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## FACTORIZATION OF THE LORENTZ TRANSFORMATIONS

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*The article shows how the factorization of an arbitrary Lorentz transformation is performed. That is, the representation of an arbitrary Lorentz transformation as a sequence of a spatial rotation and a boost or a boost and a spatial rotation. Relations are obtained that determine the required boosts and turns.*

*Keywords:* hypercomplex numbers, Lorentz group.

### 1. Introduction

In quantum electrodynamics, the most convenient and natural form of Lorentz transformations is the hypercomplex form based on 16 Dirac matrices [1]. In the hypercomplex representation, a scalar is associated with the matrix  $a\hat{1}$ , a pseudoscalar is associated with the matrix  $a\hat{i}$ , a 4-vector is associated with the matrix  $a_\alpha\gamma^\alpha$ , a 4-pseudovector is associated with the matrix  $a_\alpha\pi^\alpha$ , and an antisymmetric 4-tensor of the second rank is associated with the matrix  $a_{\alpha\beta}\sigma^{\alpha\beta}$ . Here,  $\hat{1}$  is the identity matrix  $4 \times 4$ ,

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2g^{\alpha\beta}, \quad (1)$$

$$\hat{i} = \gamma^0\gamma^1\gamma^2\gamma^3, \quad \pi^\alpha = \gamma^\alpha\hat{i}, \quad 2\sigma^{\alpha\beta} = \gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha. \quad (2)$$

As always, Greek indices take values 0, 1, 2, 3, Latin ones take values 1, 2, 3.

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Let us name the numbers

$$a\hat{1} + b\hat{i} + c_\alpha\gamma^\alpha + d_\alpha\pi^\alpha + \frac{1}{2}f_{\alpha\beta}\sigma^{\alpha\beta}. \quad (3)$$

Dirac numbers [2]. The hypercomplex system of Dirac numbers contains a subsystem based on 8 matrices  $\hat{1}$ ,  $\hat{i}$ ,  $\sigma^{\alpha\beta}$ . The numbers of this subsystem have the form

$$a\hat{1} + b\hat{i} + \frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} = a\hat{1} + b\hat{i} + L_{01}\sigma^{01} + L_{02}\sigma^{02} + L_{03}\sigma^{03} + L_{23}\sigma^{23} + L_{31}\sigma^{31} + L_{12}\sigma^{12}. \quad (4)$$

We will call these numbers the Lorentz numbers, since it is with their help that Lorentz transformations are carried out. More precisely, with the help of the matrix exponent

$$e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}} = \hat{1} + \frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} + \frac{1}{2!}\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} \times \frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} + \dots = \hat{1} + \frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} - \frac{L^2}{2!} - \frac{L^2}{3!}\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} + \frac{L^4}{4!} + \dots = \cos L + \frac{L_{\alpha\beta}\sigma^{\alpha\beta}}{2L} \sin L. \quad (5)$$

Here, we have used the equality

$$\begin{aligned} & \frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} = \\ & = -\frac{1}{2}(L^{\alpha\beta}L_{\alpha\beta} - \hat{i}L^{\alpha\beta}L_{\alpha\beta}^{\circ}) = -L^2 \end{aligned} \quad (6)$$

and the designation

$$L_{\alpha\beta}^{\circ} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}L^{\mu\nu}, \quad \varepsilon^{0123} = 1, \quad \varepsilon_{0123} = -1. \quad (7)$$

The tensor  $L_{\alpha\beta}^{\circ}$  dual to  $L_{\alpha\beta}$  has components

$$\begin{aligned} L_{01}^{\circ} &= -L_{23}, L_{02}^{\circ} = -L_{31}, L_{03}^{\circ} = -L_{12}, \\ L_{23}^{\circ} &= L_{01}, L_{31}^{\circ} = L_{02}, L_{12}^{\circ} = L_{03}. \end{aligned} \quad (8)$$

The Lorentz transformation of scalars, pseudo-scalars, vectors, pseudo-vectors, and second-rank antisymmetric tensors is performed by the operations

$$\begin{aligned} a\hat{1} &= e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}a\hat{1}e^{-\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}, \\ b\hat{i} &= e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}b\hat{i}e^{-\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}, \\ c'_{\alpha}\gamma^{\alpha} &= e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}c_{\alpha}\gamma^{\alpha}e^{-\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}, \\ d'_{\alpha}\pi^{\alpha} &= e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}d_{\alpha}\pi^{\alpha}e^{-\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}, \\ f'_{\alpha\beta}\sigma^{\alpha\beta} &= e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}f_{\alpha\beta}\sigma^{\alpha\beta}e^{-\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}. \end{aligned} \quad (9)$$

The Lorentz transformation of Dirac spinors is performed by the operation

$$\psi' = e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}\psi. \quad (10)$$

If the exponents in (9)–(10) have the form  $\frac{1}{2}L_{kl}\sigma^{kl}$ , then these formations are spatial rotations; but, if they have the form  $L_{0k}\sigma^{0k}$ , then these transformations are boosts. In the general case, transformations are neither spatial rotations nor boosts. However, any Lorentz transformation can always be represented as a sequence of a spatial rotation and a boost or a boost and a spatial rotation. Below, we obtain relations that allow us to do this in an arbitrary case.

## 2. Biquaternion Representation of the Lorentz Transformations

Hypercomplex Lorentz numbers are isomorphic with biquaternions. This makes it possible to use the well-known quaternion algebra to simplify manipulations with Lorentz transformations. To verify this isomorphism, we firstly note that

$$\sigma^{01} = -\hat{i}\sigma^{23}, \quad \sigma^{02} = -\hat{i}\sigma^{31}, \quad \sigma^{03} = -\hat{i}\sigma^{12}. \quad (11)$$

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Therefore, (4) can be written as

$$\begin{aligned} a + \hat{i}b + \frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} &= a + \hat{i}b - \hat{i}\frac{1}{2}L_{kl}^{\circ}\sigma^{kl} + \frac{1}{2}L_{kl}\sigma^{kl} = \\ &= a + \hat{i}b + \frac{1}{2}(L_{kl} - \hat{i}L_{kl}^{\circ})\sigma^{kl} = a + \hat{i}b + \frac{1}{2}l_{kl}\sigma^{kl}. \end{aligned} \quad (12)$$

Here

$$l_{kl} = L_{kl} - \hat{i}L_{kl}^{\circ}. \quad (13)$$

The algebra of matrices  $\hat{1}, \sigma^{23}, \sigma^{31}, \sigma^{12}$  is isomorphic with the algebra of quaternions:

×	$\sigma^{23}$	$\sigma^{31}$	$\sigma^{12}$
$\sigma^{23}$	$-\hat{1}$	$\sigma^{12}$	$-\sigma^{31}$
$\sigma^{31}$	$-\sigma^{12}$	$-\hat{1}$	$\sigma^{23}$
$\sigma^{12}$	$\sigma^{31}$	$-\sigma^{23}$	$-\hat{1}$

×	$i$	$j$	$k$
$i$	$-1$	$k$	$-j$
$j$	$-k$	$-1$	$i$
$k$	$j$	$-i$	$-1$

Therefore the numbers

$$a\hat{1} + \frac{1}{2}L_{kl}\sigma^{kl} \quad (14)$$

can be thought of as quaternions, and the numbers

$$\begin{aligned} a + \hat{i}b + \frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} &= \left(a\hat{1} + \frac{1}{2}L_{kl}\sigma^{kl}\right) + \\ &+ \hat{i}\left(b - \frac{1}{2}L_{kl}^{\circ}\sigma^{kl}\right) \end{aligned} \quad (15)$$

like biquaternions [3]. That is, as a system of quaternions, expanded by introducing an additional unit  $\hat{i}$ .

Mathematicians consider three possible options for introducing an additional unit: when  $\hat{i}\cdot\hat{i} = -1$ , when  $\hat{i}\cdot\hat{i} = 1$ , and when  $\hat{i}\cdot\hat{i} = 0$ . In the first case, the resulting numbers are called elliptic (ordinary) biquaternions, in the second, hyperbolic biquaternions, and, in the third, parabolic ones. Since  $\hat{i}\cdot\hat{i} = -1$ , we are dealing with elliptic (ordinary) biquaternions.

The numbers  $a\hat{1} + b\hat{i}$  are isomorphic with complex numbers and commute with  $\sigma^{\alpha\beta}$ . We will call such numbers  $\hat{i}$ -complex numbers. Accordingly, biquaternions (15) can be considered as quaternions with  $\hat{i}$ -complex coefficients. In particular, exponent (5) can be written as

$$\begin{aligned} e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}} &= e^{l\zeta} = 1 + l\zeta + \frac{1}{2!}l\zeta \cdot l\zeta + \\ &+ \frac{1}{3!}l\zeta \cdot l\zeta \cdot l\zeta + \dots = \cos l + \frac{l\zeta}{l} \sin l. \end{aligned} \quad (16)$$

Here, we use the notation

$$\begin{aligned} \mathbf{l}\zeta &\equiv \frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} = \frac{1}{2}l_{kl}\sigma^{kl} = \frac{1}{2}(L_{kl} - \hat{l}L_{kl}^{\circ})\sigma^{kl} = \\ &= (L_{23} - \hat{l}L_{23}^{\circ})\sigma^{23} + (L_{31} - \hat{l}L_{31}^{\circ})\sigma^{31} + (L_{12} - \hat{l}L_{12}^{\circ})\sigma^{12} = \\ &= (\mathbf{r} + \hat{l}\mathbf{b})\zeta = \end{aligned} \quad (17)$$

$$(r_x + \hat{l}b_x)\sigma^{23} + (r_y + \hat{l}b_y)\sigma^{31} + (r_z + \hat{l}b_z)\sigma^{12}, \quad (18)$$

$$l^2 = -\mathbf{l}\zeta \cdot \mathbf{l}\zeta = \mathbf{l} \cdot \mathbf{l} = (\mathbf{r} + \hat{l}\mathbf{b})^2 = \mathbf{r}^2 - \mathbf{b}^2 + \hat{l}2\mathbf{b} \cdot \mathbf{r}, \quad (19)$$

$$l = \sqrt{-\mathbf{l}\zeta \cdot \mathbf{l}\zeta} = \sqrt{\mathbf{l} \cdot \mathbf{l}} = \sqrt{\mathbf{r}^2 - \mathbf{b}^2 + \hat{l}2\mathbf{b} \cdot \mathbf{r}}. \quad (20)$$

If  $\mathbf{l} = \mathbf{r}$ ,  $\mathbf{b} = 0$ , then transformations (9)–(10) describe the space direct rotation of the reference frame around the  $\mathbf{r}$ -axis by an angle  $2r$ . But if  $\mathbf{l} = \hat{l}\mathbf{b}$ ,  $\mathbf{r} = 0$ , then transformations (9)–(10) describe the boost. Namely, if  $\frac{\mathbf{b}}{b} = \frac{\mathbf{v}}{v}$ ,  $\tanh 2b = \frac{v}{c}$ , then they will describe the transition to the reference frame that moves relative to the original system with a speed  $\mathbf{v}$ . In the general case, when  $\mathbf{l} = \mathbf{r} + \hat{l}\mathbf{b}$ , transformations (9)–(10) are neither spatial rotations nor boosts. However, as we will see below, they can always be represented as a sequence of a spatial rotation and a boost or a boost and a spatial rotation.

Operations with biquaternion exponents (16) are much more convenient to perform, if, instead of  $\hat{l}$ -complex vectors  $\mathbf{l}$ ,  $\hat{l}$ -complex vectors  $\boldsymbol{\lambda}$  are used as parameters of the Lorentz transformations

$$\boldsymbol{\lambda}(\mathbf{l}) = \frac{\mathbf{l}}{l} \tan l. \quad (21)$$

Obviously, in the case of a spatial rotation, when  $\mathbf{l}$  is a  $\hat{l}$ -real vector  $\mathbf{l} = \mathbf{r}$ , the parameter  $\boldsymbol{\lambda}$  is also a  $\hat{l}$ -real vector

$$\boldsymbol{\rho}(\mathbf{r}) = \frac{\mathbf{r}}{r} \tan r. \quad (22)$$

In the boost case, when  $\mathbf{l}$  is the  $\hat{l}$ -imaginary vector  $\mathbf{l} = \hat{l}\mathbf{b}$ , the parameter  $\boldsymbol{\lambda}$  is also the  $\hat{l}$ -imaginary vector

$$\boldsymbol{\beta}(\hat{l}\mathbf{b}) = \frac{\hat{l}\mathbf{b}}{b} \tanh b. \quad (23)$$

If the parameter  $\boldsymbol{\lambda}(\mathbf{l})$  is known, then the parameter  $\mathbf{l}$  is determined by the relation

$$\mathbf{l} = \frac{\boldsymbol{\lambda}}{\sqrt{\boldsymbol{\lambda} \cdot \boldsymbol{\lambda}}} \arctan \sqrt{\boldsymbol{\lambda} \cdot \boldsymbol{\lambda}} = \frac{\boldsymbol{\lambda}}{\lambda} \arctan \lambda, \quad (24)$$

$$l = \arctan \sqrt{\boldsymbol{\lambda} \cdot \boldsymbol{\lambda}} = \arctan \lambda.$$

When using the parameter  $\boldsymbol{\lambda}(\mathbf{l})$ , the exponent  $e^{l\zeta}$  takes the form

$$\begin{aligned} e^{l\zeta} &= \cos l + \frac{l\zeta}{l} \sin l = \cos l \left(1 + \frac{l\zeta}{l} \tan l\right) = \\ &= \frac{1}{\sqrt{1 + \tan^2 l}} \left(1 + \frac{l\zeta}{l} \tan l\right) = \frac{1 + \boldsymbol{\lambda}\zeta}{\sqrt{1 + \boldsymbol{\lambda} \cdot \boldsymbol{\lambda}}}, \end{aligned} \quad (25)$$

and the product of two exponents  $e^{l_2\zeta}e^{l_1\zeta}$  is of the form

$$\begin{aligned} e^{l_2\zeta}e^{l_1\zeta} &= \frac{(1 + \boldsymbol{\lambda}_2\zeta)(1 + \boldsymbol{\lambda}_1\zeta)}{\sqrt{(1 + \boldsymbol{\lambda}_2 \cdot \boldsymbol{\lambda}_2)(1 + \boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_1)}} = \\ &= \frac{1 - \boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2}{\sqrt{(1 + \boldsymbol{\lambda}_2 \cdot \boldsymbol{\lambda}_2)(1 + \boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_1)}} \times \\ &\times \left(1 + \frac{\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_2 \times \boldsymbol{\lambda}_1}{1 - \boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2} \zeta\right). \end{aligned} \quad (26)$$

Here,  $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}(l_1)$ ,  $\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}(l_2)$ . Putting

$$\boldsymbol{\lambda}(\mathbf{l}) = \frac{\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_2 \times \boldsymbol{\lambda}_1}{1 - \boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2}, \quad (27)$$

we get

$$1 + \boldsymbol{\lambda}(\mathbf{l}) \cdot \boldsymbol{\lambda}(\mathbf{l}) = \frac{(1 + \boldsymbol{\lambda}_2 \cdot \boldsymbol{\lambda}_2)(1 + \boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_1)}{(1 - \boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2)^2}. \quad (28)$$

The product of the exponents takes the form of exponent (16) with the exponent  $\mathbf{l}\zeta$ :

$$e^{l_2\zeta}e^{l_1\zeta} = \frac{1 + \boldsymbol{\lambda}\zeta}{\sqrt{1 + \boldsymbol{\lambda} \cdot \boldsymbol{\lambda}}} = e^{l\zeta}. \quad (29)$$

Relation (26) determines the rule for the composition of the parameters  $\boldsymbol{\lambda}(l_1)$  and  $\boldsymbol{\lambda}(l_2)$ , when multiplying the exponents  $e^{l_2\zeta}e^{l_1\zeta}$ . Note that, in [4], the same relation was obtained for another parameter not related to biquaternions.

### 3. Factorization of the Lorentz Transformations

Any Lorentz transformation can be represented as a sequence of a spatial rotation and a boost or a boost and a spatial rotation. Accordingly, the exponent  $e^{l\zeta}$  can be represented as a product

$$e^{l\zeta} = e^{ib\zeta}e^{r\zeta}, \quad (30)$$

or

$$e^{l\zeta} = e^{r'\zeta}e^{ib'\zeta}. \quad (31)$$

Let us take a few steps to find the parameters of boosts  $\mathbf{b}$ ,  $\mathbf{b}'$  and rotations  $\mathbf{r}$ ,  $\mathbf{r}'$ .

At the first step, we multiply the left and right parts of equalities (29)–(30) by the left and right parts of their Hermitian conjugate equalities

$$(e^{l\varsigma})^\dagger = e^{-l^*\varsigma} = e^{-r\varsigma} e^{ib\varsigma}. \quad (32)$$

or

$$(e^{l\varsigma})^\dagger = e^{-l^*\varsigma} = e^{ib'\varsigma} e^{-r'\varsigma}. \quad (33)$$

Here,  $*$  denotes the  $\hat{i}$ -complex conjugate:  $\mathbf{l} = \mathbf{r} + i\mathbf{b}$ ,  $\mathbf{l}^* = \mathbf{r} - i\mathbf{b}$ .

We multiply equality (29) by (31) from the right

$$e^{l\varsigma} e^{-l^*\varsigma} = e^{ib\varsigma} e^{r\varsigma} e^{-r\varsigma} e^{ib\varsigma} = e^{i2b\varsigma}, \quad (34)$$

and equality (30) is multiplied by (32) from the left

$$e^{-l^*\varsigma} e^{l\varsigma} = e^{ib'\varsigma} e^{r'\varsigma} e^{-r'\varsigma} e^{ib'\varsigma} = e^{ib'\varsigma} e^{ib'\varsigma} = e^{i2b'\varsigma}. \quad (35)$$

Using (26), we find the parameters  $\lambda(i2\mathbf{b})$  and  $\lambda(i2\mathbf{b}')$  corresponding to the parameters  $\mathbf{l} = i2\mathbf{b}$  and  $\mathbf{l}' = i2\mathbf{b}'$ :

$$\begin{aligned} \lambda(i2\mathbf{b}) &= \frac{\lambda(\mathbf{l}) + \lambda^*(-\mathbf{l}) + \lambda(\mathbf{l}) \times \lambda^*(-\mathbf{l})}{1 - \lambda(\mathbf{l}) \cdot \lambda^*(-\mathbf{l})} = \\ &= \frac{\lambda(\mathbf{l}) - \lambda^*(\mathbf{l}) - \lambda(\mathbf{l}) \times \lambda^*(\mathbf{l})}{1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l})}, \end{aligned} \quad (36)$$

$$\begin{aligned} \lambda(i2\mathbf{b}') &= \frac{\lambda(\mathbf{l}) + \lambda^*(-\mathbf{l}) + \lambda^*(-\mathbf{l}) \times \lambda(\mathbf{l})}{1 - \lambda(\mathbf{l}) \cdot \lambda^*(-\mathbf{l})} = \\ &= \frac{\lambda(\mathbf{l}) - \lambda^*(\mathbf{l}) - \lambda^*(\mathbf{l}) \times \lambda(\mathbf{l})}{1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l})}. \end{aligned} \quad (37)$$

At the second step, we will find the parameters  $\lambda(i\mathbf{b})$  and  $\lambda(i\mathbf{b}')$  we need. They differ from  $\lambda(i2\mathbf{b})$  and  $\lambda(i2\mathbf{b}')$  by factors

$$\frac{\tan \sqrt{i\mathbf{b} \cdot i\mathbf{b}}}{\tan \sqrt{i2\mathbf{b} \cdot i2\mathbf{b}}} \quad \text{and} \quad \frac{\tan \sqrt{i\mathbf{b}' \cdot i\mathbf{b}'}}{\tan \sqrt{i2\mathbf{b}' \cdot i2\mathbf{b}'}}. \quad (38)$$

To find these factors, we use the trigonometric equality

$$\frac{\tan z}{\tan 2z} = \frac{1}{1 + \sqrt{1 + \tan^2 2z}}. \quad (39)$$

Thus, we get

$$\frac{\tan \sqrt{i\mathbf{b} \cdot i\mathbf{b}}}{\tan \sqrt{i2\mathbf{b} \cdot i2\mathbf{b}}} = \frac{1}{1 + \sqrt{1 + \tan^2 \sqrt{i2\mathbf{b} \cdot i2\mathbf{b}}}} =$$

$$\frac{1}{1 + \sqrt{1 + \lambda(i2\mathbf{b}) \cdot \lambda(i2\mathbf{b})}}, \quad (40)$$

$$\begin{aligned} \frac{\tan \sqrt{i\mathbf{b}' \cdot i\mathbf{b}'}}{\tan \sqrt{i2\mathbf{b}' \cdot i2\mathbf{b}'}} &= \frac{1}{1 + \sqrt{1 + \tan^2 \sqrt{i2\mathbf{b}' \cdot i2\mathbf{b}'}}} = \\ &= \frac{1}{1 + \sqrt{1 + \lambda(i2\mathbf{b}') \cdot \lambda(i2\mathbf{b}')}}. \end{aligned} \quad (41)$$

Let us calculate

$$\begin{aligned} \lambda(i2\mathbf{b}) \cdot \lambda(i2\mathbf{b}) &= \lambda(i2\mathbf{b}') \cdot \lambda(i2\mathbf{b}') = \\ &= \frac{\left[ \lambda^2(\mathbf{l}) + \lambda^{*2}(\mathbf{l}) - 2\lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l}) + \right. \\ &\quad \left. + \lambda^2(\mathbf{l})\lambda^{*2}(\mathbf{l}) - [\lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l})]^2 \right]}{[1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l})]^2} = \\ &= \frac{[1 + \lambda^2(\mathbf{l})][1 + \lambda^{*2}(\mathbf{l})] - [1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l})]^2}{[1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l})]^2} = \\ &= \frac{[1 + \lambda^2(\mathbf{l})][1 + \lambda^{*2}(\mathbf{l})]}{[1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l})]^2} - 1. \end{aligned} \quad (42)$$

Respectively,

$$\begin{aligned} \frac{\tan \sqrt{i\mathbf{b} \cdot i\mathbf{b}}}{\tan \sqrt{i2\mathbf{b} \cdot i2\mathbf{b}}} &= \frac{\tan \sqrt{i\mathbf{b}' \cdot i\mathbf{b}'}}{\tan \sqrt{i2\mathbf{b}' \cdot i2\mathbf{b}'}} = \\ &= \frac{1 + \lambda(\mathbf{l}) \lambda^*(\mathbf{l})}{1 + \lambda(\mathbf{l}) \lambda^*(\mathbf{l}) + \sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*}}. \end{aligned} \quad (43)$$

Thus, the parameters  $\lambda(i\mathbf{b})$  and  $\lambda(i\mathbf{b}')$  corresponding to the parameters  $\mathbf{l} = i\mathbf{b}$  and  $\mathbf{l}' = i\mathbf{b}'$ , have the form

$$\begin{aligned} \lambda(i\mathbf{b}) &= \lambda(i2\mathbf{b}) \frac{\tan \sqrt{i\mathbf{b} \cdot i\mathbf{b}}}{\tan \sqrt{i2\mathbf{b} \cdot i2\mathbf{b}}} = \\ &= \frac{\lambda(\mathbf{l}) - \lambda^*(\mathbf{l}) - \lambda(\mathbf{l}) \times \lambda^*(\mathbf{l})}{1 + \lambda(\mathbf{l}) \lambda^*(\mathbf{l}) + \sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*}}, \end{aligned} \quad (44)$$

$$\begin{aligned} \lambda(i\mathbf{b}') &= \lambda(i2\mathbf{b}') \frac{\tan \sqrt{i\mathbf{b}' \cdot i\mathbf{b}'}}{\tan \sqrt{i2\mathbf{b}' \cdot i2\mathbf{b}'}} = \\ &= \frac{\lambda(\mathbf{l}) - \lambda^*(\mathbf{l}) + \lambda(\mathbf{l}) \times \lambda^*(\mathbf{l})}{1 + \lambda(\mathbf{l}) \lambda^*(\mathbf{l}) + \sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*}}. \end{aligned} \quad (45)$$

As it should be, the parameters  $\lambda(i\mathbf{b})$  and  $\lambda(i\mathbf{b}')$  are  $\hat{i}$ -imaginary vectors. The direction of these vectors depends on which operation – turn or boost – is performed firstly, and which is secondly. The magnitude of the vectors  $\lambda(i\mathbf{b})$  and  $\lambda(i\mathbf{b}')$  does not depend on this.

At the third step, we find the exponents describing spatial rotations. To do this, we multiply (29) by

$e^{-ib\varsigma}$  on the left side, and (30) by  $e^{-ib'\varsigma}$  on the right side:

$$e^{-ib\varsigma} e^{l\varsigma} = e^{-ib\varsigma} e^{ib\varsigma} e^{r\varsigma} = e^{r\varsigma}, \quad (46)$$

$$e^{l\varsigma} e^{-ib'\varsigma} = e^{r'\varsigma} e^{ib'\varsigma} e^{-ib'\varsigma} = e^{r'\varsigma}. \quad (47)$$

We use (26) again and, after long, but not complicated transformations, we find the parameters  $\lambda(\mathbf{r})$  and  $\lambda(\mathbf{r}')$  corresponding to the parameters  $\mathbf{r}$  and  $\mathbf{r}'$ :

$$\begin{aligned} \lambda(\mathbf{r}) &= \frac{\lambda(\mathbf{l}) + \lambda(-i\mathbf{b}) + \lambda(-i\mathbf{b}) \times \lambda(\mathbf{l})}{1 - \lambda(\mathbf{l}) \cdot \lambda(-i\mathbf{b})} = \\ &= \frac{\lambda(\mathbf{l}) - \lambda(i\mathbf{b}) + \lambda(\mathbf{l}) \times \lambda(i\mathbf{b})}{1 + \lambda(\mathbf{l}) \cdot \lambda(i\mathbf{b})} = \\ &= \left\{ \lambda(\mathbf{l}) + \lambda(\mathbf{l})[\lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l})] + \right. \\ &+ \lambda(\mathbf{l})\sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*} - \lambda(\mathbf{l}) + \lambda^*(\mathbf{l}) + \\ &+ \lambda(\mathbf{l}) \times \lambda^*(\mathbf{l}) - \lambda(\mathbf{l}) \times \lambda^*(\mathbf{l}) - \\ &\left. - \lambda(\mathbf{l})[\lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l})] + \lambda^*(\mathbf{l})[\lambda(\mathbf{l}) \cdot \lambda(\mathbf{l})] \right\} \times \\ &\times \left\{ 1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l}) + \sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*} \right\}^{-1} \times \\ &\times \left\{ 1 + \lambda(\mathbf{l}) \cdot \lambda(i\mathbf{b}) \right\}^{-1} = \\ &= \frac{\left[ \lambda(\mathbf{l})\sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*} + \right. \\ &\left. + \lambda^*(\mathbf{l})[1 + \lambda(\mathbf{l}) \cdot \lambda(\mathbf{l})] \right]}{1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l}) + \sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*}} \times \\ &\times \frac{1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l}) + \sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*}}{\left[ 1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l}) + \sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*} + \right. \\ &\left. + \lambda(\mathbf{l}) \cdot \lambda(\mathbf{l}) - \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l}) \right]} = \\ &= \frac{\lambda(\mathbf{l})\sqrt{[1 + \lambda^2(\mathbf{l})]^*} + \lambda^*(\mathbf{l})\sqrt{[1 + \lambda^2(\mathbf{l})]}}{\sqrt{[1 + \lambda^2(\mathbf{l})]^*} + \sqrt{[1 + \lambda^2(\mathbf{l})]}}, \quad (48) \end{aligned}$$

$$\begin{aligned} \lambda(\mathbf{r}') &= \frac{\lambda(\mathbf{l}) + \lambda(-i\mathbf{b}') + \lambda(\mathbf{l}) \times \lambda(-i\mathbf{b}')}{1 - \lambda(\mathbf{l}) \cdot \lambda(-i\mathbf{b}')} = \\ &= \frac{\lambda(\mathbf{l}) - \lambda(i\mathbf{b}') - \lambda(\mathbf{l}) \times \lambda(i\mathbf{b}')}{1 + \lambda(\mathbf{l}) \cdot \lambda(i\mathbf{b}')} = \end{aligned}$$

$$\begin{aligned} &\left\{ \lambda(\mathbf{l}) + \lambda(\mathbf{l})[\lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l})] + \lambda(\mathbf{l}) \times \right. \\ &\times \sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*} - \lambda(\mathbf{l}) + \lambda^*(\mathbf{l}) - \\ &\left. - \lambda(\mathbf{l}) \times \lambda^*(\mathbf{l}) + \lambda(\mathbf{l}) \times \lambda^*(\mathbf{l}) \right\} \times \end{aligned}$$

$$\begin{aligned} &- \lambda(\mathbf{l})[\lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l})] + \lambda^*(\mathbf{l})[\lambda(\mathbf{l}) \cdot \lambda(\mathbf{l})] \Big\} \times \\ &\times \left\{ 1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l}) + \sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*} \right\}^{-1} \times \\ &\times \left\{ 1 + \lambda(\mathbf{l}) \cdot \lambda(i\mathbf{b}') \right\}^{-1} = \\ &= \frac{\lambda(\mathbf{l})\sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*} + \lambda^*(\mathbf{l})[1 + \lambda(\mathbf{l}) \cdot \lambda(\mathbf{l})]}{1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l}) + \sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*}} \times \\ &\times \frac{1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l}) + \sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*}}{\left[ 1 + \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l}) + \sqrt{[1 + \lambda^2(\mathbf{l})][1 + \lambda^2(\mathbf{l})]^*} + \right. \\ &\left. + \lambda(\mathbf{l}) \cdot \lambda(\mathbf{l}) - \lambda(\mathbf{l}) \cdot \lambda^*(\mathbf{l}) \right]} = \\ &= \frac{\lambda(\mathbf{l})\sqrt{[1 + \lambda^2(\mathbf{l})]^*} + \lambda^*(\mathbf{l})\sqrt{[1 + \lambda^2(\mathbf{l})]}}{\sqrt{[1 + \lambda^2(\mathbf{l})]^*} + \sqrt{[1 + \lambda^2(\mathbf{l})]}}. \quad (49) \end{aligned}$$

As expected, the parameters  $\lambda(\mathbf{r})$  and  $\lambda(\mathbf{r}')$  are  $\hat{i}$ -real vectors. Neither the magnitude nor the direction of these vectors depend on the sequence in which a turn and a boost are performed.

Having obtained the parameters  $\lambda(i\mathbf{b})$ ,  $\lambda(i\mathbf{b}')$ ,  $\lambda(\mathbf{r})$  and  $\lambda(\mathbf{r}')$ , we can use relations (23) to pass to the parameters  $\mathbf{l}$ ,  $\mathbf{l}'$ ,  $\mathbf{r}$ ,  $\mathbf{r}'$  and represent an arbitrary Lorentz transformation in the form (29) or (30). It is even simpler to express the exponents in (29), (30) directly in terms of  $\lambda(i\mathbf{b})$ ,  $\lambda(i\mathbf{b}')$ ,  $\lambda(\mathbf{r})$ , and  $\lambda(\mathbf{r}')$  using relation (24).

#### 4. Summary

The article shows how any Lorentz transformation can be represented as a sequence of a spatial rotation and a boost or a boost and a spatial rotation. Relations are found that determine the parameters of such turns and boosts. Representing an arbitrary Lorentz transformation in the form of a rotation and a boost or a boost and a rotation makes it possible to give a physical meaning to this transformation and to analyze it.

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ГІПЕРКОМПЛЕКСНЕ ПРЕДСТАВЛЕННЯ  
ГРУПИ ЛОРЕНЦА

Досліджено гіперкомплексну структуру групи Лоренца, побудовану на матрицях Дірака. Вона подібна до кватерніонної личини групи просторових поворотів. Такий вигляд має низку переваг. По-перше, у ній перетворення різних геометричних об'єктів – векторів, антисиметричних тензорів другого рангу і біспінорів – здійснюється за допомогою тих самих операторів, бо ця личина звідна. По-друге, представлення правила композиції двох довільних перетворень Лоренца має простий вигляд. Ці переваги значно спрощують знаходження багатьох закономірностей, пов'язаних із перетвореннями Лоренца. Зокрема, вони спрощують дослідження зв'язку спіна з псевдовектором Паулі–Любанського та малою групою Вігнера.

Ключові слова: гіперкомплексні числа, група Лоренца.