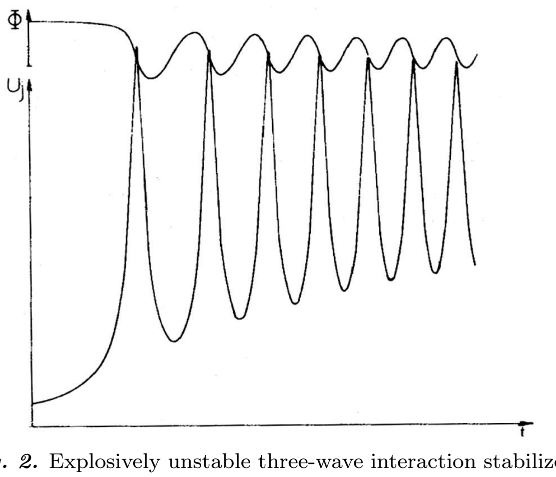

Fig. 2. Explosively unstable three-wave interaction stabilized by periodic nonlinear frequency shifts and nonlinear dissipation (From Ref. 1)

be stabilizing, thus making our saturation level in Eq. (3) correct on the average in developments as that seen in Fig. 2. This explains why the saturation level has been so successful in comparisons with experiments. However, this is true only when the $\mathbf{E}\times\mathbf{B}$ drift is entirely stabilizing, i.e., we are looking at the correlation length in the **k**-space, and we have effects which absorb the inverse cascade, thus avoiding reflections at the system size. One effect that does this is flowshear [28]. According to the Waltz rule [28], we can subtract the flowshear rate from the growth rate in Eq. (4). This is similar to a stabilizing effect of the imaginary contribution from the cubic nonliearities [1]. The full model from this approach, which includes the electron trapping, electromagnetic effects that give MHD, and kinetic ballooning modes and peeling modes, was recently used to predict ITER performance [29].

4. Analogy between Growth of Population and Nonlinear Wave Interaction

It is worth comparing the saturation mechanisms for the population growth and nonlinearly unstable interacting waves in some details. In both cases, it is due to nonlinearities which cause the periodic behavior. Although, for the population growth, this happens already in the first period. Nevertheless, we can see that the agreement with the actual population is strongly improved by the time delay of the cubic term. As it seems, one nonlinearity which simplifies the population growth is that the birth rate increases in times of a war. One can imagine that a similar effect occurs at times of strong epidemies. The surprising insensitivity to the exact location of sources for the wave interaction is partly due to the strong sensitivity to growthrates on the deviation of gradients in the temperature and density from linear thresholds. However also particle or heat pinch effects can contribute here [18, 19]. An important aspect for wave interactions is that the nonlinear three-wave systems in Refs. 12 and 13 were generalized to the turbulent case in Ref. 16. Here, the use was made of Ref 15, where a Fokker–Planck equation for turbulent collisions was derived. This is applicable to a turbulent situation, where non-Markovian effects were required for deriving the transport coefficient in Eq. (7).

5. Summary

We have here pointed out the importance of nonlinearities in both turbulent plasmas and in the population explosion. Such nonlinearities often lead to non-Markovian effects. In a plasma, we have looked at both the coherence limit represented by Refs. 12 and 13 and the turbulence case considered in Refs. 10, 15, and 16. For the turbulence kinetics, we need to go beyond quasilinear theory (strongly nonlinear case); while, for the fluids, it is sufficient to use a quasilinear approach. Although many other effects can occur both in the population growth and the nonlinear wave interaction, we note that the agreement with our predictions of the world population actually justifies the very simple model used here. For the wave interactions, we refer to Ref 1 for a wider picture. Our purpose was just to point out the most essential similarities and to motivate researchers to take more nonlinear effects into account.

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НЕЛІНІЙНІ ЕФЕКТИ У ФІЗИЦІ ТА ДЕМОГРАФІЇ

Нелінійності з'являються майже в усіх системах. Раніше ми розглядали нелінійності в плазмі, розсіюванні в іоносфері, світовій популяції. З'ясувалося, що оцінка приросту населення, яка була зроблена в 1974 році, чудово узгоджується з оцінками ООН і співпадає з нашими даними з точністю 2%. Особливо важливими для еволюції населення та взаємодії хвиль у плазмі є немарковські ефекти (вони залежать від минулих часів). Для приросту населення це пов'язано із затримкою на одне покоління впливу факторів, що обмежують популяцію, тоді як для плазми це викликано нелінійними зсувами частоти.

Ключові слова: фундаментальні нелінійності в природі, вибухові нестійкості, дослідження синтезу, демографічні дослідження.