

D. ANCHISHKIN,<sup>1,2,3</sup> V. GNATOVSKYY,<sup>2</sup> D. ZHURAVEL,<sup>1</sup> V. KARPENKO,<sup>2</sup>  
I. MISHUSTIN,<sup>3</sup> H. STÖCKER<sup>3,4</sup>

<sup>1</sup> Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine  
(14b, Metrolohichna Str., Kyiv 03680, Ukraine)

<sup>2</sup> Taras Shevchenko National University of Kyiv  
(Kyiv 03127, Ukraine)

<sup>3</sup> Frankfurt Institute for Advanced Studies  
(1, Ruth-Moufang-Strasse, 60438 Frankfurt am Main, Germany)

<sup>4</sup> Johann Wolfgang Goethe University  
(D-60438 Frankfurt am Main, Germany)

## CANONICAL ENSEMBLE VS. GRAND CANONICAL ENSEMBLE IN THE DESCRIPTION OF MULTICOMPONENT BOSONIC SYSTEMS

UDC 539

*The thermodynamics of a system of interacting bosonic particles and antiparticles in the presence of the Bose–Einstein condensate is studied in the framework of a Skyrme-like mean-field model. It is assumed that the total charge density (isospin density) is conserved at all temperatures. Two cases are explicitly considered: the zero or nonzero isospin charge of the system. A comparative analysis is carried out using the Canonical Ensemble or the Grand Canonical Ensemble. It is shown that the Grand Canonical Ensemble is not suitable for describing the bosonic systems of particles and antiparticles in the presence of a condensate, but an adequate study can be carried out within the framework of the canonical ensemble, where the chemical potential is a thermodynamic quantity that depends on the canonical free variable.*

*Key words:* relativistic bosonic system of particles and antiparticles, Bose–Einstein condensation.

### 1. Introduction

The purpose of this work is to report on the results of studies of the thermodynamic properties of the bosonic system, in particular, the nature of phase transitions during the Bose–Einstein condensation

Citation: Anchishkin D., Gnatovskyy V., Zhuravel D., Karpenko V., Mishustin I., Stöcker H. Canonical Ensemble vs. Grand Canonical Ensemble in the description of multicomponent bosonic systems. *Ukr. J. Phys.* **69**, No. 1, 3 (2024). <https://doi.org/10.15407/ujpe69.1.3>.

Цитування: Анчішкін Д., Гнатівський В., Журавель Д., Карпенко В., Мішустін І., Штокер Х. Використання канонічного та великого канонічного ансамблів для дослідження багатокомпонентних бозонних систем. *Укр. фіз. журн.* **69**, № 1, 3 (2024).

ISSN 0372-400X. *Укр. фіз. журн.* 2024. Т. 69, № 1

with conservation of the isospin (charge) density. A scalar model of the bosonic system that develops a Bose–Einstein condensate with conservation of the isospin (charge) was first studied in [1–3]. Various aspects of free and interacting systems of relativistic bosons were discussed further in Refs. [4–9]. To introduce the problem we are going to discuss here, it is appropriate to point out some features in the description of the condensate phase. To do this, we first briefly consider the Bose condensation in a non-relativistic ideal bosonic gas within the Grand Canonical Ensemble. At high temperatures, where the system is fully in the thermal phase the density of particles  $n$  is

$$n = g \int \frac{d^3k}{(2\pi)^3} f_{\text{BE}}(E_k, \mu), \quad (1)$$

where  $g$  is the degeneracy factor,  $E_k = \mathbf{k}^2/2m$ , and the Bose–Einstein distribution function  $f_{\text{BE}}(E, \mu)$  reads

$$f_{\text{BE}}(E, \mu) = \left[ \exp\left(\frac{E - \mu}{T}\right) - 1 \right]^{-1}. \quad (2)$$

Here,  $\mu$  is the chemical potential with some profile, which provides the conservation of the particle density  $n$ . In the phase, when a condensate has been developed in the system, the total particle density consists of two contributions, the density of condensed particles  $n_{\text{cond}}$  and the density of thermal (kinetic) particles  $n_{\text{th}}$ :

$$n = n_{\text{cond}}(T) + n_{\text{th}}(T)$$

with

$$n_{\text{th}}(T) = g \int \frac{d^3k}{(2\pi)^3} f_{\text{BE}}(E_k, \mu) \Big|_{\mu = E_{\text{gs}}}, \quad (3)$$

where  $E_{\text{gs}} = 0$  is the energy of the ground state. In fact, the equality  $\mu = 0$  is a necessary condition for the formation of a condensate. We consider the homogeneous bosonic system in thermodynamic equilibrium. In this case, any thermodynamic state of the system in the framework of the Grand Canonical Ensemble is determined by two canonical variables  $(T, \mu)$ . But, such a definition of the thermodynamic state cannot be made in the condensate phase, since we have one free variable  $T$  left, and the chemical potential is fixed by the condition  $\mu = 0$ . Thus, to define the thermodynamic state, we need one more variable, which can be the total particle density  $n$ . So, we define the thermodynamic state by two free variables  $(T, n)$ . But, this means that we describe the bosonic system within the framework of the Canonical Ensemble, while the Grand Canonical Ensemble is not suitable for the description in the condensed phase. Kerson Huang pointed out: “we must re-emphasize that the Bose–Einstein condensation can only occur, when the particle number is conserved” [10]. On this way the intersection of the line  $n = \text{const}$  and the critical curve  $n_{\text{th}}(T)$  determines the corresponding critical temperature  $T_c = (2\pi/m)(n/g\zeta(3/2))^{2/3}$ , which is defined using the conserved particle density  $n$  [11]. Hence, the Grand Canonical Ensemble is not very useful for describing the ideal gas of bosons in the condensate phase. And what about the use of chemi-

cal potential in the thermal phase at high temperatures? In fact, one can determine some thermodynamic state of the system by setting the specific values of  $(T, \mu)$ . However, to keep the particle density constant, it is necessary to solve Eq. (1) with respect to  $\mu$  for a given value  $n$  and for temperatures  $T > T_c$  and get the dependence (profile)  $\mu(T, n)$ . But, this means that we study the problem again in the Canonical Ensemble, where the chemical potential is a thermodynamic function depending on the free variables  $(T, n)$  (simply put, using the chemical potential as a thermodynamic quantity does not mean using the grand canonical ensemble, see Ref. [10]).

A similar situation arises when describing an ideal relativistic bosonic gas with a constant density of particles  $n$ . In this case, the expressions for the density of particles in the high-temperature thermal phase and in the condensate phase remain the same as before in the nonrelativistic approach, i.e., Eqs. (1)–(3) are valid in the relativistic sector. Only for the single-particle ground state, we have  $E_{\text{gs}} = m$ , since the dispersion relation is now relativistic  $E_k = \sqrt{m^2 + \mathbf{k}^2}$ <sup>1</sup>. Thus, it follows from the condition of condensate formation that the chemical potential is equal to the mass of the particle  $\mu = m$  in the temperature interval  $0 \leq T \leq T_c$ , that is, in the condensate phase. On the other hand, this value of the chemical potential determines the maximum density of only thermal particles for a given temperature  $T$ .

Therefore, it is not possible to determine the thermodynamic state of the system in the presence of a condensate by a pair of variables  $(T, \mu)$ ; again, an additional variable is needed. It can be the total particle density, then we describe the system by  $(T, n)$  variables, i.e., within the Canonical Ensemble.

As one can see, in both cases, with the conserved number of particles, the Grand Canonical Ensemble cannot describe the bosonic gas in the condensate phase. This is due to the fact that the chemical potential in this phase is not a free variable. Its value is fixed by the condition of condensate formation, i.e.,  $\mu = E_{\text{gs}}$ .

Next, we are going to prove that the same situation occurs in a system of interacting relativistic bosons, when the isospin (charge) is conserved.

<sup>1</sup> Here and, below, we adopt the system of units  $\hbar = c = 1$ ,  $k_B = 1$ .

## 2. The Relativistic System of Boson Particles and Antiparticles

As was stated by Kerson Huang, the real conservation law deals with the conserved quantity that is the number of particles minus the number of antiparticles [10]. That is why, any study of the Bose–Einstein condensation in a relativistic Bose gas must take antiparticles into account. Such systems of boson particles and antiparticles were first discussed in [1–3]. Our consideration of the system of interacting bosonic particles and antiparticles at finite temperatures is carried out within the framework of the thermodynamic mean-field model, which was introduced in [12, 13] and further developed in [14]. The self-consistent equations for the total particle density  $n$  and the isospin (charge) density  $n_I$  in this model are key elements for determining the thermodynamic state of the system [15, 16]. At high temperatures, when particles ( $\pi^-$  mesons) and antiparticles ( $\pi^+$  mesons) are only in the thermal (kinetic) phase, the corresponding expressions can be written as follows<sup>2</sup>:

$$n = g \int \frac{d^3k}{(2\pi)^3} [f_{\text{BE}}(E(k, n), \mu_I) + f_{\text{BE}}(E(k, n), -\mu_I)], \quad (4)$$

$$n_I = g \int \frac{d^3k}{(2\pi)^3} [f_{\text{BE}}(E(k, n), \mu_I) - f_{\text{BE}}(E(k, n), -\mu_I)]. \quad (5)$$

Here, the Bose–Einstein distribution function  $f_{\text{BE}}(E(k, n), \mu_I)$ , which is defined in (2), contains a term that describes the interaction in the system  $U(n)$ :  $E(k, n) = \omega_k + U(n)$ ,  $\omega_k = \sqrt{m^2 + \mathbf{k}^2}$ . It is worth to note that, in the case of  $\mu_I = 0$ , the number of particles and antiparticles in the system are equal to each other, and the net charge in the system (see Eq. (5)) is equal to zero,  $n_I = 0$ . This case is discussed in detail in [15], where the system of particles and antiparticles was considered using the Grand Canonical Ensemble.

Below, we consider the possibility of the Bose–Einstein condensation in a two-component system, say,  $\pi^-$  and  $\pi^+$  mesons, with a non-zero isospin (charge) density  $n_I = n^{(-)} - n^{(+)}$ . The necessary

condition for the formation of a condensate is determined by the maximum possible population of thermal states at a certain temperature; for example, for particles, it is  $m + U(n) - \mu_I = 0$ . In view of this, the phase structure of the system can be classified according to three main scenarios:

a) high temperatures – both components, i.e., particles and antiparticles, are only in the thermal (kinetic) phase:  $m + U(n) - \mu_I > 0$  and  $m + U(n) + \mu_I > 0$ , respectively.

b) Particles are in the condensate phase<sup>3</sup> and antiparticles are only in the thermal phase:  $m + U(n) - \mu_I = 0$  and  $m + U(n) + \mu_I > 0$ , respectively.

c) Both components, particles and antiparticles, are in the condensate phase ( $\mathbf{k} = 0$ ):  $m + U(n) - \mu_I = 0$  and  $m + U(n) + \mu_I = 0$ , respectively. These conditions are equivalent to the system of equations

$$\begin{cases} \mu_I = 0, \\ U(n) + m = 0. \end{cases} \quad (6)$$

We assume that the interaction between particles is described by the Skyrme-like mean field, which depends only on the total particle-number density  $n$

$$U(n) = -An + Bn^2, \quad (8)$$

where  $A$  and  $B$  are the positive model parameters. For the given mean field (8), there are two roots of Eq. (7) with respect to  $n$ :

$$n_{1,2} = \sqrt{\frac{m}{B}} \left( \kappa \mp \sqrt{\kappa^2 - 1} \right), \quad (9)$$

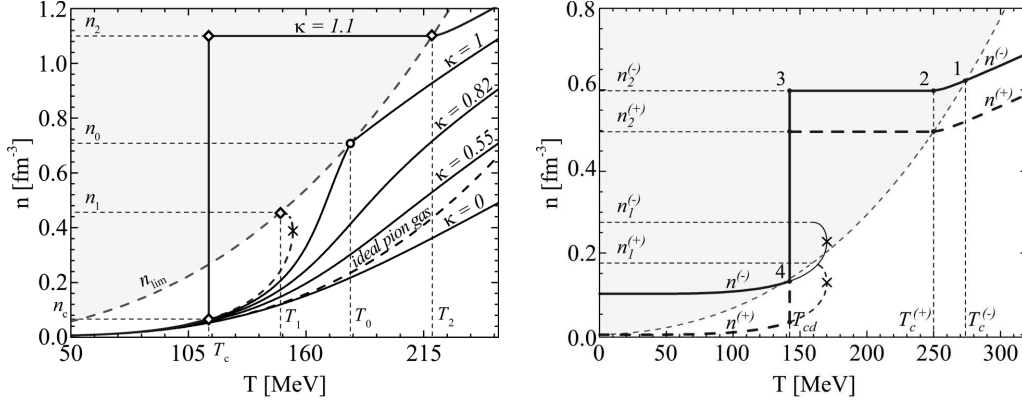
where

$$\kappa \equiv \frac{A}{2\sqrt{mB}}. \quad (10)$$

Then one can parametrize the attraction coefficient as  $A = \kappa A_c$  with  $A_c = 2\sqrt{mB}$ . The parameter  $\kappa$  determines the phase structure of the system, namely, the allowed/forbidden particle density domain. There are no real roots for the values of the parameter  $\kappa < 1$ , and we name this as a “weak” attraction. The critical value  $A_c$  is obtained, when both roots coincide, i.e.,  $\kappa = \kappa_c = 1$ , then  $A = A_c = 2\sqrt{mB}$ . The

<sup>2</sup> We simply conventionally call the particles pi-mesons. Actually, we consider bosonic particles with a mass of  $\pi$ -mesons and zero spin.

<sup>3</sup> “Particles, for example,  $\pi^-$ -mesons, are in the condensate phase” is a short name for a mixture phase, where one part of the particles is in a condensate with  $\mathbf{k} = 0$ , but another part of the same particles, i.e.,  $\pi^-$ -mesons, is in a thermal (kinetic) state.



*Left panel:* Boson system at the zero charge density  $n_I = 0$  (or  $\mu_I = 0$ ). Particle density vs. temperature at a supercritical attraction  $\kappa = 1.1$ . It is shown as a solid blue line consisting of several segments. The vertical segment (solid blue line) indicates a phase transition of the first order with the creation of the condensate. Dependencies of the particle density on the temperature at “weak” attraction  $\kappa \leq 1$  are shown as solid black lines in the thermal phase. A dashed red line is the critical curve. *Right panel:* Boson system at a finite charge density  $n_I = 0.1 \text{ fm}^{-3}$ . Temperature dependencies of the density of negative  $n^{(-)}$  and positive  $n^{(+)}$  particles are shown as a solid blue line consisting of several segments and a dashed blue line consisting of several segments, respectively. The vertical segment for both dependencies indicates a phase transition of the first order with the creation of the condensate. In the condensate phase,  $\mu_I = 0$ . A dashed red line is the critical curve  $n_{\text{lim}}(T)$ , see Eq. (12)

second interval corresponds to  $\kappa > 1$ , where Eq. (7) has two real roots. We associate this interval with a “strong” attractive interaction. The dependence of the total density of particles  $n$  on the parameter  $\kappa$  for a system with zero charge  $n_I = 0$  is shown in the figure (left panel), which was first obtained in [15] and shown here for comparison with results calculated for  $n_I \neq 0$ .

Let us consider several possibilities.

**a. Both particles and antiparticles are present only in the thermal phase.** The behavior of the particle-antiparticle boson system in the thermal phase is determined by a set of two transcendental equations (4), (5) (set (a)), which should be solved self-consistently with respect to  $n$  and  $\mu_I$  for a given temperature  $T$  and  $n_I$ . In the present study, we consider bosons with spin zero, i.e., the degeneracy factor  $g = 1$  for every boson component. We would like to point out that, in fact, we consider the many-particle system in the Canonical Ensemble, where the independent canonical variables are  $T$  and  $n_I$ . In this approach, the chemical potential  $\mu_I$  is a thermodynamic function, which depends on the canonical variables.

**b. Particles are in the condensate phase and antiparticles are only in the thermal phase.** When the particles are in the condensate phase and antiparticles are still in the thermal phase, Eqs. (4) and (5) should be generalized to include the con-

densate component of  $\pi^-$  mesons,  $n_{\text{cond}}^{(-)}$ . It should be taken into account that the particles (negatively charged component) can be in a condensed state under the necessary condition

$$m + U(n) - \mu_I = 0. \quad (11)$$

Let us look on the evolution of the particle-antiparticle system during its cooling from high temperatures, where the two components are both in the thermal phase. When the temperature decreases from high values, the density of particles  $n^{(-)}(T, n_I)$  first reaches the critical curve at the temperature  $T_c^{(-)}$ , where condition (11) is fulfilled, see Figure, right panel. This condition means that, in place of the argument  $(E(k, n) - \mu_I)/T$  of the Bose–Einstein distribution function of charge-dominant component ( $\pi^-$  mesons), we put the argument  $(\omega_k - m)/T$  and get the curve  $n_{\text{lim}}(T)$ , defined as

$$n_{\text{lim}}(T) = \int \frac{d^3k}{(2\pi)^3} f_{\text{BE}}(\omega_k, \mu_I) \Big|_{\mu_I=m}, \quad (12)$$

which coincide with the critical curve  $n_{\text{th}}(T)$  for the ideal single-component gas defined in Eq. (3). This is, indeed, a critical curve for  $\pi^-$  mesons (see the red dashed curve in the figure, right panel), since, at  $\mu_I = m$  and the temperature  $T$ , the Bose–Einstein distribution function reaches its maximum value, which

determines the maximum density of thermal particles at this particular temperature in the system of interacting bosons.

So, the crossing point of the curves  $n^{(-)}(T, n_I)$  and  $n_{\text{lim}}(T)$  determines the critical temperature  $T_c^{(-)}$  for the high-density component of the system. At  $T < T_c^{(-)}$ , the density of thermal  $\pi^-$ -mesons coincides with the critical curve (12), i.e.,  $n_{\text{th}}^{(-)} = n_{\text{lim}}(T)$ . Thus, the density  $n^{(-)}$  of  $\pi^-$  mesons consists of two parts, the density of the condensed  $\pi^-$  mesons,  $n_{\text{cond}}^{(-)}$ , and the density of thermal  $\pi^-$  mesons,  $n_{\text{th}}^{(-)}$ , or  $n^{(-)} = n_{\text{cond}}^{(-)}(T) + n_{\text{lim}}(T)$ . Therefore, in the temperature interval  $T_c^{(+)} \leq T \leq T_c^{(-)}$ , we can write the generalization of the set of Eqs. (4) and (5) (set (b)) as

$$n = n_{\text{cond}}^{(-)}(T) + n_{\text{lim}}(T) + \int \frac{d^3k}{(2\pi)^3} f_{\text{BE}}(E(k, n), -\mu_I) \Big|_{\mu_I = m + U(n)}, \quad (13)$$

$$n_I = n_{\text{cond}}^{(-)}(T) + n_{\text{lim}}(T) - \int \frac{d^3k}{(2\pi)^3} f_{\text{BE}}(E(k, n), -\mu_I) \Big|_{\mu_I = m + U(n)}. \quad (14)$$

Solving this system of equations with respect to  $n$  and  $n_{\text{cond}}^{(-)}$  for fixed values of  $(T, n_I)$  provides the function  $n^{(+)}(T, n_I)$  in the interval  $T_c^{(+)} \leq T \leq T_c^{(-)}$ , see the blue dashed curve in the figure, right panel. Obviously, only thermal  $\pi^+$  mesons contribute to the density  $n^{(+)}$ . On the other hand, two fractions of  $\pi^-$ -mesons contribute to the density  $n^{(-)}$ : condensed particles with the particle-number density  $n_{\text{cond}}^{(-)}(T)$ , and thermal particles with the density  $n_{\text{lim}}(T)$ .

**c. The temperature interval, where both particles and antiparticles are in the condensate phase.** In this case, additionally to condition (11) for  $\pi^-$  mesons, there must be a corresponding condition for  $\pi^+$  mesons to ensure that both components of the gas are in the condensate at the same temperature  $T$  and chemical potential  $\mu_I$ . As we have shown, this requirements lead to Eqs. (6) and (7). With account for these conditions, Eqs. (13) and (14) should be modified to include the condensate component  $n_{\text{cond}}^{(+)}$  of  $\pi^+$  mesons assuming that the density of thermal  $\pi^+$  mesons equals now to  $n_{\text{lim}}(T)$ , i.e., the same way as the density of thermal  $\pi^-$  mesons. So, when both

components are in the condensate, set (c) of the self-consistent equations reads

$$n = n_{\text{cond}}^{(-)}(T) + n_{\text{cond}}^{(+)}(T) + 2n_{\text{lim}}(T), \quad (15)$$

$$n_I = n_{\text{cond}}^{(-)}(T) - n_{\text{cond}}^{(+)}(T). \quad (16)$$

Indeed, since  $\mu_I = 0$ , using, for example, the root  $n = n_2$  in the argument of the Bose–Einstein distribution functions in Eqs. (13), (14) leads to  $U(n_2) = -m$ , which results in the density  $n_{\text{lim}}(T)$ . Thus, we obtain the system of equations (15), (16). The corresponding particle density of each component  $\pi^-$  or  $\pi^+$ , obtained as a result of splitting the total density, for example  $n_2$ , is equal to  $n_2^{(-)} = (n_2 + n_I)/2$  and  $n_2^{(+)} = (n_2 - n_I)/2$ , respectively.

It turns out that the solutions to cases (b) and (c) exist in the same temperature interval. We are speaking now about the temperature interval  $0 < T < T_c^{(+)}$ , see the figure, right panel. In addition to the self-consistent solutions of equations (b) for these temperatures, there are two more “condensate” branches of solutions associated with the roots  $n_1$  and  $n_2$  of the equation  $U(n) + m = 0$ . Meanwhile, in the competition between two “condensate” branches, the second branch, created by  $n_2$ , is preferable, since the pressure corresponding to these states is higher. On the other hand, the competition between solutions (b) and (c) is resolved also in the standard way according to the Gibbs criterion: the states that correspond to the larger pressure are predominant in the thermodynamic realization. Using this rule, we find the temperature  $T_{\text{cd}}$  from the equation  $p_{(b)}(T, n_I) = p_{(c)}(T, n_I)$ , where the pressure  $p_{(b)}(T, n_I)$  corresponds to solutions of the set of equations (b), and  $p_{(c)}(T, n_I)$  to the set of equations (c). For the temperatures higher than  $T_{\text{cd}}$ , pressure (c) dominates, i.e.,  $p_{(c)}(T, n_I) > p_{(b)}(T, n_I)$ . Therefore, according to the Gibbs criterion, the transition to another branch of self-consistent solutions gives rise to a phase transition of the first order at the temperature  $T_{\text{cd}}$ .

Results of the numerical solution of the sets of equations (a), (b), and (c) at  $\kappa = 1.1$  ( $B = 10mv_0^2$ ,  $m = 140 \text{ MeV}$ ,  $v_0 = 0.45 \text{ fm}^3$ ) for the particle-number densities are depicted in the figure in the right panel. The density  $n^{(-)}(T)$  of  $\pi^-$  mesons depicted as a solid blue curve that consists of several horizontal segments and one vertical segment,

which represents a first-order phase transition. The density  $n^{(+)}(T)$  of  $\pi^+$  mesons is depicted as a dashed blue line, which also consists of several horizontal segments and one vertical segment, which also represents a phase transition of the first order. Metastable and forbidden states belonging to “thermal” solutions are shown in the figure in both panels as dashed and dotted segments, resembling the extensions of thermal branches. The right panel of the figure shows that the system is actually described with a conserved isospin (charge) density, i.e., for every temperature point on the graph, the condition  $n^{(-)}(T) - n^{(+)}(T) = 0.1 \text{ fm}^{-3}$  is fulfilled. The conservation of charge leads to the splitting of roots (9) shown in the left panel. It looks like:  $n_1 \rightarrow n_1^{(-)}, n_1^{(+)}$  and  $n_2 \rightarrow n_2^{(-)}, n_2^{(+)}$ , where the difference is constant, for example,  $n_2^{(-)} - n_2^{(+)} = n_I$ .

In both panels of Figure, in the condensate phase, both systems are represented by *zero chemical potential* regardless of whether the particle-antiparticle system described in the right panel has a finite charge density, i.e.,  $n_I = 0.1 \text{ fm}^{-3}$ , while the particle-antiparticle system described in the left panel is characterized by the zero charge density, i.e.,  $n_I = 0$ . So, if one intends to study both systems, one system with a finite charge density and another one with zero charge density within the Grand Canonical Ensemble, then the canonical variables should be  $(T, \mu_I = 0)$  when describing the condensate phase in both systems.

It seems that we come to some contradiction, because, in the textbooks, it is written that the chemical potential should reflect the conservation of charge or conservation of the particle number. This contradiction is resolved by realizing that the Grand Canonical Ensemble with the canonical variables  $(T, \mu)$  is suited for the description of the thermal phase or for a description of the particles, which are in the kinetic states, but not in the condensate states. Indeed, it can be seen in the figure on the right panel in the temperature interval that corresponds to the condensate phase, that is, between points 2 and 3 on the graph, that, for every temperature from this interval, the density of thermal  $\pi^-$  mesons is equal to the density of thermal  $\pi^+$  mesons, both equal to  $n_{\text{lim}}(T)$  defined in (12). In other words, these two densities of thermal mesons, which are characterized by  $\mu_I = 0$ , are equal to the density on the critical curve  $n_{\text{lim}}$ , which is depicted as the dashed red line in

the figure (right panel). That is why the charge density, which is determined only by thermal particles, is zero, i.e.,  $n_I^{(\text{therm})} = n_{\text{lim}}(T) - n_{\text{lim}}(T) = 0$ . Therefore, the chemical potential, which corresponds to the charge of these thermal particles, is also zero. We can conclude that the chemical potential  $\mu_I$  is a useful quantity only for describing the thermal or kinetic particles. This automatically leads to the fact that the Grand Canonical Ensemble, where the chemical potential  $\mu_I$  is a free variable, can be adequate to describe the bosonic system only in the thermal phase. In the condensate phase, an adequate tool for describing this phase is the Canonical Ensemble, where the chemical potential  $\mu_I(T, n_I)$  is a thermodynamic quantity that depends on free variables. It should be noted that this statement fully corresponds to our conclusion about the description of the ideal bosonic gas in the condensate phase obtained in the Introduction.

### 3. Other Examples

One can argue that the mean field depends on the isospin density. Indeed, consider the thermodynamic mean-field model, where the mean field depends on the total particle density  $n$  and also depends on the isospin density  $n_I$  (this possibility was discussed in [16]). As shown in Ref. [13], since  $n$  and  $n_I$  are independent thermodynamic variables, the form of this mean field is as follows:  $U^{(\mp)}(n, n_I) = U(n) \mp U_I(n_I)$ , where  $U_I(n_I)$  is an odd function, for example,  $U_I(n_I) \propto n_I$ , and the field  $U^{(-)}$  acts on  $\pi^-$  mesons, while  $U^{(+)}$  acts on  $\pi^+$  mesons. Then, if  $\pi^-$  and  $\pi^+$  mesons are in the condensate phase, two necessary conditions must be fulfilled:  $m + U(n) - U_I(n_I) - \mu_I = 0$  and  $m + U(n) + U_I(n_I) + \mu_I = 0$ . From here, we get the equivalent equations:  $m + U(n) = 0$  and  $\mu_I = -U_I(n_I)$ . Therefore, the chemical potential is fixed by the condition of condensate formation and is determined by the isospin density, which remains constant. Hence, when the mean interaction in the system depends on the isospin (charge) density, we again conclude that  $\mu_I$  cannot be a free variable in the presence of a condensate. Hence, the Grand Canonical Ensemble is not applicable in the condensate phase.

When describing the same many-boson system at a finite charge density in the field-theoretic approach [17], we encounter exactly the same paradox. Indeed, for the development of a condensate by both parti-

cles and antiparticles, two conditions must be met:  $M - \mu_I = 0$  and  $M + \mu_I = 0$ , where  $M$  is the effective mass of quasiparticles. By the complete analogy with case (c) discussed above, these conditions lead to two equations:  $M = 0$  and  $\mu_I = 0$ . So, it turns out that the system with a finite charge density  $n_I \neq 0$  is characterized by zero value of the chemical potential. On the other hand, we see that, in the presence of a condensate, the density of thermal particles is the same in the negatively and positively charged components of the system, i.e.,  $n_{\text{th}}^{(-)}(T) = n_{\text{th}}^{(+)}(T)$ . Hence, the problem can be resolved by accepting that the chemical potential is responsible only for thermal (kinetic) particles.

The picture obtained becomes even more striking, when we study the conservation of charge in a relativistic ideal boson gas of particles and antiparticles at  $n_I \neq 0$ . Indeed, if we assume that particles and antiparticles are simultaneously in the condensate phase, then two conditions must be satisfied simultaneously:  $m - \mu_I = 0$  and  $m + \mu_I = 0$ , where  $\mu_I$  is the isospin chemical potential, which corresponds to  $n_I$ . This leads to two equations:  $m = 0$  and  $\mu_I = 0$ . As we can see, the first equation is not possible or unphysical. That is, only one condition can be fulfilled, for example,  $m - \mu_I = 0$ . So, we can formulate the theorem: *In a relativistic bosonic ideal gas of particles and antiparticles with a conserved isospin (charge)  $n_I \neq 0$ , only one component of the system can form the condensate phase. In the case of zero isospin  $n_I = 0$ , the system of particles and antiparticles does not form a condensate.* The dashed black line in the figure (left panel) can be seen as an illustration of the second statement.

Almost the same features in the behavior of the ideal relativistic particle-antiparticle bosonic gas were noticed by H.E. Haber and H.A. Weldon in 1981 in Ref. [1]: below  $T_c$  the chemical potential determines only the charge density of the excited states  $\rho - \rho_0$ , where  $\rho_0$  is the charge density of the ground state. This actually means that the boson system in the condensate phase cannot be adequately described by a Grand Canonical Ensemble, but can be described by a Canonical Ensemble, where  $\mu(T, n_I)$  (according to our notation:  $\rho = n_I$ ,  $\rho_0 = n_{\text{cond}}$ ). In [3], the authors wrote that “since  $n_I$  is a physical quantity, and  $\mu_I$  is a derived quantity”, Eq. (5) is, in fact, an implicit formula for  $\mu_I$  as a function of  $n_I$  and  $T$ . This means that the Canonical Ensemble is used.

In addition, in the thermodynamic mean-field model considered above, we have similar conditions for the formation of a condensate by both components:  $m + U(n) - \mu_I = 0$  and  $m + U(n) + \mu_I = 0$ . When the effective interaction in the system vanishes, i.e.,  $U(n) \rightarrow 0$ , these conditions are the reason for the same statement that only one component of the bosonic particle-antiparticle system can develop a condensate. As we have seen above, both components can be in a condensed state only with the help of the attractive mean field, when its value is equal to the mass of a particle, i.e.  $m = |U(n)|$ .

#### 4. Conclusion

We have shown how the relativistic interacting system of Bose particles and antiparticles can be described at zero and finite isospin (charge) densities. In both cases, the meson system develops a first-order phase transition for sufficiently strong attractive interactions via forming a Bose condensate with releasing the latent heat. The model predicts that the condensed phase is characterized by a constant density of particles.

We have demonstrated that the Grand Canonical Ensemble is *not suitable* for describing a multicomponent bosonic system in the presence of the condensate phase. In particular, it cannot describe the condensate state in the system of particles and antiparticles. The reason is that the chemical potential is not a free parameter in the condensate phase, and its values are determined by the necessary condition for condensate formation. As we have shown, these statements are valid in interacting bosonic systems, as well as in an ideal bosonic gas.

Details of the calculations presented in this work can be found in [18].

*D.A. is very grateful to J. Steinheimer and O. Philipsen for useful discussions and comments and greatly appreciates the warm hospitality and support provided by FIAS administration and the scientific community. The work of D.Zh. and D.A. was supported by the Simons Foundation and by the Program “The structure and dynamics of statistical and quantum-field systems” of the Department of Physics and Astronomy of the NAS of Ukraine. I.M. thanks FIAS for the support and hospitality. H.St. thanks for the support from the J.M. Eisenberg Professor Laureatus of the Fachbereich Physik.*

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Received 06.12.23

Д. Анчішкін, В. Гнатовський,  
Д. Журавель, В. Карпенко, І. Мішустін, Х. Штокер

#### ВИКОРИСТАННЯ КАНОНІЧНОГО ТА ВЕЛИКОГО КАНОНІЧНОГО АНСАМБЛІВ ДЛЯ ДОСЛІДЖЕННЯ БАГАТОКОМПОНЕНТНИХ БОЗОННИХ СИСТЕМ

В роботі досліджено термодинамічні властивості системи взаємодіючих бозонних частинок і античастинок за наявності бозе-ейнштейнівського конденсату. В рамках моделі середнього поля розглядається система за умови збереження повної густини заряду (ізоспінової густини) в усьому діапазоні температур. Порівняльний аналіз виконано за допомогою канонічного та великого канонічного ансамблів. Показано, що за наявності конденсату великий канонічний ансамбль не дає адекватного опису систем бозе-частинок і античастинок. Коректний опис таких систем можна провести в рамках канонічного ансамблю, в якому хімічний потенціал є термодинамічною величиною, що залежить від відповідних канонічних змінних.

*Ключові слова:* релятивістські бозонні системи, конденсація Бозе–Ейнштейна.