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Sufficient Condition for Partial Optimality for (max, +)-labeling Problems and its Usage

Для (max,+)-задач разметки сформулированы достаточные условия оптимальности метки в каждом пикселе изображения. Описан алгоритм, позволяющий определить оптимальные метки в некоторых пикселах и тем самым существенно снизить сложность исходной задачи.

Sufficient conditions for the optimal label detection in every pixel are formulate. An algorithm is described which makes it possible to define the optimal labels in some pixels and to decrease essentially the complexity of the original problem.

Для (max, +)-задач розмітки сформульовано достатні умови оптимальності мітки у кожному пікселі зображення. Описано алгоритм, що дозволяє визначити оптимальні мітки у деяких пікселах, завдяки чому суттєво зменшується складність вихідної задачі.

Introduction

A labeling problems play a significant role in computer vision. In the present paper labeling problems are formulated as (max, +) problems. Many image recognition problems lead to (max, +) problems. For instance, energy minimization, image segmentation by texture features, three-dimensional object reconstruction on the basis of stereoimages, noisy image restoration and other.

For the first time the (max, +) problem was formulated in the paper [1] in 1976. In the papers [1] and [2] an algorithm based on the substitution of the (max, +) problem by an auxiliary linear programming problem is suggested. The algorithm takes the decision itself whether the solution of the auxiliary linear programming problem leads to the solution of the (max, +) problem or not. Thus, the algorithm returns either the solution of (max, +) problem or the answer «no answer». In [3] and [4] a subclass of the solvable (max, +) problems is determined and it is shown that for this subclass the algorithm suggested in [1, 2] always finds the solution (the answer «no answer» never appears). These problems belong to the supermodular subclass of (max, +) problems.

A new branch of algorithms for solving (max, +) problems was formed in works [5–7]. These algorithms are based on a reduction of the initial problem to a min-cut problem. At that the class of solvable (max, +) problems was not expanded: only a subset of supermodular (max, +) problems is solvable by proposed algorithms.

Thus, a rather well investigated subclass of the (max, +) problems that can be solved in polynomial time arises. Namely, a subset of supermodular (max, +) problems. At the same time there are practically significant subclasses of the (max, +) problems that are known to be NP-hard. Therefore, a number of investigations were devoted to a searching for approximative algorithms [8–11]. However, the exact solution of the problem in some separately taken pixels is of interest as well.

This paper presents sufficient conditions for making decision about the optimal labeling in each pixel individually. At that the special label "no label" is allowed. Sufficient conditions are based on auxiliary supermodular labeling problems. At that proof that given supermodular labeling problem is auxiliary problem is much easier then its construction. The construction of auxiliary problem may be based on any heuristics and may incorporate any additional knowledge about the problem, but result of auxiliary labelling problem usage will always be exact. There are some examples of auxiliary problem construction in the article.

The basic definitions

Basic definitions are introduced in this section. Later we operate with such notions as ***a vision filed, a pixel, a labeling, a label set, a structure of the vision field, an order of the structure and neighboring pixels.*** After the definition of ***a labeling quality*** the (max, +) problem is formulated in general form as the problem of searching for the labeling with an optimal quality.

Let ***a vision field*** T be an arbitrary finite set. The elements of the vision field are called ***pixels.*** One of the most frequently encountered examples of a vision field is a rectangular area of a two-dimensional integer lattice $\{(i, j) \mid 0 \le i < I, 0 \le$

$\leq j < J\}$. Let ***a labeling*** of the vision field T be a function $k_T : T \to \{1,2,...,l\}$. The set $\{1,2,...,l\}$ is called ***a label set*** and denoted by the symbol L. A restriction of this function on a subset $\tau \subseteq T$ of the vision field is denoted by $k_\tau (k_\tau : \tau \to L)$, and the value of the function k_T in the pixel t is denoted by k_t. Let ***a structure of the vision field*** T be a set $\Psi \subseteq 2^T$ of subsets of the vision field T. Note that Ψ does not necessarily contain all subsets of the vision field. Let ***the order of the structure*** be a maximum cardinality of the elements of the structure Ψ, i.e. $\max_{\tau \in \Psi} |\tau|$. Usually but not necessarily the order of the structure is two. Pixels t and t' are called neighboring according to the structure Ψ if there exists a subset $\tau \in \Psi$ that contains both pixels $\{t, t'\} \subseteq \tau$. Let us denote the set of all labelings of the part τ of the vision field by L^τ. A function $g_\tau : L^\tau \to R$ is given for every subset $\tau \in \Psi$ of the structure. This function assigns a real number to every labeling $k_\tau : \tau \to L$. Let ***the quality of the labeling*** $k_T : T \to L$ be the number

$$Q(k_T) = \sum_{\tau \in \Psi} g_\tau (k_\tau). \qquad (1)$$

The (max, +) problem consists in maximization of the quality function $Q(*)$ and determination of the appropriate labeling $k_T^* = \arg\max_{k_T} Q(k_T) = = \arg\max_{k_T} \sum_{\tau \in \Psi} g_\tau (k_\tau)$.

Supermodular functions forms a polynomially solvable subclass of (max, +) problems

Let us suppose that the label set L is a completely ordered set: $1 < 2 < ... < l$. A partial ordering can be defined on the set of all labelings. For each pair of labelings k_T and k'_T we denote by $k_T \bigcup k'_T$ their maximum and by $k_T \bigcap k'_T$ their minimum.

A function $Q : L^T \to R$ is called ***supermodular*** if the following condition is fulfilled for arbitrary labelings k_T and k'_T:

$$Q(k_T) + Q(k'_T) \leq Q(k_T \bigcup k'_T) + Q(k_T \bigcap k'_T). \quad (2)$$

We call a (max, +) problem supermodular if its quality function is supermodular.

An equivalent definition of supermodular function $Q : L^T \to R$ is based on the notion of discrete derivation: $\forall k_T : T \to L, k_t < l$:

$$Q'_t(k_T) = Q(k_{T \setminus t}, k_t + 1) - Q(k_T). \qquad (3)$$

The function $Q : L^T \to R$ is ***supermodular*** if and only if its second derivative $Q''_{tt'}(k_T) \geq 0$ for every two neighboring pixels t and t' ($t \neq t'$) and an arbitrary labeling $k_T : T \to L$ ($k_t < l, k_{t'} < l$).

From the supermodularity condition (2) it follows that if k_T^* and k_T^{**} are solutions of a supermodular (max, +) problem then $k_T^* \bigcup k_T^{**}$ and $k_T^* \bigcap k_T^{**}$ are also solutions of the same problem. Hence, one can define ***the highest*** and ***the lowest*** optimal labelings in the following way:

$$k_T^{high} = \bigcup_{k_T^* = \arg\max_{k_T} Q(k_T)} k_T^*, \quad k_T^{low} = \bigcap_{k_T^* = \arg\max_{k_T} Q(k_T)} k_T^*. \quad (4)$$

It can be shown that computational complexity of searching for the lowest as well as the highest optimal labelings is the same as for an arbitrary optimal labeling.

The following lemma describes a property of the supermodular (max, +) problems.

***Lemma* 1.** *Let* $k_T^{low} = \bigcap_{k_T^* = \arg\max_{k_T} Q(k_T)} k_T^*$ *be the lowest optimal labeling for some supermodular problem and* k_T *be an arbitrary labeling satisfying the condition*:

$$k_T \bigcap k_T^{low} \neq k_T^{low}. \qquad (5)$$

Then the quality of the labeling k_T *is strictly less than the quality of the maximum of the labelings* k_T *and* k_T^{low}:

$$Q(k_T) < Q(k_T \bigcup k_T^{low}). \qquad (6)$$

Proof. Let us rewrite the inequality (2) for labelings k_T and k_T^{low}

$$Q(k_T) + Q(k_T^{low}) \leq Q(k_T \bigcup k_T^{low}) + Q(k_T \bigcap k_T^{low}). \quad (7)$$

The condition (5) together with the definition of the lowest optimal labeling leads to the inequality:

$$Q\left(k_T \cap k_T^{low}\right) < Q\left(k_T^{low}\right). \quad (8)$$

We obtain the statement of the lemma by adding the inequalities (7) and (8).

The main result

The following lemma defines some trivial feature of arbitrary (max, +) problem.

***Lemma* 2.** *Let* k_T^{low} *be a fixed labeling. If any labeling* k_T *such that*

$$k_T \cap k_T^{low} \neq k_T^{low} \quad (9)$$

satisfies the inequality

$$Q\left(k_T\right) < Q\left(k_T \cup k_T^{low}\right) \quad (10)$$

then any optimal labeling k_T^* *of the problem is bounded from the bottom with* k_T^{low}:

$$\forall t \in T : k_t^* \geq k_t^{low}. \quad (11)$$

Proof of the theorem is trivial. The existence of lower even in one pixel optimal labeling contradicts the inequality (10).

Combining lemma 1 and lemma 2 we obtain the following sufficient condition for arbitrary (max, +) problem optimality.

***Theorem* 1. (*Sufficient conditions for arbitrary* (*max,* +) *problem optimality*)** *Let* Q *– defines an arbitrary* (*max*, +) *problem. Let* $k_T^{low} = \bigcap_{k_T^* = \arg\max_{k_T} \widehat{Q}(k_T)} k_T^*$ *be the lowest optimal labeling for some supermodular problem* $\widehat{Q}$. *If for arbitrary* k_T *such that*

$$k_T \cap k_T^{low} \neq k_T^{low} \quad (12)$$

the inequality holds

$$Q\left(k_T \cup k_T^{low}\right) - Q\left(k_T\right) \geq \widehat{Q}\left(k_T \cup k_T^{low}\right) - \widehat{Q}\left(k_T\right) \quad (13)$$

then optimal labeling k_T^* *of initial arbitrary* (*max*, +) *problem* Q *satisfies the condition* (11).

Proof. Conditions (8) and (13) prove condition (10) and according to lemma 2, condition (11) holds.

Let us call supermodular problem from the theorem 1 ***an auxiliary problem***. Though the theorem may look simple the construction of the auxiliary problems is not. First of all it depends upon ordering of labels in each object, then we have to construct supermodular problem, which satisfy condition (13). The remaining part of the article will present methods of auxiliary problems construction.

Auxiliary problem construction for the Potts model

For the Potts model an order of the structure is 2 and quality function $Q()$ has the form

$$Q\left(k_T\right) = \sum_{\{t\}\in\Psi} q_t\left(k_t\right) + \sum_{\{t,t'\}\in\Psi} g_{t,t'}\left(k_t, k_{t'}\right), \quad (14)$$

where $q_t\left(k_t\right)$ are arbitrary values assigned to pixel labels and

$$g_{t,t'}\left(r, r'\right) = \begin{cases} C_{t,t'} > 0, & r = r', \\ 0, & r \neq r'. \end{cases} \quad (15)$$

This task is supermodular only for the case when the number of labels is two. It is shown in [11] that for three and more labels the problem is NP-hard.

An auxiliary problem will be constructed in the following way. We select one label $s \in L$ and reorder labels in each pixel so that the label s becomes the highest and independently in each pixel one of the rest labels l_t^s whose quality $q_t\left(l_t^s\right)$ is maximal ($l_t^s = \arg\max_{l \in L\setminus\{s\}} q_t\left(l\right)$) becomes the lowest. Quality function of the auxiliary problem is

$$Q^s\left(k_T\right) = \sum_{\{t\}\in\Psi} q_t\left(k_t\right) + \sum_{\{t,t'\}\in\Psi} g_{t,t'}^s\left(k_t, k_{t'}\right), \quad (16)$$

where

$$g_{t,t'}^s\left(r, r'\right) = \begin{cases} C_{t,t'}, & r = s, r' = s, \\ C_{t,t'}, & r \neq s, r' \neq s, \\ 0, & otherwise. \end{cases} \quad (17)$$

Fig. 1 shows transformation (17) of initial functions $g_{t,t'}$. The constructed problem is supermodular and its lowest solution $\widehat{k}_T^s$ can be found in polynomial time. Let us show that it also satisfies condition (13) of the theorem. Note that lowest solution in every pixel t may take only two values s and l_t^s. Also note that $g_{t,t'}^s\left(r, r'\right) \geq g_{t,t'}\left(r, r'\right)$ for arbitrary pair $\left(r, r'\right)$ and $g_{t,t'}^s\left(r, r'\right) = g_{t,t'}\left(r, r'\right)$ if r or r' equals to s. Now to prove inequality (13) we just need to prove it separately for each $g_{t,t'}$ and $g^s_{t,t'}$:

$$g_{t,t'}\left(k_t \cup \hat{k}^s_t, k_{t'} \cup \hat{k}^s_{t'}\right) - g_{t,t'}\left(k_t, k_{t'}\right) \geq \\ \geq g^s_{t,t'}\left(k_t \cup \hat{k}^s_t, k_{t'} \cup \hat{k}^s_{t'}\right) - g^s_{t,t'}\left(k_t, k_{t'}\right). \tag{18}$$

Here we have to consider 4 cases for $\hat{k}^s_t$ and $\hat{k}^s_{t'}$: (s, s), (s, $l^s_{t'}$), (l^s_t, s) and (l^s_t, $l^s_{t'}$). For each of the cases we use simplification $k_t \cup s = s$ and $k_t \cup l^s_t = k_t$:

$$\begin{aligned} &g_{t,t'}(s,s) - g_{t,t'}(k_t,k_{t'}) \geq g^s_{t,t'}(s,s) - g^s_{t,t'}(k_t,k_{t'}),\\ &g_{t,t'}(s,k_{t'}) - g_{t,t'}(k_t,k_{t'}) \geq g^s_{t,t'}(s,k_{t'}) - g^s_{t,t'}(k_t,k_{t'}),\\ &g_{t,t'}(k_t,s) - g_{t,t'}(k_t,k_{t'}) \geq g^s_{t,t'}(k_t,s) - g^s_{t,t'}(k_t,k_{t'}),\\ &g_{t,t'}(k_t,k_{t'}) - g_{t,t'}(k_t,k_{t'}) \geq g^s_{t,t'}(k_t,k_{t'}) - g^s_{t,t'}(k_t,k_{t'}). \end{aligned} \tag{19}$$

First three cases are equivalent to $g^s_{t,t'}(r,r') \geq \geq g_{t,t'}(r,r')$ and fourth is just the identity $0 = 0$.

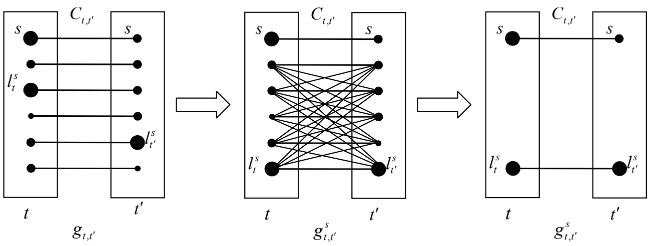

Fig. 1. Auxiliary problem construction for the Potts model. Constructed auxiliary problem is equivalent to the problem with only two labels

One against all approach for binary auxiliary problem construction for arbitrary (max, +) problem of the second order

The results formulated in the previous section may be generalized on the arbitrary (max, +) problems of the second order. Let the quality function is defined by the formula (14) without any restrictions on functions q_t and $g_{t,t'}$. Then we will construct auxiliary problem which will poses the following features.

- Labels in each pixel are ordered so that the lowest solution may take only two labels – the lowest and the highest. To achieve this we fix the highest label and order the rest labels so that the best of the rest labels becomes the lowest.
- Let us denote the highest label in pixel t as s_t and the lowest as l_t. Then the quality function of the auxiliary problem will have the form

$$\hat{Q}(k_T) = \sum_{\{t\}\in\Psi} q_t(k_t) + \sum_{\{t,t'\}\in\Psi} \hat{g}_{t,t'}(k_t,k_{t'}), \tag{20}$$

where

$$\hat{g}_{t,t'}(r,r') = \begin{cases} a_{t,t'}, & r = s_t, r' = s_{t'},\\ b_{t,t'}, & r = s_t, r' \neq s_{t'},\\ c_{t,t'}, & r \neq s_t, r' = s_{t'},\\ d_{t,t'}, & r \neq s_t, r' \neq s_{t'}. \end{cases} \tag{21}$$

Fig. 2 presents transformation of function $g_{t,t'}$ to $\hat{g}_{t,t'}$.

- We select values $a_{t,t'}$, $b_{t,t'}$, $c_{t,t'}$ and $d_{t,t'}$ in such a way that constructed task is supermodular. For this we require that

$$a_{t,t'} - b_{t,t'} - c_{t,t'} + d_{t,t'} \geq 0. \tag{22}$$

Also we require that these values satisfy inequality (18). It leads to

$$\begin{cases} g_{t,t'}(s_t \cup r, s_{t'} \cup r') - g_{t,t'}(r,r') \geq \\ \quad \geq \hat{g}_{t,t'}(s_t \cup r, s_{t'} \cup r') - \hat{g}_{t,t'}(r,r'), & r,r' \in L,\\ g_{t,t'}(s_t \cup r, l_{t'} \cup r') - g_{t,t'}(r,r') \geq \\ \quad \geq \hat{g}_{t,t'}(s_t \cup r, l_{t'} \cup r') - \hat{g}_{t,t'}(r,r'), & r,r' \in L,\\ g_{t,t'}(l_t \cup r, s_{t'} \cup r') - g_{t,t'}(r,r') \geq \\ \quad \geq \hat{g}_{t,t'}(l_t \cup r, s_{t'} \cup r') - \hat{g}_{t,t'}(r,r'), & r,r' \in L,\\ g_{t,t'}(l_t \cup r, l_{t'} \cup r') - g_{t,t'}(r,r') \geq \\ \quad \geq \hat{g}_{t,t'}(l_t \cup r, l_{t'} \cup r') - \hat{g}_{t,t'}(r,r'), & r,r' \in L. \end{cases} \tag{23}$$

Simplification of the system (23) is

$$\begin{cases} g_{t,t'}(s_t, s_{t'}) - g_{t,t'}(r,r') \geq \\ \quad \geq \hat{g}_{t,t'}(s_t, s_{t'}) - \hat{g}_{t,t'}(r,r'), & r,r' \in L,\\ g_{t,t'}(s_t, r') - g_{t,t'}(r,r') \geq \\ \quad \geq \hat{g}_{t,t'}(s_t, r') - g^s_{t,t'}(r,r'), & r,r' \in L,\\ g_{t,t'}(r, s_{t'}) - g_{t,t'}(r,r') \geq \\ \quad \geq \hat{g}_{t,t'}(r, s_{t'}) - \hat{g}_{t,t'}(r,r'), & r,r' \in L. \end{cases} \tag{24}$$

We obtain further simplification of system (24) by splitting cases $r = s$, $r \neq s$, $r' = s$ and $r' \neq s$:

$$\begin{cases} g_{t,t'}(s_t,s_{t'}) - g_{t,t'}(s_t,r') \geq \\ \quad \geq \widehat{g}_{t,t'}(s_t,s_{t'}) - \widehat{g}_{t,t'}(s_t,r'), & r = s_t, r' \neq s_{t'}, \\ g_{t,t'}(s_t,s_{t'}) - g_{t,t'}(r,s_{t'}) \geq \\ \quad \geq \widehat{g}_{t,t'}(s_t,s_{t'}) - \widehat{g}_{t,t'}(r,s_{t'}), & r \neq s_t, r' = s_{t'}, \\ g_{t,t'}(s_t,s_{t'}) - g_{t,t'}(r,r') \geq \\ \quad \geq \widehat{g}_{t,t'}(s_t,s_{t'}) - \widehat{g}_{t,t'}(r,r'), & r \neq s_t, r' \neq s_{t'}, \\ g_{t,t'}(s_t,r') - g_{t,t'}(r,r') \geq \\ \quad \geq \widehat{g}_{t,t'}(s_t,r') - \widehat{g}_{t,t'}(r,r'), & r \neq s_t, r' \neq s_{t'}, \\ g_{t,t'}(r,s_{t'}) - g_{t,t'}(r,r') \geq \\ \quad \geq \widehat{g}_{t,t'}(r,s_{t'}) - \widehat{g}_{t,t'}(r,r'), & r \neq s_t, r' \neq s_{t'}. \end{cases} \quad (25)$$

Now we can use definition (21) and rewrite (25) so:

$$\begin{cases} a_{t,t'} - b_{t,t'} \leq g_{t,t'}(s_t,s_{t'}) - \\ \quad - g_{t,t'}(s_t,r'), & r = s_t, r' \neq s_{t'}, \\ a_{t,t'} - c_{t,t'} \leq g_{t,t'}(s_t,s_{t'}) - \\ \quad - g_{t,t'}(r,s_{t'}), & r \neq s_t, r' = s_{t'}, \\ a_{t,t'} - d_{t,t'} \leq g_{t,t'}(s_t,s_{t'}) - \\ \quad - g_{t,t'}(r,r'), & r \neq s_t, r' \neq s_{t'}, \\ b_{t,t'} - d_{t,t'} \leq g_{t,t'}(s_t,r') - \\ \quad - g_{t,t'}(r,r'), & r \neq s_t, r' \neq s_{t'}, \\ c_{t,t'} - d_{t,t'} \leq g_{t,t'}(r,s_{t'}) - \\ \quad - g_{t,t'}(r,r'), & r \neq s_t, r' \neq s_{t'}. \end{cases} \quad (26)$$

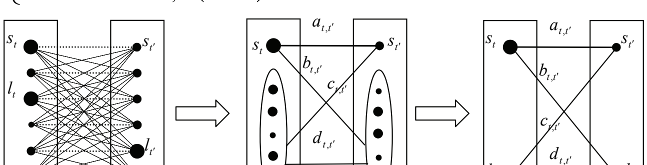

Fig. 2. One against all approach for binary auxiliary problem construction for arbitrary (max, +) problem of the second order. A constructed auxiliary problem is equivalent to the problem with only two labels

From (22) and (26) follows that $a_{t,t'}$, $b_{t,t'}$, $c_{t,t'}$ and $d_{t,t'}$ may be defined up to the constant summand. That is why we just fix

$$a_{t,t'} = g_{t,t'}(s_t,s_{t'}). \quad (27)$$

Restrictions on $b_{t,t'}$, $c_{t,t'}$, and $d_{t,t'}$ becomes as follows

$$\begin{cases} a_{t,t'} - b_{t,t'} - c_{t,t'} + d_{t,t'} \geq 0, \\ b_{t,t'} \geq g_{t,t'}(s_t,r'), r \neq s_{t'}, \\ c_{t,t'} \geq g_{t,t'}(r,s), r \neq s_t, \\ d_{t,t'} \geq g_{t,t'}(r,r'), r \neq s_t, r' \neq s_{t'}, \\ b_{t,t'} - d_{t,t'} \leq g_{t,t'}(s_t,r') - g_{t,t'}(r,r'), r \neq s_t, r' \neq s_{t'}, \\ c_{t,t'} - d_{t,t'} \leq g_{t,t'}(r,s_{t'}) - g_{t,t'}(r,r'), r \neq s_t, r' \neq s_{t'}. \end{cases} \quad (28)$$

One of possible solutions of the system (28) is

$$\begin{cases} a_{t,t'} = g_{t,t'}(s_t,s_{t'}), \\ b_{t,t'} = \max\limits_{r \neq s} g_{t,t'}(s_t,r), \\ c_{t,t'} = \max\limits_{r \neq s} g_{t,t'}(r,s_{t'}), \\ d_{t,t'} = \max\Big\{b_{t,t'} + c_{t,t'} - a_{t,t'}, \\ \max\limits_{r \neq s_t, r' \neq s_{t'}} \Big(g_{t,t'}(r,r') + \max\big\{b_{t,t'} - \\ \quad - g_{t,t'}(s_t,r'), c_{t,t'} - g_{t,t'}(r,s_{t'})\big\}\Big)\Big\}. \end{cases} \quad (29)$$

• The constructed supermodular problem is an auxiliary problem because requirements of the theorem 1 are satisfied automatically due to the construction method.

Many against many approach to construct binary auxiliary problem for arbitrary (max, +)-problem of the second order

Results of the previous section may be generalized in the following way. We split labels in each pixel onto two parts and then order the parts so that one part becomes higher then other and the labels with the best quality in each part become the lowest. See figure 3. We denote s_t and l_t the lowest labels of the higher and lower set respectively.

The quality function has the form (14) without any restrictions and the auxiliary problem has the form (20) with the following form of the function $\widehat{g}_{t,t'}$ (see figure 3 and compare with (21)):

$$\widehat{g}_{t,t'}(r,r') = \begin{cases} a_{t,t'}, & r \geq s_t, r' \geq s_{t'}, \\ b_{t,t'}, & r \geq s_t, r' < s_{t'}, \\ c_{t,t'}, & r < s_t, r' \geq s_{t'}, \\ d_{t,t'}, & r < s_t, r' < s_{t'}. \end{cases} \quad (30)$$

We select values $a_{t,t'}$, $b_{t,t'}$, $c_{t,t'}$ and $d_{t,t'}$ that satisfy (18) and (22). Rewrite them together:

$$\begin{cases} a_{t,t'} - b_{t,t'} - c_{t,t'} + d_{t,t'} \ge 0, \\ g_{t,t'}\left(r \cup l, r' \cup l'\right) - g_{t,t'}\left(r, r'\right) \ge \\ \quad \ge \hat{g}_{t,t'}\left(r \cup l, r' \cup l'\right) - \hat{g}_{t,t'}\left(r, r'\right), \\ \quad r, r' \in L,\ l \in \{s_t, l_t\}, l' \in \{s_{t'}, l_{t'}\}. \end{cases} \tag{31}$$

To solve system of inequalities (31) we consider 16 cases. Namely we consider $r \ge s_t$ and $r < s_t$, $r' \ge s_{t'}$ and $r' < s_{t'}$, $l = s_t$ and $l = l_t$, $l' = s_{t'}$ and $l' = l_{t'}$. After we remove all identity cases the following system remains

$$\begin{cases} a_{t,t'} - b_{t,t'} - c_{t,t'} + d_{t,t'} \ge 0, \\ g_{t,t'}(r, s_{t'}) - g_{t,t'}(r, r') \ge a_{t,t'} - b_{t,t'}, r \ge s_t, r' < s_{t'}, \\ g_{t,t'}(s_t, r') - g_{t,t'}(r, r') \ge a_{t,t'} - c_{t,t'}, r < s_t, r' \ge s_{t'}, \\ g_{t,t'}(s_t, s_{t'}) - g_{t,t'}(r, r') \ge a_{t,t'} - d_{t,t'}, r < s_t, r' < s_{t'}, \\ g_{t,t'}(s_t, r') - g_{t,t'}(r, r') \ge b_{t,t'} - d_{t,t'}, r < s_t, r' < s_{t'}, \\ g_{t,t'}(r, s_{t'}) - g_{t,t'}(r, r') \ge c_{t,t'} - d_{t,t'}, r < s_t, r' < s_{t'}. \end{cases} \tag{32}$$

Again the solution of the system (32) is possible only up to the constant summand. That is why we put $a_{t,t'} = 0$ and then obtain one of the possible solutions:

$$\begin{cases} a_{t,t'} = 0, \\ b_{t,t'} = \max\limits_{r \ge s_t, r' < s_{t'}} \left(g_{t,t'}\left(r, r'\right) - g_{t,t'}\left(r, s_{t'}\right)\right), \\ c_{t,t'} = \max\limits_{r < s_t, r' \ge s_{t'}} \left(g_{t,t'}\left(r, r'\right) - g_{t,t'}\left(s_t, r'\right)\right), \\ d_{t,t'} = \max\Big\{b_{t,t'} + c_{t,t'}, \max\limits_{r < s_t, r' < s_{t'}} \big(g_{t,t'}\left(r, r'\right) - \\ \quad - g_{t,t'}\left(s_t, s_{t'}\right)\big), \max\limits_{r < s_t, r' < s_{t'}} \big(b_{t,t'} + g_{t,t'}\left(r, r'\right) - \\ \quad - g_{t,t'}\left(s_t, r'\right)\big), \\ \quad \max\limits_{r < s_t, r' < s_{t'}} \left(c_{t,t'} + g_{t,t'}\left(r, r'\right) - g_{t,t'}\left(r, s_{t'}\right)\right)\Big\}. \end{cases} \tag{33}$$

Therefore we can construct an auxiliary problem for arbitrary splitting of vertices in each pixels on two parts.

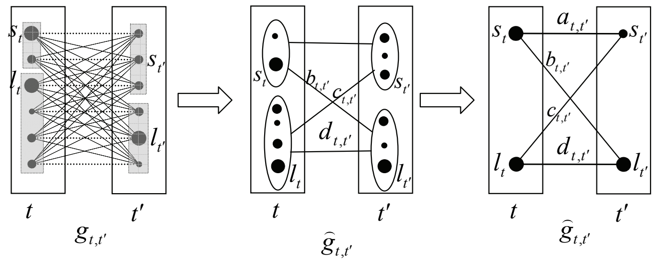

Fig. 3. Many against many approach for binary auxiliary problem construction for arbitrary (max, +) problem of the second order. A constructed auxiliary problem is equivalent to the problem with only two labels

Construction of the auxiliary problems for the arbitrary (max, +)-problem of the second order

Further generalization of the method described in the previous section is also possible. Given problem (14) we construct the auxiliary problem of the form (20) without restriction (21) on the functions $\hat{g}_{t,t'}$. But again as in previous section we require that functions $\hat{g}_{t,t'}$ satisfy inequality (18) and are supermodular. It is enough for constructed problem to be an auxiliary one. The inequality (18) together with supermodularity condition may be written so:

$$\begin{cases} \hat{g}_{t,t'}\left(r \cup l, r' \cup l'\right) - \hat{g}_{t,t'}\left(r, r'\right) \le \\ \quad \le g_{t,t'}\left(r \cup l, r' \cup l'\right) - g_{t,t'}\left(r, r'\right), \\ \quad \forall r, r', l, l' \in L, \\ \hat{g}_{t,t'}\left(r+1, r'\right) + \hat{g}_{t,t'}\left(r, r'+1\right) \le \\ \quad \le \hat{g}_{t,t'}\left(r+1, r'+1\right) + \hat{g}_{t,t'}\left(r, r'\right), \\ \quad \forall r, r', r+1, r'+1 \in L. \end{cases} \tag{34}$$

Here we assume that the solution of the constructed auxiliary problem is not known. That is why we require first inequality to be true for the arbitrary pair (l, l'). System (34) is equivalent to the following

$$\begin{cases} \hat{g}_{t,t'}\left(l, l'\right) - \hat{g}_{t,t'}\left(r, r'\right) \le g_{t,t'}\left(l, l'\right) - g_{t,t'}\left(r, r'\right), \\ \quad \forall l, l', r, r' \in L : l \ge r, l' \ge r', \\ \hat{g}_{t,t'}\left(r+1, r'\right) + \hat{g}_{t,t'}\left(r, r'+1\right) \le \\ \quad \le \hat{g}_{t,t'}\left(r+1, r'+1\right) + \hat{g}_{t,t'}\left(r, r'\right), \\ \quad \forall r, r', r+1, r'+1 \in L. \end{cases} \tag{35}$$

Again solution of the system may be found up to the constant summand and we may fix

$$\widehat{g}_{t,t'}(s,r)=g_{t,t'}(s,r) \text{ and}$$
$$\widehat{g}_{t,t'}(r,s)=g_{t,t'}(r,s) \text{ for all } r\in L. \quad (36)$$

Now we rewrite system of inequalities (31) so that $g^{s}_{t,t'}(r,r')$ depends only on $g^{s}_{t,t'}(l,l')$ with $l\geq r$ and $l'\geq r'$.

$$\begin{cases} \widehat{g}_{t,t'}(r,r')\geq g_{t,t'}(r,r')+\widehat{g}_{t,t'}(l,l')- \\ \quad -g_{t,t'}(l,l'), \forall l,l',r,r'\in L: l\geq r, l'\geq r', \\ \widehat{g}_{t,t'}(r,r')\geq \widehat{g}_{t,t'}(r+1,r')+\widehat{g}_{t,t'}(r,r'+1)- \\ \quad -\widehat{g}_{t,t'}(r+1,r'+1), \forall r,r'\in L\setminus\{s\}. \end{cases} \quad (37)$$

One of possible solutions for (37) is described by recursion formula

$$\begin{cases} g^{s}_{t,t'}(r,s)=g_{t,t'}(r,s), \forall r\in L, \\ g^{s}_{t,t'}(s,r')=g_{t,t'}(s,r'), \forall r'\in L, \\ g^{s}_{t,t'}(r,r')= \\ =\max\left\{ \max\limits_{\substack{l\geq r,\\ l'\geq r'}}\left(g_{t,t'}(r,r')+g^{s}_{t,t'}(l,l')-g_{t,t'}(l,l')\right), \right. \\ \left. g^{s}_{t,t'}(r+1,r')+g^{s}_{t,t'}(r,r'+1)-g^{s}_{t,t'}(r+1,r'+1)\right\}, \\ \quad \forall r,r'\in L\setminus\{s\}. \end{cases} \quad (38)$$

The solution (38) allows constructing huge number of auxiliary problems by changing labels ordering in each pixel independently. In the next section we will try to improve this method by reducing set of possible labels which may take lowest optimal labelling of the auxiliary problem.

Iterative construction of the auxiliary problems for arbitrary (max, +) problem of the second order

The method described in the previous section produces weak auxiliary problems because it requires correctness of many redundant inequalities. We do not really need the first inequality in system (30) to be correct for all $l,l'\in L$ though we do not know solution of auxiliary problem in advance. We will construct sequence of supermodular problems with less and less restrictions on l, l' and obtain auxiliary problem as a result. Namely we require first inequality in system (30) to be true only for $(l,l')\in\Omega_{t,t'}\subseteq L\times L$. At the beginning sets $\Omega_{t,t'}$ are empty. Now we construct supermodular (max, +) problem:

$$\begin{cases} \widehat{g}_{t,t'}(r,s)=g_{t,t'}(r,s), \forall r\in L, \\ \widehat{g}_{t,t'}(s,r')=g_{t,t'}(s,r'), \forall r'\in L, \\ \widehat{g}_{t,t'}(r,r')= \\ =\max\left\{ \max\limits_{(l,l')\in\Omega_{t,t'}}\left(g_{t,t'}(r,r')+\widehat{g}_{t,t'}(l,l')-g_{t,t'}(l,l')\right), \right. \\ \left. \widehat{g}_{t,t'}(r+1,r')+\widehat{g}_{t,t'}(r,r'+1)-\widehat{g}_{t,t'}(r+1,r'+1)\right\}, \\ \quad \forall r,r',r+1,r'+1\in L. \end{cases} \quad (39)$$

After we find solution of this problem we verify whether inequality

$$g^{s}_{t,t'}(r,r')\geq g_{t,t'}(r,r')+g^{s}_{t,t'}(\widehat{k}_t,\widehat{k}_{t'})-g_{t,t'}(\widehat{k}_t,\widehat{k}_{t'}), \quad \forall r,r'\in L \quad (40)$$

holds for every $\{t,t'\}\in\Psi$. If yes then constructed problem is auxiliary one. Otherwise we increase every set $\Omega_{t,t'}$ with the pair $(\widehat{k}_t,\widehat{k}_{t'})$ and construct the next problem in the sequence using formula (39).

Construction of the auxiliary problems for two dimensional shifts (optical flow)

Two dimensional shifts problem possess the following properties which allow construction auxiliary problems in a better way.

- Vision field $T=\{(i,j)\,|\,0\leq i<I, 0\leq j<J\}$ is a rectangular area of two-dimensional integer lattice.
- Label set $L=\{1,2,...,l\}\times\{1,2,...,l\}$ have also two dimensional structure and defines for every pixel of one image its shift on the second image.
- Quality function has the form (14) with the following meaning of values q_t and $g_{t,t'}$.
- $q_t(i,j)$ defines how similar are area of pixel t on the first image and area of pixel $t+(i,j)$ on the second image.
- $g_{t,t'}(i,j,i',j')$ induces smoothness restrictions.

We also require that $g_{t,t'}(i,j,i',j')=g^{x}_{t,t'}(i,i')+$ $+g^{y}_{t,t'}(j,j')$ and are supermodular. For instance

$$g_{t,t'}(i,j,i',j')=-(i-i')^2-(j-j')^2 \quad (41)$$

or $g_{t,t'}(i,j,i',j') = -|i-i'| - |j-j'|$. (42)

Let us consider the quality function of the problem

$$Q(k_T) = \sum_{\{t\}\in\Psi} q_t(k_t) + \sum_{\{t,t'\}\in\Psi} g_{t,t'}(k_t,k_{t'}) =$$

$$= \sum_{\{t\}\in\Psi} q_t(i_t,j_t) + \sum_{\{t,t'\}\in\Psi} g_{t,t'}(i_t,j_t,i_{t'},j_{t'}) =$$

$$= \sum_{\{t\}\in\Psi} q_t(i_t,j_t) + \sum_{\{t,t'\}\in\Psi} g^x_{t,t'}(i_t,i_{t'}) + \sum_{\{t,t'\}\in\Psi} g^y_{t,t'}(j_t,j_{t'}) \quad (43)$$

The quality function (43) may be also understood as the quality function for differently formulated (max, +) problem. Let vision field consists of two layers T_x and T_y. Every pixel t of initial vision field T will be split onto two parts. First part t_x belongs to T_x and second part t_y to T_y. Label set is one-dimensional set $L = \{1,2,...,l\}$. Labelling k_T also will be split onto two parts $i_T : T_x \to L$ and $j_T : T_y \to L$. Now quality function (43) takes the form

$$Q(k_T) = Q(i_T, j_T) = \sum_{\{t\}\in\Psi} q_t\left(i_{t_x}, j_{t_y}\right) + \\ + \sum_{\{t,t'\}\in\Psi} g^x_{t,t'}\left(i_{t_x}, i_{t'_x}\right) + \sum_{\{t,t'\}\in\Psi} g^y_{t,t'}\left(j_{t_y}, j_{t'_y}\right). \quad (44)$$

Latest formula defines also new vision field structure. Auxiliary problems for the task (44) may be constructed using approach described in the previous section.

Conclusion

We have shown in this paper that the exact solution can be found at least partially even for NP-hard (max, +) problems. Sufficient conditions for problem simplification are formulated. The sufficient conditions are not constructive and it is not evident how to use them. That is why some methods of auxiliary problems construction are presented as well. The suggested methods for auxiliary problems construction are not quite determinant. The auxiliary problem efficiency essentially depends upon ordering of the labels in each pixel, upon selected labels splitting etc. These questions remain open and we answer them using heuristics which were not described here. Important is that independently upon heuristic selection we obtain exact answer about problem simplification and about restriction on the optimal labeling.

We do not show any examples. Some examples of auxiliary problems usage for Potts model may be found in [12, 13]. They show that simplification rate various from task to task and may reach 99%. Sometimes we remove half of labels. Sometimes auxiliary problems produces no simplification at all. After all we try to restrict exact solution for NP-hard problems.

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