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A New Method of Minimization of Logical Functions in the Polynomial Set-theoretical Format. 2. Minimization of Complete and Incomplete Functions

Рассмотрен новый метод минимизации логических функций от n переменных в полиномиальном теоретико-множественном формате, основанный на процедуре расщепления заданных минтермов и обобщенных теоретико-множественных правилах упрощения конъюнктермов разных рангов. Преимущества метода иллюстрируют примеры.

A new minimization method of the logic functions of n variables in the polynomial set-theoretical format is considered. The method is based on the splitting procedure of the given minterms and on the generalized of the set-theoretical simplify rules of the conjuncterms of different ranks. The advantages of the method are illustrated by the examples.

Розглянуто новий метод мінімізації логічних функцій від n змінних у поліноміальному теоретико-множинному форматі, що ґрунтується на процедурі розчеплення заданих мінтермів та узагальнених теоретико-множинних правилах спрощення кон'юнктермів різних рангів. Переваги методу ілюструють приклади.

The suggested method of minimization of the completely specified (complete) and incompletely specified (incomplete) logic functions in the polynomial set-theoretical format is based on the idea of splitting of given minterms of a function $f(x_1, x_2, ..., x_n)$ in the disjunctive format [27–29]. The difference consists in the procedure of splitting conjuncterms reading and formation of a minimal PSTF $Y^{\oplus}$ of a given function *f*.

2.1. Algorithm of minimization of complete functions. Examples of minimization

The algorithm of minimization of a complete function *f* in the polynomial set-theoretical format is realized on two stages:

1-*st stage*: the procedure of minterms splitting of a given function *f* is carried out and a set of covering of a matrix of splitting is recieved;

2-*nd stage*: the procedure of iterative conjuncterms simplification of a set of covering (got on the 1-st stage) based on the generalized rules of the theorems 1, 2 і 3 (п. 1.2) and formation of a minimal PSTF $Y^{\oplus}$ of a given function *f*.

Let us consider each stage of the algorithm in details: the 1-st stage is realized by the sequence of such steps:

Step 1: the given binary minterms $m_1, m_2, ...m_k$ of the perfect PSTF $Y^{\oplus} = \{m_1, m_2, ...m_k\}^{\oplus}$ of the function *f* are split (operator $\overset{S}{\Rightarrow}$) with the help of the matrix-column of the masks of literals of $r \geq n - \log_2 k$, $r = 1, 2, ..., n$ rank, as a result of this a matrix of splitting M_n^r of $C_n^r \times k$ dimension is formed, where $C_n^r = \dfrac{n!}{(n-r)!r!}$; for example, let $n = 5$; if the number *k* of minterms is $8 \leq k < 16$, then we use the matrix of masks of $r = 2$ rank, and as a result the matrix M_5^2 of the dimension $C_5^2 \times k$ is formed;

Step 2: in the matrix M_n^r (in our example M_5^2) for carring out the procedure of covering (operator $\overset{C}{\Rightarrow}$) the conjuncterms-copies of *r*-rank, the number of which $2^{n-r-1} < k_r \leq 2^{n-r}$ (i.e. $4 < k_r \leq 8$); are highlighted by underlining; priority is given to the conjuncterms-copies the number of which $k_r = 2^{n-r}$ (i.e. $k_r = 8$); if $k = k_r$, then the matrix is covered with a conjuncterm-copy of *r*-rank; if $k_r < 2^{n-r}$ (i.e. $k_r < 8$), then covering of the matrix will be made of the conjuncterms-copies the number of which

$2^{n-r-1} < k_r < 2^{n-r}$, and if there are not enough of them then – together with generating minterms of the matrix M_n^r; if $k_r < 2^{n-r-1}$, then transition to step 1 is done for realization of analogical procedures with application of the matrix of masks of the rank $r = 3$ and etc. up to getting in the covering of the matrix M_n^r of the minterms, splitting of which secures its covering, if such minterms > 2, then transition to step 1 is done.

The 1-st stage of algorithm is completed when there are not only minterms in the set of covering of the matrix M_n^r or when the split elements do not secure its covering. Then the cost of the function *f* realization in the set of covering is determined by the interrelation $k_\theta / k_l / k_{in}$, where k_θ – a number of conjuncterms, k_l – a number of literals, k_{in} – a number of inverted literals (k_{in} is determined only in the case of digital devices that do not have inverse entrances).

The 1-st step of the considered algorithm has been described for the case of consequent splitting minterms [27–28]. However, the 1-st step can also be realized in the procedure of minterms parallel splitting [29], when the matrix-column of the mask 1-, 2-, …, *n*-ranks, is applied as a result of this the matrix M_n^r, $r = 1, 2, ..., n$, of splitting conjuncterms of respective ranks is formed.

Before description of the 2-nd stage of the algorithm, we will consider the procedure of the 1-st stage in detail. For this we will consider the function $f(x_1, x_2, x_3, x_4)$, that has the perfect STF $Y^1 = \{1, 2, 4, 6, 7, 8, 9, 10, 15\}^1$, on the example of which the author [32, p. 211] illustrates his own method of minimization in the polynomial format on the basis of *K*-map (*constracted Reed-Muller transform method*) in combination with analytical transformations (*convertional method*). We will illustrate the 1-st stage of algorithm using the procedure of minterms parallel splitting [29]. In this case the matrix-column will consist of masks of literals starting with 1-, 2-, 3- and 4-ranks, as this function *f* has $k = 9$ minterms ($8 \le k < 16$). Splitting the last minterms of the perfect PSTF $Y^\oplus = \{1, 2, 4, 6, 7, 8, 9, 10, 15\}^\oplus$, we got the splitting matrix M_4^r, $r = 1, 2, 3, 4$:

$$Y^\oplus = \{(0001), (0010), (0100), (0110), (0111), (1000), (1001), (1010), (1111)\}^\oplus \overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow}\begin{bmatrix} l{-}{-}{-} \\ {-}l{-}{-} \\ {-}{-}l{-} \\ {-}{-}{-}l \\ \hline ll{-}{-} \\ l{-}l{-} \\ l{-}{-}l \\ {-}ll{-} \\ {-}l{-}l \\ {-}{-}ll \\ \hline lll{-} \\ ll{-}l \\ l{-}ll \\ {-}lll \\ \hline llll \end{bmatrix} = \left[\begin{array}{ccccccccc}
\underline{0{-}{-}{-}} & \underline{0{-}{-}{-}} & \underline{0{-}{-}{-}} & \underline{0{-}{-}{-}} & \underline{0{-}{-}{-}} & 1{-}{-}{-} & 1{-}{-}{-} & 1{-}{-}{-} & 1{-}{-}{-} \\
\underline{{-}0{-}{-}} & \underline{{-}0{-}{-}} & {-}1{-}{-} & {-}1{-}{-} & {-}1{-}{-} & \underline{{-}0{-}{-}} & \underline{{-}0{-}{-}} & \underline{{-}0{-}{-}} & {-}1{-}{-} \\
{-}{-}0{-} & \underline{{-}{-}1{-}} & {-}{-}0{-} & \underline{{-}{-}1{-}} & \underline{{-}{-}1{-}} & {-}{-}0{-} & {-}{-}0{-} & \underline{{-}{-}1{-}} & \underline{{-}{-}1{-}} \\
{-}{-}{-}1 & \underline{{-}{-}{-}0} & \underline{{-}{-}{-}0} & \underline{{-}{-}{-}0} & {-}{-}{-}1 & \underline{{-}{-}{-}0} & {-}{-}{-}1 & \underline{{-}{-}{-}0} & {-}{-}{-}1 \\
\hline
00{-}{-} & 00{-}{-} & \underline{\mathbf{01}{-}{-}} & \underline{\mathbf{01}{-}{-}} & \underline{\mathbf{01}{-}{-}} & \underline{\mathbf{10}{-}{-}} & \underline{\mathbf{10}{-}{-}} & \underline{\mathbf{10}{-}{-}} & 11{-}{-} \\
0{-}0{-} & \underline{0{-}1{-}} & 0{-}0{-} & \underline{0{-}1{-}} & \underline{0{-}1{-}} & 1{-}0{-} & 1{-}0{-} & 1{-}1{-} & 1{-}1{-} \\
0{-}{-}1 & \underline{0{-}{-}0} & \underline{0{-}{-}0} & \underline{0{-}{-}0} & 0{-}{-}1 & 1{-}{-}0 & 1{-}{-}1 & 1{-}{-}0 & 1{-}{-}1 \\
\underline{{-}00{-}} & {-}01{-} & {-}10{-} & \underline{{-}11{-}} & \underline{{-}11{-}} & \underline{{-}00{-}} & \underline{{-}00{-}} & {-}01{-} & \underline{{-}11{-}} \\
{-}0{-}1 & \underline{{-}0{-}0} & {-}1{-}0 & {-}1{-}0 & {-}1{-}1 & \underline{{-}0{-}0} & {-}0{-}1 & \underline{{-}0{-}0} & {-}1{-}1 \\
{-}{-}01 & \underline{{-}{-}10} & {-}{-}00 & \underline{{-}{-}10} & {-}{-}11 & {-}{-}00 & {-}{-}01 & \underline{{-}{-}10} & {-}{-}11 \\
\hline
000{-} & 001{-} & 010{-} & \underline{011{-}} & \underline{011{-}} & \underline{100{-}} & \underline{100{-}} & 101{-} & 111{-} \\
00{-}1 & 00{-}0 & \underline{01{-}0} & \underline{01{-}0} & 01{-}1 & \underline{10{-}0} & 10{-}1 & \underline{10{-}0} & 11{-}1 \\
0{-}01 & \underline{0{-}10} & 0{-}00 & \underline{0{-}10} & 0{-}11 & 1{-}00 & 1{-}01 & 1{-}10 & 1{-}11 \\
\underline{{-}001} & \underline{{-}010} & {-}100 & {-}110 & \underline{{-}111} & {-}000 & \underline{{-}001} & \underline{{-}010} & \underline{{-}111} \\
\hline
\mathbf{0001} & \mathbf{0010} & 0100 & 0110 & 0111 & 1000 & 1001 & 1010 & \mathbf{1111}
\end{array}\right]. \quad (25)$$

The conjuncterms-copies the number of which $2^{4-r-1} < k_r \leq 2^{4-r}$ for $r = 1,2,3$ are underlined in the matrix M_4^r. From all possible coverings the greatest number of conjuncterms-copies, as we see, has the submatrix M_4^2, in which the conjuncterms of 2-rank $(01--)$ and $(10--)$, the number of which $k_r < 2^2$, i.e. $k_r = 3$, are highlighted in bold font. They can be elements of the matrix covering M_4^r if they are completed with the absent minterms that belong to them, these are: $\big((01--),(0101)\big)$ and $\big((10--),(1011)\big)$. Except for the last ones, the minterms (0001), (0010) and (1111) will also enter the matrix covering M_4^r. So, the matrix covering M_4^r (step 2) will be composed of the set:

$$Y^{\oplus} \overset{C}{\Rightarrow} \big\{\big((01--),(0101)\big),\big((10--),(1011)\big),(0001),(0010),(1111)\big\}^{\oplus}.$$

In the obtained set there are more than two minterms here $4 \leq k < 8$, but application of the matrix M_4^2 does not give any positive result if compared with the matrix M_4^3:

$$\{(0001),(0010),(0101),(1011),(1111)\}^{\oplus} \overset{S}{\Rightarrow} \begin{bmatrix} lll- \\ ll-l \\ l-ll \\ -lll \end{bmatrix} = \begin{bmatrix} 000- & 001- & 010- & 101- & 111- \\ 00-1 & 00-0 & 01-1 & 10-1 & 11-1 \\ \underline{0-01} & 0-00 & \underline{0-01} & \underline{1-11} & \underline{1-11} \\ -001 & -010 & -101 & -011 & -111 \end{bmatrix} \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow} \{l-ll\} = \{(0-01),(1-11),(0010)\}^{\oplus}.$$

So, in the considered case the 1-st stage of the algorithm is completed with the set of the covering

$$Y^{\oplus} = \{(01--),(10--),(0-01),(1-11),(0010)\}^{\oplus}, \tag{26}$$

the cost of realization of which reflects the interrelation $k_\theta / k_l / k_{in} = 5/14/7$.

In case of application of the procedure of the consequent conjuncterms splitting, the 1-st step of the algorithm has been done with the help of the submatrix M_4^1 (25), as $k = 9$. Then the matrix covering M_4^1 (step 2), for example for the mask $\{--l-\}$, will be made of the set:

$$Y^{\oplus} \overset{C}{\Rightarrow} \big\{\big((--1-),(0011),(1011),(1110)\big),(0001),(0100),(1000),(1001)\big\}^{\oplus}.$$

As in the obtained set there are more than two minterms here $4 \leq k < 8$, then with them the procedure of splitting (step1) with the help of the matrix M_4^2 and its covering (step 2) will be done:

$$\{(0011),(1011),(1110),(0001),(0100),(1000),(1001)\}^{\oplus} \overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow} \begin{bmatrix} ll-- \\ l-l- \\ l--l \\ -ll- \\ -l-l \\ --ll \end{bmatrix} = \begin{bmatrix} 00-- & \underline{10--} & 11-- & 00-- & 01-- & \underline{10--} & \underline{10--} \\ 0-1- & 1-1- & 1-1- & 0-0- & 0-0- & 1-0- & 1-0- \\ 0--1 & 1--1 & 1--0 & 0--1 & 0--0 & 1--0 & 1--1 \\ -01- & -01- & -11- & \underline{-00-} & -10- & \underline{-00-} & \underline{-00-} \\ \underline{\mathbf{-0-1}} & \underline{\mathbf{-0-1}} & -1-0 & \underline{\mathbf{-0-1}} & -1-0 & -0-0 & \underline{\mathbf{-0-1}} \\ --11 & --11 & --10 & --01 & --00 & --00 & --01 \end{bmatrix} \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow} \{-l-l\} = \{(-0-1),(1110),(0100),(1000)\}^{\oplus}.$$

So, the 1-st stage of the algorithm in the case of the procedure of consequent conjuncterms splitting will be completed with the set of covering

$$Y^{\oplus} \overset{C}{\Rightarrow} \{(--1-),(-0-1),(1110),(0100),(1000)\}^{\oplus}, \tag{27}$$

which has the interrelation $k_{\theta} / k_{l} / k_{in} = 5/15/8$.

The 2-nd stage of the algorithm $Y^{\oplus}$ – the procedure of iterative simplification – is done with the conjuncterms of the set of the covering in sequence of the following steps:

Step 1: for every pair with $d=1$ (pairs with $d=0$ are not taken into account) either the rule (2) of a theorem 1, or the rule (6) of a theorem 2; are applied; after the respective replacement transition to the 1-st is done, if there are not such pairs then to the step 2;

Step 2: for every pair with $d=2$ we apply either one from the sets of the rule (3) of a theorem 1, or the rule (7) of a theorem 2; after the respective replacement transition to the 1-st step is done and if there are not such pairs, then to the 3-rd step;

Step 3: for every pair with $d=3$ we apply either one from the sets of the rule (4) of a theorem 1, or one from the sets of the rules (8), (9) or (10) of a theorem 2, or one from the rules (15), (16) or (17) of a theorem 3; after the respective replacement transition to the 1-st step is done and if there are not such pairs, then to the 4-th step;

Step 4: for every pair with $d=4$ we apply one from the sets of the rule (5) of a theorem 1, or one from the rules (8), (9) or (10) of a theorem 2, or one from the sets of the rules (18)–(23) of a theorem 3; after the respective replacement transition to the 1-st step is done and if there are not such pairs, then to the 5-th step;

Step 5: if further transformation does not lead to simplification of the set of conjuncterms, then this set is the searched minimal PSTF $Y^{\oplus}$ of the given function *f*, the cost of realization of which is determined by the interrelation $k_{\theta}^{*} / k_{l}^{*} / k_{in}^{*}$.

Let us consider the 2-nd stage of the algorithm on the example of our function first for the case of the minterms parallel splitting. The least distance $d=2$ have the pairs in the set (26) $\begin{pmatrix} 01-- \\ 10-- \end{pmatrix}$ and $\begin{pmatrix} 0-01 \\ 1-11 \end{pmatrix}$. For them we apply the rule (3) of a theorem 1: $\begin{pmatrix} 01-- \\ 10-- \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} 0--- \\ \hline -0-- \end{pmatrix}, \begin{pmatrix} 1--- \\ \hline\hline -1-- \end{pmatrix} \right\}$ and $\begin{pmatrix} 0-01 \\ 1-11 \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} 0--1 \\ \hline --11 \end{pmatrix}, \begin{pmatrix} 1--1 \\ \hline\hline --01 \end{pmatrix} \right\}$. In these sets we highlight the pairs with the distance $d=1$, for which we apply the rule (6) of a theorem 2. Now $Y^{\oplus} = \left\{ \begin{Bmatrix} 1.\,(0--0),(--11),(-0--) \\ 2.\,(1--0),(--01),(-1--) \end{Bmatrix}, (0010) \right\}^{\oplus}$, so, the given function *f* has two solutions of minimization that reflect the minimal PSTF

1. $Y^{\oplus} = \{(0--0),(--11),(-0--),(0010)\}^{\oplus}$;
2. $Y^{\oplus} = \{(1--0),(--01),(-1--),(0010)\}^{\oplus}$.

The cost of realization of the first solution is $k_{\theta}^{*} / k_{l}^{*} / k_{in}^{*} = 4/9/6$, the second – $k_{\theta}^{*} / k_{l}^{*} / k_{in}^{*} = 4/9/5$.

The second solution corresponds to [32] of the minimized function $f = x_2 \oplus x_1 \overline{x}_4 \oplus \overline{x}_3 x_4 \oplus \overline{x}_1 \overline{x}_2 x_3 \overline{x}_4$. Let us consider the 2-nd stage of the algorithm for the set (27), which is obtained in the procedure of consequent splitting. The least distance $d=2$ is in three pairs of the minterms $\begin{pmatrix} 1110 \\ 0100 \end{pmatrix}$, $\begin{pmatrix} 1110 \\ 1000 \end{pmatrix}$ and

$\begin{pmatrix}0100\\1000\end{pmatrix}$. Applying the rule (3) of a theorem 1, for example, to the first pair, $\begin{pmatrix}1110\\0100\end{pmatrix}\overset{\oplus}{\Rightarrow}$ $\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}11-0\\-100\end{pmatrix},\begin{pmatrix}-110\\01-0\end{pmatrix}\right\}$ we get $Y^{\oplus}=\left\{(--1-),(-0-1),\begin{Bmatrix}(11-0),(-100)\\(-110),(01-0)\end{Bmatrix},(1000)\right\}^{\oplus}$, where the pairs $\begin{pmatrix}11-0\\1000\end{pmatrix}$ and $\begin{pmatrix}-100\\1000\end{pmatrix}$ have the distance $d=2$. If further transformation is done, for example of the first pair, then according to the rule (7) of a theorem 2 we get $\begin{pmatrix}11-0\\1000\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}1--0\\1010\end{pmatrix}$. Now $Y^{\oplus}=\{(--1-),(-0-1),(1--0),(\underline{1010}),(\underline{-100})\}^{\oplus}$, in which for the pair that has $d=3$, we apply the rule (10) of a theorem 2: $\begin{pmatrix}1010\\-100\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}-1-0\\--10\\0010\end{pmatrix},\begin{pmatrix}-0-0\\--00\\0010\end{pmatrix}\right\}$. Taking for further transformation, for example, the first set, we get

$$Y^{\oplus}=\{(\underline{--1-}),(\underline{\underline{-0-1}}),(1--0),(\underline{\underline{-1-0}}),(\underline{--10}),(0010)\}^{\oplus}\Rightarrow$$

$$\Rightarrow\{(\underline{--11}),(\underline{---1}),(-1--),(1--0),(0010)\}^{\oplus}\Rightarrow\{(--01),(-1--),(1--0),(0010)\}^{\oplus}.$$

As we see the obtained minimal PSTF $Y^{\oplus}$ coincides with the solution 2 of the previous case.

Let us note that for this function other possible variants of choice of sets on different transformation steps will give the analogical result. The given further examples illustrate the suggested method of minimization of complete functions.

Example 6. To minimize by minimization method the function in the polynomial format given in SOP $f(a,b,c,d)=ab\vee\overline{a}c\vee b\overline{d}$ (*this function is borrowed from* [33, p. 318]).

Solution. Having transformed SOP of the given function *f* into the perfect PSTF $Y^{\oplus}$ [29], we get:

$$Y^{\oplus}=\{(0010),(0011),(0100),(0110),(0111),(1100),(1101),(1110),(1111)\}^{\oplus}\overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow}\begin{bmatrix}l---\\-l--\\--l-\\---l\end{bmatrix}=\begin{bmatrix}0--- & 0--- & 0--- & 0--- & 0--- & 1--- & 1--- & 1--- & 1---\\ -0-- & -0-- & \underline{\underline{-1--}} & \underline{\underline{-1--}} & \underline{\underline{-1--}} & \underline{\underline{-1--}} & \underline{\underline{-1--}} & \underline{\underline{-1--}} & \underline{\underline{-1--}}\\ --1- & --1- & --0- & --1- & --1- & --0- & --0- & --1- & --1-\\ ---0 & ---1 & ---0 & ---0 & ---1 & ---0 & ---1 & ---0 & ---1\end{bmatrix}\overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow}\{--l-\}=\{\big((-1--),(0101)\big),(0010),(0011)\}^{\oplus}=\{(-1--),(0101),(001-)\}^{\oplus}.$$

The given function *f* has the minimal PSTF $Y^{\oplus}=\{(-1--),(001-),(0101)\}^{\oplus}$, that corresponds to $f=b\oplus\overline{a}\overline{b}c\oplus\overline{a}b\overline{c}d$. The cost of its realization $k_{\theta}^{*}/k_{l}^{*}/k_{in}^{*}=3/8/4$, analogically [33].

Example 7. To minimize in the polynomial format with the help of splitting method the function $f(x_1,x_2,x_3,x_4)$, given by the perfect STF $Y^1=\{0,6,14,15\}^1$ (*this function is borrowed from* [21, p. 28]).

Solution. $Y^{\oplus}=\{(0000),(0110),(1110),(1111)\}^{\oplus}\overset{S}{\Rightarrow}$

$$\overset{S}{\Rightarrow}\begin{bmatrix} ll-- \\ l-l- \\ l--l \\ -ll- \\ -l-l \\ --ll \end{bmatrix} = \begin{bmatrix} 00-- & 01-- & 11-- & 11-- \\ 0-0- & 0-1- & 1-1- & 1-1- \\ 0--0 & 0--0 & 1--0 & 1--1 \\ -00- & \underline{-11-} & \underline{-11-} & \underline{-11-} \\ -0-0 & -1-0 & -1-0 & -1-1 \\ --00 & --10 & --10 & --11 \end{bmatrix} \overset{C}{\Rightarrow} \{-ll-\} = \left\{\left((-11-),(0111)\right),(0000)\right\}^{\oplus}.$$

We apply the rule (4) of a theorem 1 to the minterms (0000) and (0111):

$$\begin{pmatrix} 0000 \\ 0111 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 000- \\ 00-1 \\ 0-11 \end{pmatrix}, \begin{pmatrix} 000- \\ 01-1 \\ 0-01 \end{pmatrix}, \begin{pmatrix} 001- \\ 00-0 \\ 0-11 \end{pmatrix}, \underline{\begin{pmatrix} 011- \\ 00-0 \\ 0-10 \end{pmatrix}}, \begin{pmatrix} 010- \\ 01-1 \\ 0-00 \end{pmatrix}, \underline{\begin{pmatrix} 011- \\ 01-0 \\ 0-00 \end{pmatrix}} \right\}^{\oplus}.$$

After putting the underlined sets in the set of covering, we get two equal as to the cost of realization solutions of minimization of the given function which is reflected by the minimal PSTF:

$$Y^{\oplus} = \{(-11-),(0000),(0111)\}^{\oplus} \Rightarrow \left\{ (\underline{-11-}),(\underline{011-}), \begin{Bmatrix} 1.(00-0),(0-10) \\ 2.(01-0),(0-00) \end{Bmatrix} \right\}^{\oplus} \Rightarrow$$

$$\Rightarrow \left\{ (111-), \begin{Bmatrix} 1.(00-0),(0-10) \\ 2.(01-0),(0-00) \end{Bmatrix} \right\}^{\oplus}.$$

The cost of realization of the minimized function is a better result $k_{\theta}^{*} / k_{l}^{*} / k_{in}^{*} =$ 3/9/5 than in [21], where it is equal to 3/10/5, namely: $Y^{\oplus} = \{(-11-),(0000),(0111)\}^{\oplus}$.

Example 8. To minimize in the polynomial format with the help of splitting method the function $f(x_1,x_2,x_3,x_4)$, given by the perfect STF $Y^1 = \{0,2,4,7,9,10,12,13\}^1$ (*this function is borrowed from* [2, p. 300] and [35]).

Solution. $Y^{\oplus} = \{(0000),(0010),(0100),(0111),(1001),(1010),(1100),(1101)\}^{\oplus} \overset{S}{\Rightarrow}$

$$\overset{S}{\Rightarrow}\begin{bmatrix} l--- \\ -l-- \\ --l- \\ ---l \end{bmatrix} = \begin{bmatrix} 0--- & 0--- & 0--- & 0--- & 1--- & 1--- & 1--- & 1--- \\ -0-- & -0-- & -1-- & -1-- & -0-- & -0-- & -1-- & -1-- \\ \underline{--0-} & --1- & \underline{--0-} & --1- & \underline{--0-} & --1- & \underline{--0-} & \underline{--0-} \\ \underline{---\mathbf{0}} & \underline{---\mathbf{0}} & \underline{---\mathbf{0}} & ---1 & ---1 & \underline{---\mathbf{0}} & \underline{---\mathbf{0}} & ---1 \end{bmatrix} \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow}\{---l\} = \left\{\left((---0),(0110),(1000),(1110)\right),(0111),(1001),(1101)\right\}^{\oplus}.$$

We will do the procedure of splitting for the minterms of this set with the help of the matrix M_4^2:

$$\{(0110),(0111),(1000),(1001),(1101),(1110)\}^{\oplus} \overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow}\begin{bmatrix} ll-- \\ l-l- \\ l--l \\ -ll- \\ -l-l \\ --ll \end{bmatrix} = \begin{bmatrix} 01-- & 01-- & 10-- & 10-- & 11-- & 11-- \\ 0-1- & 0-1- & \underline{1-0-} & \underline{1-0-} & \underline{1-0-} & 1-1- \\ 0--0 & 0--1 & 1--0 & 1--1 & 0--1 & 1--0 \\ \underline{-11-} & \underline{-11-} & -00- & -00- & -10- & \underline{-11-} \\ -1-0 & -1-1 & -0-0 & -0-1 & -1-1 & -1-0 \\ --10 & --11 & --00 & --01 & --01 & --10 \end{bmatrix} \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow}\begin{Bmatrix} l-l- \\ -ll- \end{Bmatrix} = \{((-11-),(1111)),((1-0-),(1100))\}^{\oplus}.$$

So, the covering of the given function *f* is made of the set

$$\overset{C}{\Rightarrow}\{(---0),(-11-),(1-0-),(\underline{1100}),(\underline{1111})\}^{\oplus}.$$

After transformation of the underlined minterms pair according to the rule (3) of a theorem 1, namely $\begin{pmatrix}1100\\1111\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}11-0\\111-\end{pmatrix},\begin{pmatrix}110-\\11-1\end{pmatrix}\right\}$, we get two solutions of minimization of the given function *f*, that reflects PSTF $Y^{\oplus}$:

$$Y^{\oplus} = \left\{(---0),(\underline{-11-}),(\underline{\underline{1-0-}}),\begin{Bmatrix}1.(11-0),(\underline{111-})\\2.(\underline{\underline{110-}}),(11-1)\end{Bmatrix}\right\}^{\oplus}.$$

After application to the underlined pairs the rules (6) of a theorem 2 we get the minimal PSTF $Y^{\oplus}$ of both solutions:

1. $Y^{\oplus} = \{(---0),(011-),(1-0-),(11-0)\}^{\oplus}\pi$;
2. $Y^{\oplus} = \{(---0),(-11-),(100-),(11-1)\}^{\oplus}$.

The cost of realization of the solution 1 is $k_{\theta}^{*}/k_{l}^{*}/k_{in}^{*} =$ 4/9/4, and solution 2 is 4/9/3, that is better than all possible variants of covering of matrices and better than in [2, 34], where the cost of realization is 4/9/7, namely $Y^{\oplus} = \{(--0-),(-0-0),(000-),(01-1)\}^{\oplus}$.

Example 9. To minimize in the polynomial format with the help of splitting method the function $f(a,b,c,d) = \overline{a}\overline{c} \oplus \overline{a}b\overline{c}\overline{d} \oplus ab \oplus a\overline{c}d$ (*this function is borrowed from* [15, p. 6]) (look at the example 5).

Solution. $Y^{\oplus} = \{(0000),(0001),(0101),(1001),(1100),(1110),(1111)\}^{\oplus} \overset{S}{\Rightarrow}$

$$\overset{S}{\Rightarrow}\begin{bmatrix} ll-- \\ l-l- \\ l--l \\ -ll- \\ -l-l \\ --ll \end{bmatrix} = \begin{bmatrix} 00-- & 00-- & 01-- & 10-- & \underline{\mathbf{11--}} & \underline{\mathbf{11--}} & \underline{\mathbf{11--}} \\ \underline{0-0-} & \underline{0-0-} & \underline{0-0-} & 1-0- & 1-0- & 1-1- & 1-1- \\ 0--0 & 0--1 & 0--1 & 1--1 & 0--0 & 1--0 & 1--1 \\ \underline{-00-} & \underline{-00-} & -10- & \underline{-00-} & -10- & -11- & -11- \\ -0-0 & -0-1 & -1-1 & -0-1 & -1-0 & -1-0 & -1-1 \\ --0 & \underline{\mathbf{--01}} & \underline{\mathbf{--01}} & \underline{\mathbf{--01}} & --00 & --10 & --11 \end{bmatrix} \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow}\begin{Bmatrix} ll-- \\ --ll \end{Bmatrix} = \{(0000),((--01),(1101)),((11--),(1101))\}^{\oplus} \Rightarrow \{(0000),(--01),(11--)\}^{\oplus}.$$

The cost of realization of the minimized function is equal to $k_{\theta}^{*}/k_{l}^{*}/k_{in}^{*} =$ 3/8/5, that corresponds to [15].

2.2. Algorithm of minimization of incomplete functions. Examples of minimization

As it is known [27, 28], the incomplete function $f(x_1, x_2, ..., x_n)$ can be given by the perfect STF $\{Y^1, Y^{\sim}\}$, if it is inpredeterminated function *f*, i.e. $|Y^{\sim}| \le |Y^0 \cup Y^1|$, or by the perfect STF $\{Y^1, Y^0\}$, if it is weakly determinated function *f*, i.e. $|Y^{\sim}| > |Y^0 \cup Y^1|$, where Y^1, Y^0, $Y^{\sim}$ – the subsets of the complete set $\mathbf{E}_2^n$, on which the function *f* takes the value respectively 1, 0, ~ (~~so called don't care,~~ i.e. unspecified value of function *f*). In the polynomial set-theoretical format to the sets Y^1, Y^0, $Y^{\sim}$ correspond the sets $Y^{\oplus}$, $Y^{\overline{\oplus}}$, $Y^{\tilde{\oplus}}$, the elements of which are numeric minterms of the perfect PSTF of this or that incomplete function *f*, namely: inpredeterminated function *f* given by the perfect PSTF $\{Y^{\oplus}, Y^{\tilde{\oplus}}\}$, and weakly determinated function *f* given by the perfect PSTF $\{Y^{\oplus}, Y^{\bar{\oplus}}\}$.

Analogically as in the disjunctive format [27], the procedure of splitting of conjuncterms is realized by the splitting matrix M_n^r, which consists of basic submatrix $Mr^{\oplus}$ and additional submatrix $Mr^{\tilde{\oplus}}$ or $Mr^{\bar{\oplus}}$. The matrix M_n^r of the inpredeterminated function *f* will be designated as $Mr^{\oplus} \vdots Mr^{\tilde{\oplus}}$, and of the weakly determinated function *f* will be designated as $Mr^{\oplus} \vdots Mr^{\bar{\oplus}}$, where $Mr^{\oplus}$ contains the splitting elements of the subset $Y^{\oplus}$, and $Mr^{\tilde{\oplus}}$ and $Mr^{\bar{\oplus}}$ contains the splitting elements of the corresponding subsets $Y^{\tilde{\oplus}}$ and $Y^{\bar{\oplus}}$; here $\vdots$ that is a symbol of separation of the matrix M_n^r. The sets of the conjuncterms obtained as a result of covering of the matrix M_n^r of the mentioned functions will be designated respectively $Y^{\oplus} \vdots Y^{\tilde{\oplus}}$ and $Y^{\oplus} \vdots Y^{\bar{\oplus}}$.

An algorithm of minimization of an incomplete function in the polynomial set-theoretical format is realized in the same way as for a complete function (see p. 2.1) in two stages. On the 1-st stage the minterms of the perfect PSTF $\{Y^{\oplus}, Y^{\tilde{\oplus}}\}$ of the inpredeterminated function *f* or the perfect PSTF $\{Y^{\oplus}, Y^{\bar{\oplus}}\}$ of the weakly determinated function *f* perform the procedure of splitting with the help of the splitting matrix M_n^r. In both cases the main role in covering the matrix M_n^r play the elements of the basic submatrix $Mr^{\oplus}$. If compared with the algorithm of a complete function the only difference consists in the way of selection the elements of the submatrices covering $Mr^{\tilde{\oplus}}$ and $Mr^{\bar{\oplus}}$. Whereas the 2-nd stage of the algorithm of minimization of an incomplete function is realized in an analogical way as p. 2.1.

First of all let us consider the case for an inpredeterminated function *f* in detail.

In this case the procedure of splitting (the 1-st stage) is done with the help of the splitting matrix M_n^r, the rank *r* of which is determined on the ground of the data of the set $Y^{\oplus}$ (look at p. 2.1). Here the elements of the matrix covering M_n^r can be, except for the conjuncterms-copies of the submatrix $Mr^{\oplus}$, the elements of submatrix covering $Mr^{\tilde{\oplus}}$, if they do not reduce at least one of the parameters of the interrelation, in the opposite case such elements of the submatrix $k_\theta^* / k_l^* / k_{in}^*$, are not taken into account $Mr^{\tilde{\oplus}}$.

Let us illustrate the suggested method on the example of the inpredeterminated function $f(x_1, x_2, x_3, x_4)$ given by the perfect STF $\begin{cases} Y^1 = \{1,4,7,8,11\}^1 \\ Y^{\sim} = \{3,5,6,15\}^{\sim} \end{cases}$ (*this function is borrowed from* [35, p. 20]). The given function *f* has the perfect PSTF $\begin{cases} Y^{\oplus} = \{(0001),(0100),(0111),(1000),(1011)\}^{\oplus} \\ Y^{\tilde{\oplus}} = \{(0011),(0101),(0110),(1111)\}^{\tilde{\oplus}} \end{cases}$. We will do the 1-st stage of minimization with the minterms of the set $\{Y^{\oplus} \vdots Y^{\tilde{\oplus}}\}$ the procedure of splitting with the help of the matrix M_4^1 and its covering (the elements of covering are highlighted in bold font):

$$\{Y^{\oplus} \vdots Y^{\tilde{\oplus}}\} = \{(0001),(0100),(0111),(1000),(1011) \vdots (0011),(0101),(0110),(1111)\} \overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow} \begin{bmatrix} l--- \\ -l-- \\ --l- \\ ---l \end{bmatrix} = \left[\begin{array}{ccccc:cccc} \underline{0---} & \underline{0---} & \underline{0---} & 1--- & 1--- & \underline{0---} & \underline{0---} & \underline{0---} & 1--- \\ \underline{-0--} & -1-- & -1-- & \underline{-0--} & \underline{-0--} & \underline{-0--} & -1-- & -1-- & -1-- \\ \underline{--0-} & \underline{--0-} & --1- & \underline{--0-} & --1- & --1- & \underline{--0-} & --1- & --1- \\ \underline{\underline{---\mathbf{1}}} & ---0 & \underline{\underline{---\mathbf{1}}} & ---0 & \underline{\underline{---\mathbf{1}}} & \underline{\underline{---\mathbf{1}}} & \underline{\underline{---\mathbf{1}}} & ---0 & \underline{\underline{---\mathbf{1}}} \end{array}\right] \overset{C}{\Rightarrow}$$

$$\stackrel{C}{\Rightarrow}\{---l\}=\left\{\left((---1),(\underline{1001}),(\underline{1101})\right),(0100),(1000)\vdots(0110)\right\}^{\oplus}.$$

We will remove the minterm (0110) of the set $Y^{\tilde{\oplus}}$, from the obtained set as its participation in any variants of the transformation has no success. So, on the 1-st stage of minimization after the transformation (2) of the underlined minterms, i.e. $\begin{pmatrix}1001\\1101\end{pmatrix}\stackrel{\oplus}{\Rightarrow}(1-01)$, we will get PSTF

$$Y^{\oplus}=\{(---1),(1-01),(0100),(1000)\}^{\oplus}.$$

The cost of realizationn of the formed set of the 1-st stage reflects the interrelation $k_{\theta}/k_{l}/k_{in}=$ $=4/12/7$.

On the 2-nd stage of minimization after application of the rule (3) of a theorem 1 to the minterms PSTF $Y^{\oplus}$ $\begin{pmatrix}0100\\1000\end{pmatrix}\stackrel{\oplus}{\Rightarrow}\left\{\begin{pmatrix}0-00\\-000\end{pmatrix},\begin{pmatrix}-100\\1-00\end{pmatrix}\right\}$ and further simplification of the formed set (look at underlined elements) the minimal PSTF $Y^{\oplus}$ of the given function *f* is formed, namely:

$$Y^{\oplus}=\left\{(---1),(\underline{1-01}),\left\{\begin{matrix}(0-00),(-000)\\(-100),(\underline{1-00})\end{matrix}\right\}\right\}^{\oplus}\Rightarrow\{(---1),(1-0-),(-100)\}^{\oplus},$$

The cost of realization of which $k_{\theta}^{*}/k_{l}^{*}/k_{in}^{*}=3/6/3$ corresponds to [35], where $f(x_1,x_2,x_1,x_4)=$ $=x_4\oplus x_2\overline{x}_3\oplus x_1\overline{x}_3\overline{x}_4$.

Example 10. To minimize in the polynomial format with the help of the splitting method of the inpredeterminated function $f(x_1,x_2,x_3,x_4)$ given by the perfect STF $\begin{cases}Y^{1}=\{3,5,6,9,12,15\}^{1}\\Y^{\sim}=\{1,2,8,11\}^{\sim}\end{cases}$ (*this function is borrowed from* [37, p. 460]).

Solution. $\begin{cases}Y^{\oplus}=\{(0011),(0101),(0110),(1001),(1100),(1111)\}^{\oplus}\\Y^{\tilde{\oplus}}=\{(0001),(0010),(1000),(1011)\}^{\tilde{\oplus}}\end{cases}\stackrel{S}{\Rightarrow}$

$$\stackrel{S}{\Rightarrow}\begin{bmatrix}ll--\\l-l-\\l--l\\-ll-\\-l-l\\--ll\end{bmatrix}=\left[\begin{array}{cccccc:cccc}00-- & 01-- & 01-- & 10-- & 11-- & 11-- & 00-- & 00-- & 10-- & 10--\\ \underline{\mathbf{0-1-}} & 0-0- & \underline{\mathbf{0-1-}} & \underline{\mathbf{1-0-}} & \underline{\mathbf{1-0-}} & 1-1- & 0-0- & \underline{\mathbf{0-1-}} & \underline{\mathbf{1-0-}} & 1-1-\\ \underline{0--1} & \underline{0--1} & 0--0 & \underline{1--1} & 1--0 & \underline{1--1} & \underline{0--1} & 0--0 & 1--0 & \underline{1--1}\\ -01- & \underline{-00-} & -11- & \underline{-00-} & -10- & -11- & \underline{-00-} & -01- & \underline{-00-} & -01-\\ \underline{-0-1} & -1-1 & -1-0 & \underline{-0-1} & -1-0 & -1-1 & \underline{-0-1} & -0-0 & -0-0 & \underline{-0-1}\\ \underline{--11} & \underline{--01} & --10 & \underline{--01} & --00 & \underline{--11} & \underline{--01} & --10 & --00 & \underline{--11}\end{array}\right]\stackrel{C}{\Rightarrow}$$

$$\stackrel{C}{\Rightarrow}\{l-l-\}=\left\{\left((0-1-),(\underline{0111})\right),\left((1-0-),(\underline{\underline{1101}})\right),(\underline{\underline{0101}}),(\underline{1111})\vdots(0001),(1011)\right\}^{\oplus}\Rightarrow$$

$$\Rightarrow\{(0-1-),(\underline{-111}),(1-0-),(\underline{-101})\}^{\oplus}\Rightarrow\{(\underline{0-1-}),(-1-1),(\underline{1-0-})\}^{\oplus}.$$

After the transformation (3) $\begin{pmatrix}0-1-\\1-0-\end{pmatrix}\stackrel{\oplus}{\Rightarrow}\begin{pmatrix}--1-\\1---\end{pmatrix}$ we will get the final minimal PSTF

$$Y^{\oplus}=\{(--1-),(1---),(-1-1)\}^{\oplus},$$

that corresponds to STF $Y^{\oplus}=\{(2,3,6,7,10,11,14,15),(8,9,10,11,12,13,14,15),(5,7,13,15)\}^{\oplus}=$

$=\{\mathbf{2},3,5,6,\mathbf{8},9,12,15)\}^{\oplus}$ (the highlighted in bold font elements belong to $Y^{\sim}$). The cost of realization of the given function is equal to 3/4/0. If compared with [36] it is a better result, where $Y^{\oplus}=\{(-11-),(11--),(---1)\}^{\oplus}$, the cost of realization of which is equal to 3/5/0.

Example 11. To minimize in the polynomial format with the help of method of splitting the inpredeterminated function $f(x_1,x_2,x_3,x_4)$ given by the STF $\begin{cases} Y^1=\{(110-),(0-11),(1110)\}^1 \\ Y^{\sim}=\{(0-10),(10-1)\}^{\sim} \end{cases}$ (*this function is borrowed from* [20, p. 16]).

Solution. $\begin{cases} Y^{\oplus}=\{(0011),(0111),(1100),(1101),(1110)\}^{\oplus} \\ Y^{\tilde{\oplus}}=\{(0010),(0110),(1001),(1011)\}^{\tilde{\oplus}} \end{cases} \overset{S}{\Rightarrow}$

$$\overset{S}{\Rightarrow}\begin{bmatrix} ll-- \\ l-l- \\ l--l \\ -ll- \\ -l-l \\ --ll \end{bmatrix}=\left[\begin{array}{ccccc|cccc} 00-- & 01-- & \underline{\mathbf{11--}} & \underline{\mathbf{11--}} & \underline{\mathbf{11--}} & 00-- & 01-- & 10-- & 10-- \\ \underline{0-1-} & \underline{0-1-} & 1-0- & 1-0- & 1-1- & \underline{0-1-} & \underline{0-1-} & 1-0- & 1-1- \\ 0--1 & 0--1 & 1--0 & 1--1 & 1--0 & 0--0 & 0--0 & 1--1 & 1--1 \\ -01- & -11- & -10- & -10- & -11- & -01- & -11- & -00- & -01- \\ -0-1 & -1-1 & -1-0 & -1-1 & -1-0 & -0-0 & -1-0 & -0-1 & -0-1 \\ \underline{\mathbf{--11}} & \underline{\mathbf{--11}} & --00 & --01 & --10 & --10 & --10 & --01 & \underline{\mathbf{--11}} \end{array}\right]\overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow}\begin{Bmatrix} ll-- \\ --ll \end{Bmatrix}=\{((--11),(1111)),((11--),(1111))\}^{\oplus}\Rightarrow\{(--11),(11--)\}^{\oplus}.$$

Answer. The given function has the minimal PSTF $Y^{\oplus}=\{(--11),(11--)\}^{\oplus}$, the cost of realization of which $k_{\theta}^{*}/k_{l}^{*}/k_{in}^{*}=$ 2/4/0, that corresponds to [20].

Now let us consider minimization of the weakly determinated function in the polynomial format with the help of the splitting method. Contrary to the inpredeterminated function, here, only the procedure of consequent splitting starting with the matrix M_n^{n-1} is applied. Here the elements of the additional matrix $Mr^{\bar{\oplus}}$, which belong to the set $Y^{\bar{\oplus}}$, cannot make the covering of the matrices M_n^{n-1}, M_n^{n-2}, …, M_n^r. Their role is to determine the elements of basic matrix $Mr^{\oplus}$ which can or cannot make the covering of the matrices M_n^{n-1}, M_n^{n-2}, …, M_n^r. If some element of the submatrix $Mr^{\oplus}$ has a copy in $Mr^{\bar{\oplus}}$ for the given mask, it cannot belong to the set of covering but only its generating element. Respectively, the elements of covering of splitting matrices of a weakly determinated function can be only those elements of the submatrix $Mr^{\oplus}$, which do not have any copies in the submatrix $Mr^{\bar{\oplus}}$.

Further given example illustrates the peculiarities of minimization of a weakly determinated function.

Example 12. To minimize in the polynomial format with the help of the splitting method the weakly determinated function $f(x_1,x_2,x_3,x_4,x_5)$, given by perfect STF $\begin{cases} Y^1=\{1,2,10,15,22,27\}^1 \\ Y^0=\{6,8,12,17,23\}^0 \end{cases}$ (*this function is borrowed from* [38, p. 55]).

Solution. Having transformed the perfect STF of weakly determinated function *f* into the perfect PSTF $\begin{cases} Y^{\oplus}=\{(00001),(00010),(01010),(01111),(10110),(11011)\}^{\oplus} \\ Y^{\bar{\oplus}}=\{(00110),(01000),(01100),(10001),(10111)\}^{\bar{\oplus}} \end{cases}$, we do the procedure of consequent splitting of the given minterms starting with the matrix M_5^4:

$$\overset{S}{\Rightarrow}\begin{bmatrix} lll- \\ lll-l \\ ll-ll \\ l-lll \\ -llll \end{bmatrix} = \left[\begin{array}{cccccc|ccccc} 0000- & 0001- & 0101- & 0111- & \underline{\underline{1011-}} & 1101- & 0011- & 0100- & 0110- & 1000- & \underline{\underline{1011-}} \\ 000-1 & 000-0 & \underline{\underline{010-0}} & 011-1 & 101-0 & 110-1 & 001-0 & \underline{\underline{010-0}} & 011-0 & 100-1 & 101-1 \\ 00-01 & \underline{\underline{00-10}} & 01-10 & 01-11 & 10-10 & 11-11 & \underline{\underline{00-10}} & \underline{\underline{01-00}} & \underline{01-00} & 10-01 & 10-11 \\ \mathbf{0-001} & \mathbf{0-010} & \mathbf{0-010} & \mathbf{0-111} & \mathbf{1-110} & \mathbf{1-011} & \mathbf{0-110} & \mathbf{0-000} & \mathbf{0-100} & \mathbf{1-001} & \mathbf{1-111} \\ -0001 & -0010 & -1010 & -1111 & \underline{\underline{-0110}} & -1011 & \underline{\underline{-0110}} & -1000 & -1100 & -0001 & -0111 \end{array}\right].$$

Covering this matrix, for example, with the conjuncterms of the mask $\{l-lll\}$, we get the set

$$\overset{C}{\Rightarrow}\{(0-001),(0-010),(0-111),(1-110),(1-011)\vdots(0-110),(0-000),(0-100),(1-001),(1-111)\}^{\oplus}.$$

The splitting procedure will be done with the obtained conjuncterms with the help of M_5^3:

$$\overset{S}{\Rightarrow}\begin{bmatrix} l-ll- \\ l-l-l \\ l--ll \\ --lll \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} \underline{\underline{0-00-}} & 0-01- & \underline{\underline{0-11-}} & \underline{\underline{1-11-}} & 1-01- & \underline{\underline{0-11-}} & \underline{\underline{0-00-}} & 0-10- & 1-00- & \underline{\underline{1-11-}} \\ 0-0-1 & \underline{\underline{0-0-0}} & 0-1-1 & 1-1-0 & \underline{\underline{1-0-1}} & \underline{\underline{0-1-0}} & \underline{\underline{0-0-0}} & \underline{0-1-0} & \underline{1-0-1} & 1-1-1 \\ 0--01 & \underline{\underline{0--10}} & 0--11 & 1--10 & \underline{\underline{1--11}} & \underline{\underline{0--10}} & \underline{0--00} & \underline{0--00} & 1--01 & \underline{\underline{1--11}} \\ \underline{\underline{--001}} & --010 & \underline{\underline{--111}} & \underline{\underline{--110}} & --011 & \underline{\underline{--110}} & --000 & --100 & \underline{\underline{--001}} & \underline{\underline{--111}} \end{array}\right]\begin{matrix} 1) \\ 1) \\ 2) \\ 2) \end{matrix}$$

We remove from the matrix M_5^3 the underlined with two lines elements and from its submatrix $Mr^{\bar{\oplus}}$ beside this also underlined with one line elements. Then the procedure of covering the matrix M_5^3 will be conveniently realized by: 1) uniting of the masks $\{l-ll-\}\cup\{l-l-l\}$ or 2) uniting of the masks $\{l--ll\}\cup\{--lll\}$.

Let us consider the case 1) separately for the masks $\{l-ll-\}$ and $\{l-l-l\}$. Particularly, for $\{l-ll-\}$ we have:

$$\overset{C}{\Rightarrow}\{l-ll-\}=\{(0-01-),(1-01-)\vdots(0-10-),(1-00-)\}^{\oplus}.$$

We will do the next step of the procedure of splitting of the elements of this set with the help of the matrix M_5^2 and its covering:

$$\overset{S}{\Rightarrow}\begin{bmatrix} l-l-- \\ l--l- \\ --ll- \end{bmatrix} = \left[\begin{array}{cc|cc} 0-0-- & \underline{\underline{1-0--}} & 0-1-- & \underline{\underline{1-0--}} \\ 0--1- & 1--1- & 0--0- & 1--0- \\ \underline{--01-} & \underline{--01-} & --10- & --00- \end{array}\right]\overset{C}{\Rightarrow}(--01-).$$

Doing the analogical procedures for the set formed by the mask $\{l-l-l\}$, we get:

$$\overset{C}{\Rightarrow}\{l-l-l\}=\{(0-0-1),(0-1-1),(1-1-0)\vdots(1-1-1)\}^{\oplus}\overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow}\begin{bmatrix} l-l-- \\ l---l \\ --l-l \end{bmatrix} = \left[\begin{array}{ccc|c} 0-0-- & 0-1-- & \underline{\underline{1-1--}} & \underline{\underline{1-1--}} \\ \underline{0---1} & \underline{0---1} & 1---0 & 1---1 \\ --0-1 & \underline{\underline{--1-1}} & --1-0 & \underline{\underline{--1-1}} \end{array}\right]\overset{C}{\Rightarrow}(0---1),(1---0).$$

After uniting these sets we get $Y^{\oplus}=\{(--01-),(\underline{0---1}),(\underline{1---0})\}^{\oplus}$, which can be simplified according to the rule (3) of a theorem 1: $\begin{pmatrix} 0---1 \\ 1---0 \end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix} ----1 \\ 1---- \end{pmatrix}$. We should mark that the same result, namely $Y^{\oplus}=\{(--01-),(----1),(1----)\}^{\oplus}$, is got for the case 2 ~~too~~.

Answer. The given functionn *f* has the minimal PSTF $Y^{\oplus}=\{(--01-),(----1),(1----)\}^{\oplus}$, the cost of realization of which is equal to $k_{\theta}^{*}/k_{l}^{*}/k_{in}^{*}=$ 3/4/1, that is a better result than in [38], where $k_{\theta}^{*}/k_{l}^{*}/k_{in}^{*}=$ 3/6/3. With the aim of verification the result we write down the obtained PSTF in the numeric expression $Y^{\oplus} = \{\mathbf{1},\mathbf{2},5,7,9,\mathbf{10},13,\mathbf{15},16,19,20,\mathbf{22},24,\mathbf{27},28,30,31\}^{\oplus}$, where we see in bold font the decimal minterms of the set $Y^{\tilde{\oplus}}$, by which the perfect PSTF $Y^{\oplus}$ of the given function *f* is predetermined.

Conclusion

A minimization method in the polynomial set-theoretical format of complete and incomplete logic functions with *n* variables has been suggested. It is based on the splitting procedure of given minterms and iterative simplification of pairs of conjuncterms according to the set-theoretical rules described in the author's previous article (1. Generalized rules of conjuncterms simplification). Efficiency of the method has been proved by numerous examples borrowed from well-known publications with a better result in most cases. The last is explained by the following: as these rules can be applied to pairs of conjuncterms with Hamming distance $d \geq 3$, in a set of conjuncterms of different ranks, probability of their efficient simplification is increased and respectively, the cost of the minimized function realization is reduced. Beside this, due to the procedure of the splitting minterms on the covering of splitting matrices in the polynomial format a way to obtain the searched result is reduced.

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