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A Method of Integral Equations for a Structure in the Form of Thin Elastic Plates

Рассмотрена вторая основная задача динамики тонких упругих пластин в рамках модели Кирхгофа. С помощью динамического аналога потенциала простого слоя задача сводится к системе нестационарных граничных уравнений. Получены численные решения этих систем. Исследованы сходимости метода дискретных особенностей для прямоугольной пластины.

The second basic dynamic problem for thin elastic plates in Kirchhoff model is under consideration. The problem reduces to system of the non-stationary boundary equations by means of dynamic analogue of a single layer potential. The numerical solutions of these systems have been obtained. The investigation of convergence of method of discrete singularities for the plate of the rectangular form have been carried out.

Розглянуто другу основну задачу динаміки тонких пружних пластин у рамках моделі Кірхгофа. За допомогою динамічного аналога потенціалу простого шару задача зводиться до системи нестаціонарних граничних рівнянь. Одержано чисельні розв'язки цих систем. Досліджено збіжності методу дискретних особливостей для прямокутної пластини.

Introduction. Thin elastic plates are elements of numerous structures being used in aerospace and electronic engineering and many other industries. That is why the development of the methods for a calculation of strains arising in a plate is a very important problem. The potential theory method is suggested in the article to reduce the problems to the systems of non-stationary boundary equations. The plate of the rectangular form and the plate with a square hole are considered for the further numerical calculation. The method of the research is based on the scheme developed in [1–4] for the problems of elastodynamics and in [5–7] for the transient diffraction problems for acoustic and electromagnetic waves. The modern studied problems of mathematical modeling of thin elastic plates also are directed to researches of floating structures such as an ice cover of ocean, supertankers, floating platform (Megafloat), etc [8–13].

Notation and statement of the problem

A thin elastic plate of thickness $h=\text{const}$ occupying a domain $\Omega\times\left[-\frac{h}{2};\frac{h}{2}\right]$, is considered, where Ω is bounded by a closed C^2-curve Γ. From the standard Kirchhoff hypotheses it follows that the displacement vector at $(x;x_3)$, where $x=(x_1;x_2)$, has a form $(-x_3\partial_1 u(x,t);\ -x_3\partial_2 u(x,t);\ u(x,t))$. $u(x,t)$ is the displacement of the middle plane of the plate, $\partial_i=\frac{\partial}{\partial x_i}$, $i=1,2$. For the sake of simplicity we consider only the homogeneous case, i.e. the case of zero loading and zero initial data. This does not lead to a loss of generality since the existing of non-homogeneities can be transferred to the boundary conditions. The function $u(x,t)$ satisfies the problem

$$\begin{gathered}\rho h\partial_t^2 u(x,t)+\tilde{D}\Delta^2 u(x,t)=0,\ (x,t)\in\Omega\times R_+;\\ \begin{cases}u(x,0)=0,\\ \partial_t u(x,0)=0,\end{cases} x\in\Omega;\\ \begin{cases}(Qu)(x,t)=g_1(x,t),\\ (-Mu)(x,t)=g_2(x,t),\end{cases} (x,t)\in\Gamma\times R_+;\end{gathered} \tag{1}$$

where

$$Qu=-D\left(\partial_n\Delta u+(1-\nu)\partial_\tau[n_1 n_2(\partial_2^2 u-\partial_1^2 u)+ +(n_1^2-n_2^2)\partial_1\partial_2 u]\right),$$

$$Mu=-D\left(\Delta u+(1-\nu)\times \times[2n_1 n_2\partial_1\partial_2 u-n_2^2\partial_1^2 u-n_1^2\partial_2^2 u]\right)$$

are the operations that correspond to the generalized cutting force and the bending moment, $R_+=(0;+\infty)$, $D=\frac{\tilde{D}}{\rho h}$, $\partial_t=\frac{\partial}{\partial t}$, ∂_n is the normal derivative, $n(x)=(n_1,n_2)$ is a unit outward normal to Γ, ρ is the surface density of the plate, $\tilde{D}$ is its cylindrical stiffness, ∂_τ is the tangent derivative to Γ and unit vector τ is obtained by turning n on the angle $\frac{\pi}{2}$ against the hour-hand.

Later on let us consider the interior II^+ and exterior II^- problems in interior domains $\Omega^+=\Omega$ and $\Omega^-=R^2\setminus\overline{\Omega^+}$, respectively.

Function spaces

We introduce the function spaces by the scheme used in [15]. We choose and fix $\kappa > 0$. Denote $C_\kappa = \{p \in C\colon \Re ep > \kappa\}$. Let $H_{L;m,k,\kappa}(\Omega)$ and $H_{L;m,k,\kappa}(\Gamma)$ be the spaces of functions $U(p) = u(x,p),\ x \in \Omega$, $F(p) = f(x,p),\ x \in \Gamma$, $p \in C_\kappa$, respectively, are homeomorphisms from C_κ to standard Sobolev spaces $H_m(\Omega)$, $H_m(\Gamma)$ with their finite norms

$$\|u\|^2_{m,k,\kappa;\Omega} = \sup_{\sigma>\kappa} \int_{-\infty}^{\infty} (1+|p|)^{2k} \|u\|^2_{m,p;\Omega}\, d\tau,$$
$$\|f\|^2_{m,k,\kappa;\Gamma} = \sup_{\sigma>\kappa} \int_{-\infty}^{\infty} (1+|p|)^{2k} \|f\|^2_{m,p;\Gamma}\, d\tau, \qquad p = \sigma + i\tau.$$

Let $G = \Omega \times R_+$, $\Sigma^+ = \Gamma \times R_+$ and let L be the Laplace transformation operator. The spaces $H_{r;m,k,\kappa}(G)$ and $H_{r;m,k,\kappa}(\Sigma^+)$ consist of the inverse Laplace transformations $u(x,t) = L^{-1}u(x,p)$ and $f(x,t) = L^{-1} f(x,p)$ of elements $u(x,p) \in H_{L;m,k,\kappa}(\Omega)$ and $f(x,p) \in H_{L;m,k,\kappa}(\Gamma)$ respectively. The norms of these spaces are defined by $\|u\|_{m,k,\kappa;G} = \|Lu\|_{m,k,\kappa;\Omega}$, $\|f\|_{m,k,\kappa;\Sigma_+} = \|Lf\|_{m,k,\kappa;\Gamma}$.

Let $G^\pm = \Omega^\pm \times R_+$. Denote by $\vec{\chi}^\pm$ the trace operators that are continuous from $H_{r;m,k,\kappa}(G^\pm)$ to $H_{r;m-1/2,k,\kappa}(\Sigma^+) \times H_{r;m-3/2,k,\kappa}(\Sigma^+)$ for $m > 3/2, k \in R$. The element $\vec{\chi}^\pm u$ is the couple consisting of the traces of $u(x,t)$ and $\partial_n u$ on Σ_+.

The solution of the problems $\text{II}^\pm$ is the element $u(x,t) \in H_{r;2,0,\kappa}(G^\pm)$ that satisfies

$$\int_0^\infty a_\pm(u,v)dt - \int_{G^\pm} \partial_t u \overline{\partial_t v} dx dt = \pm \int_0^\infty \left\langle \vec{g}, \vec{\chi}^\pm v \right\rangle dt, \quad (2)$$

for an arbitrary $v(x,t) \in C^\infty(\overline{G^\pm})$ with the compact support. In (2) $\vec{g} = (g_1, g_2)$ and $\left\langle \vec{g}, \vec{\chi}^\pm v \right\rangle_{0,\Gamma}$ is the inner $\left[L^2(\Gamma)\right]^2$-product.

Theorem 1. *For any* $\vec{g} = (g_1, g_2) \in H_{r;-3/2,k,\kappa}(\Sigma^+) \times H_{r;-1/2,k,\kappa}(\Sigma^+)$, $k \in R, \kappa > 0$, *the problems* $II^\pm$ *have the unique solutions* $u(x,t) \in H_{r;2,k-1,\kappa}(G^\pm)$ *for all* $k \ge 1, \kappa > 0$. *The estimates*

$$\|u\|_{2,k-1,\kappa;G^\pm} \le c\left(\|g_1\|_{-3/2,k,\kappa;\Sigma^+} + \|g_2\|_{-1/2,k,\kappa;\Sigma^+}\right)$$

hold, where c *is some positive constant.*

The dynamic single layer potentials

Let $\Phi(x,t)$ be the fundamental solution of the equation of oscillations that satisfies

$$\partial_t^2 \Phi(x,t) + D\Delta^2 \Phi(x,t) = \delta(x,t),$$
$$\Phi(x,t) = 0, \quad t < 0,$$

where $\delta(x,t)$ is the Dirac function.

It is easy to verify that

$$\Phi(x,t) = \frac{\theta(t)}{4\pi\sqrt{D}}\, si\left(\frac{|x|^2}{4t\sqrt{D}}\right),$$

where $si(z) = -\int_z^\infty \frac{\sin\mu}{\mu} d\mu$ and $\theta(t)$ is the characteristic function of $(0;+\infty)$. The dynamic single layer potential with a defined on $\Gamma \times R$ two-component density $\vec{\alpha}(x,t)$ is introduced by

$$(V\vec{\alpha})(x,t) = \int_R \int_\Gamma \Phi(x-y, t-\tau)\alpha_1(y,\tau) + \partial_{n,y}\Phi(x-y,t-\tau)\alpha_2(y,\tau) ds_y d\tau,$$

where $\partial_{n,y}$ is the normal derivative with respect to y.

Obviously at least for the smooth finite densities on $\Gamma \times R$ the potential satisfies in $\Omega \times R_+$ the homogeneous oscillation equation. If the densities vanish as $t < 0$ then the potential satisfy the zero initial data. The single layer potential and its first derivatives are continuous when a point goes across the boundary curve. The jump formulae for the single layer potentials has the form

$$\begin{aligned}(QV\vec{\alpha})^\pm(x,t) &= \pm\alpha_1(x,t) + (QV\vec{\alpha})^0(x,t),\\ (-MV\vec{\alpha})^\pm(x,t) &= \pm\alpha_2(x,t) + (-MV\vec{\alpha})^0(x,t),\end{aligned} \qquad (x,t) \in \Gamma \times R_+, \quad (3)$$

where the superscripts «±» denote the limiting value of the corresponding functions when (x,t) tends to $\Gamma \times R_+$ from $\Omega^+ \times \mathrm{R}_+$ and the superscript «0» denotes the direct value of the corresponding integral.

The representation of the solutions of the problem (1) by the single layer potential yields the systems

$$\begin{cases} (QV\vec{\alpha})^{\pm}(x,t) = g_1(x,t), \\ (-MV\vec{\alpha})^{\pm}(x,t) = g_2(x,t), \end{cases} \quad (x,t) \in \Gamma \times R_+ . \qquad (4)$$

The solvability of the system (4) is proved in a one-parameter scale of Sobolev-type function in [14].

Theorem 2. For all $\vec{g} \in \mathrm{H}_{r;-3/2,k,\kappa}\left(\Sigma^+\right) \times H_{r;-1/2,k,\kappa}\left(\Sigma^+\right)$, $k \geq 1, \kappa > 0$ the potentials $(V\vec{\alpha})(x,t)$, with densities $\vec{\alpha}(x,t)$ that are the solutions of (4), are the solutions $u \in H_{r;2,k-1,\kappa}(G^{\pm})$ of the problems $II^{\pm}$.

Systems of the boundary equations

The plate of the rectangular form and the plate with a square hole are considered (fig. 1, 2).

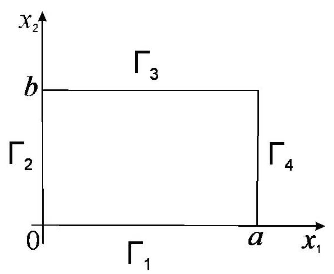

Fig. 1

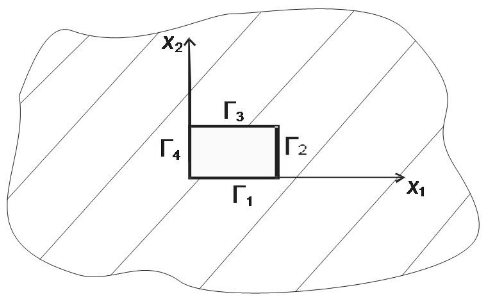

Fig. 2

The obvious kind of the boundary equations system (4) is received, taking into account the jump formulae (3). For example, the equations on $\Gamma_1:\ x = (x_1;0)$, $0 < x_1 < a$, look like

$$\pm\frac{1}{2}\alpha_1(x_1,0,t) +$$

$$+\int_{\Gamma_1}\left(\alpha_2(s,0,t)\left[\frac{\theta(t)(1+\nu)}{4D\pi(x_1-s)^2} + Q_{12}(x_1-s,t)\right]\right)ds +$$

$$+\sum_{i=2}^{4}\left(\int_{\Gamma_i}\alpha_1(s,t)Q_{i1}(s,x_1,t) + \alpha_2(s,t)Q_{i2}(s,x_1,t)\right)ds +$$

$$+\int_0^{\infty}\left(\int_{\Gamma_1}\frac{\alpha_2(s,0,t)-\alpha_2(s,0,\tau)}{(x_1-s)^2(t-\tau)^2}Q_{14}ds +\right.$$

$$+\sum_{i=2}^{4}\int_{\Gamma_i}\left(\frac{\alpha_1(s,t)-\alpha_1(s,\tau)}{(t-\tau)^2}Q_{i3} +\right.$$

$$\left.\left.+\frac{\alpha_2(s,t)-\alpha_2(s,\tau)}{(t-\tau)^2}Q_{i4}\right)ds\right)d\tau = g_1(x_1,t).$$

$$\pm\frac{1}{2}\alpha_2(x_1,0,t) - \left[\int_{\Gamma_1}\alpha_1(s,0,t)M_{11}(s,x_1,t)ds +\right.$$

$$+\sum_{i=2}^{4}\left(\int_{\Gamma_i}\alpha_1(s,t)M_{i1}(s,x_1,t) + \alpha_2(s,t)M_{i2}(s,x_1,t)\right)ds +$$

$$+\int_0^{\infty}\left(\int_{\Gamma_1}\frac{\alpha_1(s,0,t)-\alpha_1(s,0,\tau)}{(t-\tau)}M_{13}ds +\right.$$

$$+\sum_{i=2}^{4}\int_{\Gamma_i}\left(\frac{\alpha_1(s,t)-\alpha_1(s,\tau)}{(t-\tau)}M_{i3} +\right.$$

$$\left.\left.\left.+\frac{\alpha_2(s,t)-\alpha_2(s,\tau)}{(t-\tau)^2}M_{i4}\right)ds\right)d\tau\right] = g_2(x_1,t).$$

M_{ij}, Q_{ij}, $i = \overline{1,4}$, $j = 1,2$ are integrated on *t*, and have no features on a spatial variable. The functions

$$M_{i1} = -\theta(t)\, m_{i1}(x_1,s)\, t\sin\frac{w_{i1}(x_1,s)}{t} +$$

$$+p_{i1}\int_0^{\infty}\theta(t-\tau)\sin\frac{w_{i1}(x_1,s)}{(t-\tau)}d\tau,$$

$$M_{i2} = \theta(t)m_{i2}(x_1,s)\cos\frac{w_{i2}(x_1,s)}{t} +$$

$$+\theta(t)\,p_{i2}(x_1,s)t\sin\frac{w_{i2}(x_1,s)}{t},$$

$$Q_{i1} = \theta(t)q_{i1}(x_1,s)\cos\frac{v_{i1}(x_1,s)}{t} +$$

$$+\theta(t)h_{i1}(x_1,s)\cdot t\sin\frac{v_{i1}(x_1,s)}{t},$$

$$Q_{22} = \theta(t) q_{i2}(x_1, s) \frac{1}{t} \cdot \sin \frac{v_{i2}(x_1, s)}{t} +$$

$$+ \theta(t) h_{i2} \cos \frac{v_{i2}(x_1, s)}{t} - \theta(t) z_{i2}(x_1, s) \cdot t \sin \frac{v_{i2}(x_1, s)}{t}.$$

M_{ij}, Q_{ij}, $i = \overline{1,4}$, $j = 3,4$ have no singularities. The results turn out similar on other sides of the border.

Calculation results

Let us consider the interior problem. The plate has the sizes 1 × 1 × 0,1 (m), the Poison coefficient $\nu = 0,3$, the density $\rho = 7800\, kg/m^3$, Yuong module $E = 2,1 \cdot 10 MPa$. The boundaries of a plate are free. Thus the problem

$$\rho h \partial_t^2 u(x,t) + \tilde{D} \Delta^2 u(x,t) = q(x,t),$$
$$(x,t) \in (0;1) \times (0;1) \times R_+$$
$$\begin{cases} u(x,0) = 0, \\ \partial_t u(x,0) = 0, \end{cases} x \in (0;1) \times (0;1) \quad (5)$$
$$\begin{cases} (Qu)(x,t) = 0, \\ (-Mu)(x,t) = 0, \end{cases} (x,t) \in \Gamma \times R_+$$

is solved. Let's consider various kinds of loads. The point displacements of medial plane of a plate under loads $q = q_0 \sin t$ and $q = q_0 (t-2)^2 (t-9)^2$ are represented in fig. 3 and fig. 4, respectively ($x_1 = x_2 = 0,5$).

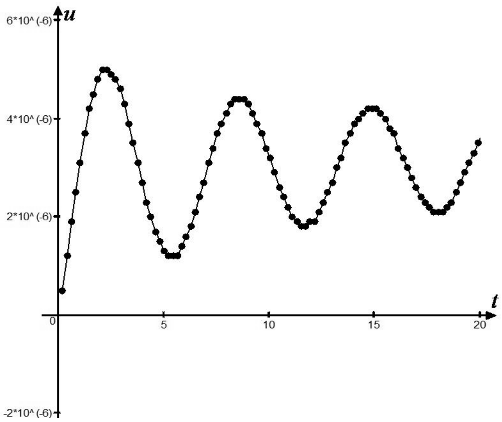

Fig. 3

Also let us solve an exterior problem. The plate with the square hole of the sizes 1 × 1 × 0,1 (m) is under consideration for Poison coefficient $\nu = 0,3$, plate density $\rho = 7800\, kg/m^3$, Yuong module $E =$ $= 2,1 \cdot 10 Mpa$. The boundaries of a plate are free.

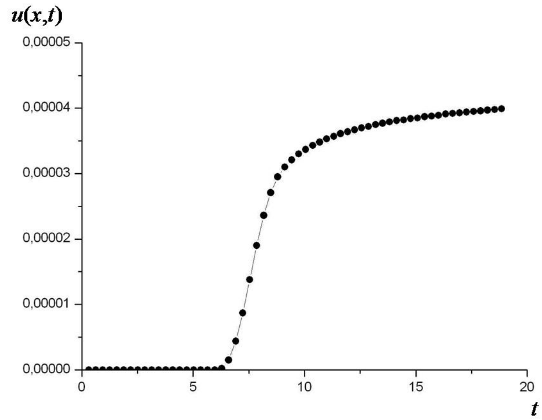

Fig. 4

The problem

$$\rho h \partial_t^2 u(x,t) + \tilde{D} \Delta^2 u(x,t) = q(x,t), \ (x,t) \in$$
$$\in \big((-\infty;0) \cup (1;\infty)\big) \times \big((-\infty;0) \cup (1;\infty)\big) \times R_+$$
$$\begin{cases} u(x,0) = 0, \\ \partial_t u(x,0) = 0, \end{cases} x \in \big((-\infty;0) \cup (1;\infty)\big) \times \big((-\infty;0) \cup (1;\infty)\big)$$
$$\begin{cases} (Qu)(x,t) = 0, \\ (-Mu)(x,t) = 0, \end{cases} (x,t) \in \Gamma \times R_+$$

is solved. Let us consider various kinds of loads. The point displacements of a medial plane of plate under loads $q = q_0 \sin t$ and $q = q_0 (t-6)^2 (t-9)^2$ are represented in fig. 5 and fig. 6, respectively ($x_2 = 0,5$).

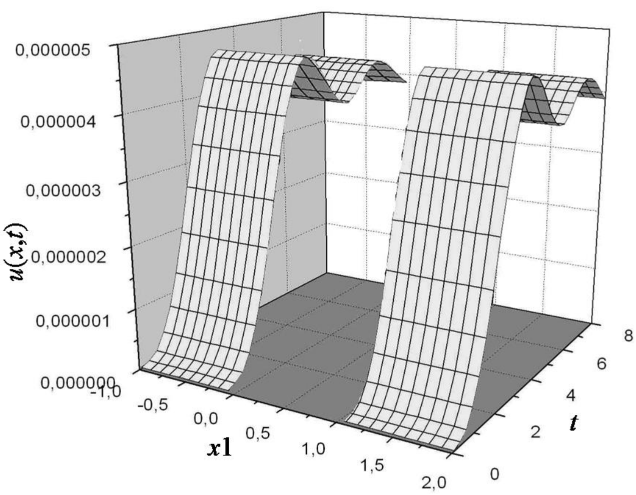

Fig. 5

Later on we consider the plate point displacements in various time moments under load $q = q_0 \sin t$ for the interior (fig. 7) and the exterior (fig. 8) problems.

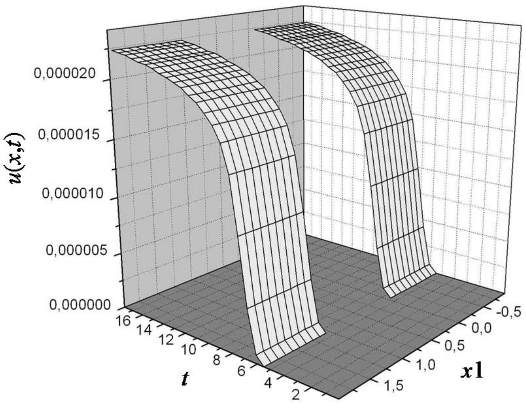

Fig. 6

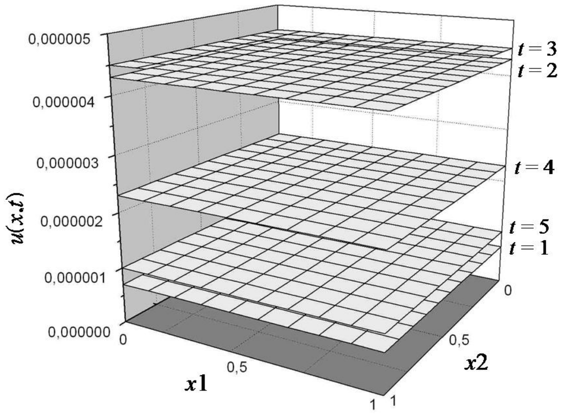

Fig. 7

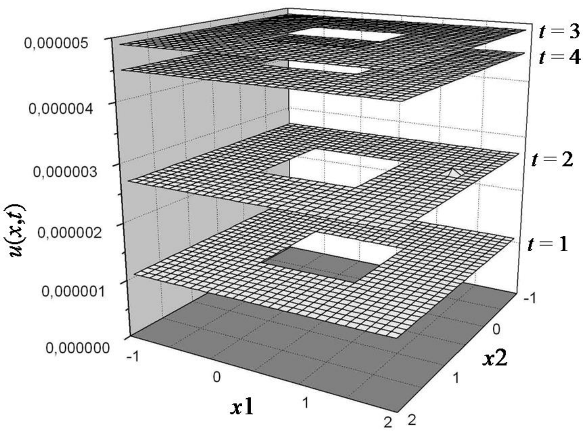

Fig. 8

Investigation of the convergence of the method of discrete singularities

The grid on space and time variable is changed to research the method convergence. The problem (5) is considered. The displacement in a plate center is considered for the grid with a time step $\Delta t = \dfrac{\pi}{6}$, $\Delta t = \dfrac{\pi}{10}$, $\Delta t = \dfrac{\pi}{15}$ (fig. 9) and for the grid with a space step $\Delta x_i = \dfrac{1}{8}$, $\Delta x_i = \dfrac{1}{12}$, $\Delta x_i = \dfrac{1}{15}$. The close results are received. So, it allows to make a conclusion about the method convergence.

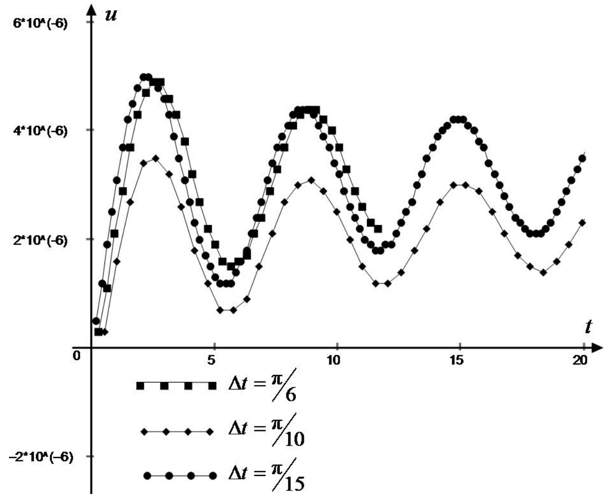

Fig. 9

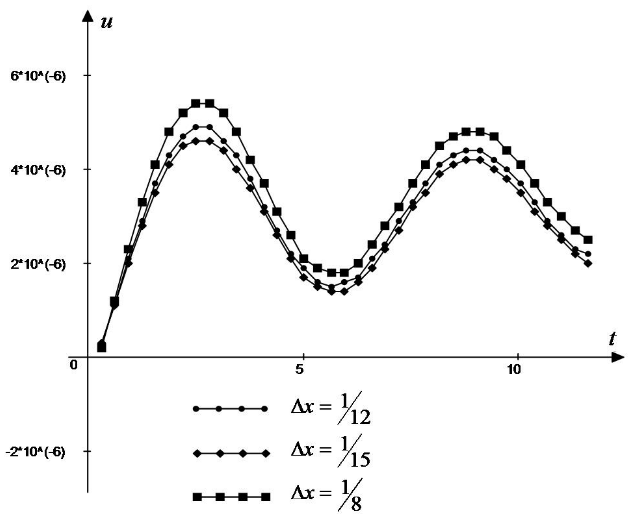

Fig. 10

Conclusion

The second basic initial–boundary– value problems for thin elastic plate are under considerations. Its purpose is to prove the results about unique solvability of boundary equations systems that appear as a result one solves the corresponding mixed initial–boundary–value problems by the potential theory methods. The potential theory allowed finding the unknown quantities on domain boundary without any calculations in the whole domain and also makes it possible to study the uniformity internal and external problems.

In the work the solutions of dynamic problems are represented by the surface potentials of a single layer. These surface potentials are constructed on the basis of a fundamental solution of a thin elastic plate oscillations equation. The representation by surface potentials leads to a boundary system with respect to the unknown densities of a potential. To prove the unique solvability of these boundary systems, the Laplase transformation for a time variable is used in the boundary systems and also in the corresponding initial problems. The results about solvability of elliptic problems with parameters are used and the boundary operators properties which arise in stationary systems with a parameter following that transformation are investigated. The study of the dependence of the boundary operators on the Laplace transformation parameter, their bijectivity and holomorphic ones in a right-hand half-plane of the complex plane, after returning to the original space makes it possible to prove a theorem about the solvability of the initial boundary equation systems in one-parameter scale of functional spaces of a Sobolev type.

The obtained results create the sound base for constructing the corresponding convergent numerical methods.

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