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A New Method of the Logical Functions Minimization in the Polynomial Set-Theoretical Format. 3. Minimization of Function System

Рассмотрен новый метод минимизации системы логических функций от *n* переменных в полиномиальном теоретико-множественном формате, основанный на процедуре расщепления системных минтермов и обобщенных теоретико-множественных правилах упрощения множества конъюнктермов разных рангов. Преимущества метода иллюстрируются примерами.

A new minimization method of the logic functions system of *n* variables in polynomial set-theoretical format is considered. The method is based on the splitting procedure of minterms system and on the generalizing the conjuncterms set of the different ranks by the set-theoretical simplifing rules. The advantages of the suggested method are illustrated by the examples.

Розглянуто новий метод мінімізації системи логікових функцій від *n* змінних у поліноміальному теоретико-множинному форматі, що ґрунтується на процедурі розчеплення системних мінтермів та узагальнених теоретико-множинних правилах спрощення множин кон'юнктермів різних рангів. Переваги методу ілюструються прикладами.

As it is known [27–29], in general case the system $F(X)$ of the functions $f_i(x_1, x_2, ..., x_n)$, $i = 1, 2, ..., s$, in the disjunctive set-theoretical format is reflected by the PSTF $\{Y_i^1, Y_i^*\}$ of the following expression:

$$F(X) = \begin{cases} Y_1^1 = \{\theta_{11}, \theta_{12}, ..., \theta_{1k_1}\}^1, \ Y_1^* = \{\theta_{k_1+1}, \theta_{k_1+2}, ..., \theta_{2^n-k_1-v_1}\}^* \\ Y_2^1 = \{\theta_{21}, \theta_{22}, ..., \theta_{2k_2}\}^1, \ Y_2^* = \{\theta_{k_2+1}, \theta_{k_2+2}, ..., \theta_{2^n-k_2-v_2}\}^* \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ Y_s^1 = \{\theta_{s1}, \theta_{s2}, ..., \theta_{sk_s}\}^1, \ Y_s^* = \{\theta_{k_s+1}, \theta_{k_s+2}, ..., \theta_{2^n-k_s-v_s}\}^* \end{cases}, \quad v_i < 2^n - k_i, \tag{27}$$

where θ_{ij} are conjuncterms of the ranks $r \in \{1, 2, ..., n\}$; the mark * changes the symbol ~ or 0 depending on the fact which part of *n*-dimension space of the given system $F(X)$ belongs to indefinite values of the functions f_i for all $i = 1, 2, ..., s$, namely: if $Y_i^* \equiv Y_i^\sim$, here $|Y_i^\sim| \le |Y_i^0 \cup Y_i^1|$, then (27) is the system of inpredeterminated (incomplete) functions reflected by the STF $\{Y_i^1, Y_i^\sim\}$, if $Y_i^* \equiv Y_i^0$, here $|Y_i^\sim| > |Y_i^0 \cup Y_i^1|$, then (27) then (27) is the system of weakly determinated (incomplete) functions reflected by the STF $\{Y_i^1, Y_i^0\}$, and if $Y_i^* \equiv \varnothing$, then (27) is the system of complete functions reflected by the STF $\{Y_i^1\}$.

According to the suggested method of minimization in the polynomial set-theoretical format the system $F(X)$ of the functions $f_i(x_1, x_2, ..., x_n)$ is given by the minterms m_{ij} (i.e. conjuncterms of *n*-rank) of the perfect PSTF $\{Y_i^\oplus, Y_i^{\oplus *}\}$, where $Y_i^\oplus \equiv Y_i^1$ and $Y_i^{\oplus *} \equiv Y_i^*$ in case of the perfect STF $\{Y_i^1, Y_i^*\}$ (27), $i = 1, 2, ..., s$:

$$F(X) = \begin{cases} Y_1^\oplus = \{m_{11}, m_{12}, ..., m_{1k_1}\}^\oplus, \ Y_1^{\oplus *} = \{m_{k_1+1}, m_{k_1+2}, ..., m_{2^n-k_1-v_1}\}^{\oplus *} \\ Y_2^\oplus = \{m_{21}, m_{22}, ..., m_{2k_2}\}^\oplus, \ Y_2^{\oplus *} = \{m_{k_2+1}, m_{k_2+2}, ..., m_{2^n-k_2-v_2}\}^{\oplus *} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ Y_s^\oplus = \{m_{s1}, m_{s2}, ..., m_{sk_s}\}^\oplus, \ Y_s^{\oplus *} = \{m_{k_s+1}, m_{k_s+2}, ..., m_{2^n-k_s-v_s}\}^{\oplus *} \end{cases}, \quad v_i < 2^n - k_i. \tag{28}$$

The minimization of the complete and incomplete functions system in the polynomial set-theoretical format will be considered on the basis of compatible realization when the search of the minimal PSTF $\{Y_i^{\oplus}, Y_i^{\oplus *}\}$ of the system $F(X)$ (28) is done with maximal use of equal conjuncterms of its function. For this similarly as in the case of compatible minimization of the system in the disjunctive format [27–29] with minterms of the perfect PSTF $\{Y_i^{\oplus}, Y_i^{\oplus *}\}$ (28) a set of so called *system minterms* $(m)_{1,2,...,s'}$, $s' \in \{1,2,...,s\}$, is formed the indices of which $1,2,...,s'$ mark belonging of equal minterms of the system (28) to its certain functions.

The algorithm of the system minimization $F(X)$ (28) is realized in the similar way as for one function in two stages (see p. 2.1). On the first stage we do the splitting procedure of the system minterms with the help of splitting matrix, covering of which results in formation of a set of system conjuncterms of different ranks. On the second stage after distribution of these conjuncterms in functions of the system $F(X)$ the procedure of iterative simplification, as a result of which the minimal PSTF $\{Y_i^{\oplus}, Y_i^{\oplus *}\}$ is formed taking into account compatible realization of the system.

3.1 Algorithm of minimization of a complete functions system. Examples of minimization

The algorithm of the compatible minimization system $F(X)$ of complete functions given by the perfect PSTF $\{Y_i^{\oplus}\}$, $Y_i^{*} \equiv \varnothing$, is realized in the following way. On the first stage the set $\{Y_I^{\oplus}\}$, $I \in \{1,2,...,s\}$, of the system minterms $(m)_{1,2,...,s'}$, which are split with the help of the matrix M_n^r, is formed from the given minterms. The minimal covering of the M_n^r differs from the analogical procedure in case of one function (p. 2.1). It is connected with the fact that generative elements of the matrix M_n^{n-1} are the system minterms $(m)_{1,2,...,s'}$ and consequently every element of the M_n^{n-1} is a splitting system conjuncterm of $(n-1)$-rank $(\theta_i^{n-1})_{1,2,...,s'}$ will have the same index with subsets of functions of the given system as its generative element. The elements of minimal covering of the matrix M_n^{n-1} in the same way as in case of one function will be identical to the system conjuncterms-copies of $(n-1)$-rank. But among them a decisive role for realization of compatible minimization of the system will play those ones the indices of which contain the greatest quantity of numbers. So, if $(\theta_i^{r-1})_{1,2,...,s'}$ and $(\theta_j^{r-1})_{1,2,...,s''}$, $s', s'' \in \{1,2,...,s\}$, these are identical system conjuncterms-copies of $(r-1)$-rank of the matrix M_n^r, $r = 1,2,...,n-1$, then they can be elements of its covering if the indices of their generative form $(\theta_i^r)_{1,2,...,s'}$ and $(\theta_j^r)_{1,2,...,s''}$ form a not empty intersection, i.e. $\{1,2,...,s'\} \cap \{1,2,...,s''\} \neq \varnothing$. For example, let the system minterms $(100)_{1,2,4}$, $(110)_{1,3}$, $(010)_{1,2,3}$ be generative elements of the matrix M_n^{n-1}. For the mask $\{I-I\}$ the identical system conjuncterms-copies will be $(1-0)_1$ and $(1-0)_1$, the index of which determines the intersection $\{1,2,4\} \cap \{1,3\} = \{1\}$, and for the mask $\{-II\}$ will be $(-10)_{1,3}$ and $(-10)_{1,3}$ because $\{1,3\} \cap \{1,2,3\} = \{1,3\}$. So, in this case for covering of the matrix M_n^{n-1} it is advisable to choose the second pair of the mask $\{-II\}$. The system conjuncterms-copies of the matrix M_n^r as well as for one function will be underlined

We should note that it is unadvisable to use a compatible minimization of a system with the complete functions in the case if the given the perfect PSTF (28) contains less than two minterms or it does not contain any common minterms at all. Then, instead of formation of a set of system minterms, independent minimization of system is used when every its function is minimized separately. After this a possibility to form the more the better common system conjuncterms is searched. In this sense the splitting procedure is quite convenient as the matrices of splitting can be covered for identical masks of literals.

The cost of the compatibly minimized system realization $F(X)$, similarly as in the case of minimization of one function, will be determined on the basis of the interrelation $k_\theta^* / k_l^* / k_{in}^*$, where numeric parameters k_θ, k_l and k_{in} have something to do with different system conjuncterms of the minimal system of the PSTF $\{Y_I^\oplus\}$.

The described algorithm of the system minimization of the complete functions with the help of splitting method in the polynomial set-theoretical format will be considered for the system $F(X)$, $f_i(x_1, x_2, x_3)$, $i = 1,2$, of the perfect STF $\begin{cases} Y_1^1 = \{(001),(010),(101),(111)\}^1 \\ Y_2^1 = \{(010),(111)\}^1 \end{cases}$, on the example of which the authors [20, example10, p. 391] illustrate efficiency of their own *exorlink method*.

First of all, having changed the upper indices 1 on ⊕ in the given system of the perfect STF $\{Y_{1,2}^1\}$ we get the system of the perfect PSTF $\{Y_{1,2}^\oplus\}$. The set of the given system minterms $Y_{1,2}^\oplus = \{(001)_1,(010)_{1,2},(101)_1,(111)_{1,2}\}^\oplus$ will be formed from the minterms of the last one and the splitting procedure with the help of the matrix M_3^1 will be done:

$$Y_{1,2}^\oplus = \{(001)_1,(010)_{1,2},(101)_1,(111)_{1,2}\}^\oplus \overset{S}{\Rightarrow} \begin{bmatrix} l-- \\ -l- \\ --l \end{bmatrix} = \begin{bmatrix} \overset{001_1}{0--} & \overset{010_{1,2}}{0--} & \overset{101_1}{1--} & \overset{111_{1,2}}{1--} \\ -0- & \underline{-\mathbf{1}-}_{1,2} & -0- & \underline{-\mathbf{1}-}_{1,2} \\ \underline{--\mathbf{1}}_1 & --0 & \underline{--\mathbf{1}}_1 & --1_{1,2} \end{bmatrix}.$$

This procedure corresponds to the first step on the first stage of the algorithm (see p. 2.1). In the second step we do covering of the matrix by singled out identical conjuncterms-copies M_3^1:

$$\overset{C}{\Rightarrow}\{((--1)_1,(011)_1,(111)_1),((-1-)_{1,2},(011)_{1,2},(110)_{1,2})\}^\oplus.$$

Having distributed system conjuncterms in the function, we get the system of the PSTF $\{Y_{1,2}^\oplus\}$, with the underlined elements of which we will do for every system function. The procedures of simplification according to the rules (2), (3) and (6) of the respective theorems (p. 2.1):

$$\begin{cases} Y_1^\oplus = \{(--1),(\underline{111}),(-1-),(\underline{110})\}^\oplus \Rightarrow \{(--1),(\underline{-1-}),(\underline{11-})\}^\oplus \Rightarrow \{(--1),(01-)\}^\oplus \\ Y_2^\oplus = \{(-1-),(\underline{011}),(\underline{110})\}^\oplus \Rightarrow \{(\underline{-1-}),(-11),(\underline{11-})\}^\oplus \Rightarrow \{(-11),(01-)\}^\oplus \end{cases}.$$

So, the minimal system of the PSTF $\begin{cases} Y_1^\oplus = \{(--1),(01-)\}^\oplus \\ Y_2^\oplus = \{(-11),(01-)\}^\oplus \end{cases}$, cost of realization, which is reflected by the interrelation $k_\theta^* / k_l^* / k_{in}^* = 3/5/1$, and corresponds to [20].

Example 13. To minimize the system $F(X)$ of complete functions $f_i(x_1, x_2, x_3)$, $i = 1,2,3$, given by the perfect STF $\begin{cases} Y_1^1 = \{0,2,5,6\}^1 \\ Y_2^1 = \{1,3,5\}^1 \\ Y_3^1 = \{0,1,2,3\}^1 \end{cases}$, in the polynomial set-theoretical format with the help of splitting method [21, p. 35] on the example of this system the author illustrates efficiency of *xlinking method.*

Solution. Having transformed the given system of the perfect STF $\{Y_{1,2,3}^1\}$ in to the system of the perfect PSTF $\{Y_{1,2,3}^\oplus\}$ and having formed from it a set of system minterms we will do the splitting procedure with the help of the matrix and the coveting procedure of the last one M_3^1:

$$Y_{1,2,3}^\oplus = \{(000)_{1,3},(001)_{2,3},(010)_{1,3},(011)_{2,3},(101)_{1,2},(110)_1\}^\oplus \overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow}\begin{bmatrix} l-- \\ -l- \\ --l \end{bmatrix} = \begin{bmatrix} \underline{0--}_{1,3} & \underline{0--}_{2,3} & \underline{0--}_{1,3} & \underline{0--}_{2,3} & 1--_{1,2} & 1--_{1} \\ -0- & -0- & -1- & -1- & -0- & -1- \\ --0 & --1 & --0 & --1 & --1 & --0 \end{bmatrix} \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow}\{((0--)_{1,3},(001)_{1,3},(011)_{1,3}),((0--)_{2,3},(000)_{2,3},(010)_{2,3}),(101)_{1,2},(110)_{1}\}^{\oplus}.$$

We will do the splitting procedure with system minterms of the formed set with the help of the matrix M_3^2 and the covering procedure of the last one:

$$(001)_{1,3},(011)_{1,3},(000)_{2,3},(010)_{2,3},(101)_{1,2},(110)_{1}\overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow}\begin{bmatrix} ll- \\ l-l \\ -ll \end{bmatrix} = \begin{bmatrix} 00- & 01- & 00- & 01- & 10- & 11- \\ \underline{0-1}_{1,3} & \underline{0-1}_{1,3} & \underline{0-0}_{2,3} & \underline{0-0}_{2,3} & 1-1 & 1-0 \\ -01 & -11 & -00 & -10 & -01 & -10 \end{bmatrix} \overset{C}{\Rightarrow}\{(0-1)_{1,3},(0-0)_{2,3},(101)_{1,2},(110)_{1}\}^{\oplus}.$$

Having distributed system conjuncterms in the functions we get the system of the PSTF $\{Y_{1,2,3}^{\oplus}\}$, with the underlined elements, which we transformation according to the rules (2), (3) and (7) of the corresponding theorems:

$$\begin{cases} Y_1^{\oplus} = \{(\underline{0--}),(\underline{0-1}),(\underline{\underline{101}}),(\underline{\underline{110}})\}^{\oplus} \Rightarrow \{(\underline{0-0}),(\underline{1-1}),(11-)\}^{\oplus} \Rightarrow \{(0--),(--1),(11-)\}^{\oplus} \\ Y_2^{\oplus} = \{(\underline{0--}),(\underline{0-0}),(101))\}^{\oplus} \Rightarrow \{(\underline{0-1}),(\underline{101})\}^{\oplus} \Rightarrow \{(--1),(111)\}^{\oplus} \\ Y_3^{\oplus} = \{(\underline{0-1}),(\underline{0-0})\}^{\oplus} \Rightarrow \{(0--)\}^{\oplus} \end{cases}.$$

Answer. Minimal system of the PSTF $\{Y_{1,2,3}^{\oplus}\}$ looks like:

$$\begin{cases} Y_1^{\oplus} = \{(0--),(--1),(11-)\}^{\oplus} \equiv \{(0,1,2,3),(1,3,5,7),(6,7)\}^{\oplus} = \{0,2,5,6\}^{\oplus} \\ Y_2^{\oplus} = \{(--1),(111)\}^{\oplus} \equiv \{(1,3,5,7),(7)\}^{\oplus} = \{1,3,5\}^{\oplus} \\ Y_3^{\oplus} = \{(0--)\}^{\oplus} \equiv \{0,1,2,3\}^{\oplus} \end{cases},$$

A cost of realization reflects the interrelation $k_{\theta}^{*}/k_{l}^{*}/k_{in}^{*} =$ 4/7/1. If compared with [21] it is a better result, where this system is minimized in a compatible way by *xlinking method* with the interrelation $k_{\theta}^{*}/k_{l}^{*}/k_{in}^{*} =$ 4/9/4, namely:

$$\begin{cases} Y_1^{\oplus} = \{(0--),(0-1),(110),(101)\}^{\oplus} \equiv \{(0,1,2,3),(1,3),(6),(5)\}^{\oplus} = \{0,2,5,6\}^{\oplus} \\ Y_2^{\oplus} = \{(0-1),(101)\}^{\oplus} \equiv \{(1,3),(5)\}^{\oplus} = \{1,3,5\}^{\oplus} \\ Y_3^{\oplus} = \{(0--)\}^{\oplus} \equiv \{0,1,2,3\}^{\oplus} \end{cases}.$$

Example 14. To minimize the system $F(X)$ of complete functions $f_i(x_1,x_2,x_3,x_4)$, $i=1,2$, given by the perfect PSTF $\begin{cases} Y_1^{\oplus} = \{3,5,6,8,9,12,15\}^{\oplus} \\ Y_2^{\oplus} = \{1,2,4,7,10,11,12,13\}^{\oplus} \end{cases}$, in the polynomial set-theoretical format with the help of splitting method , the author [33, p. 223] illustrates a *conventional K-maps method*.

Solution. We apply independent minimization to the given system as its the perfect PSTF $\{Y_{1,2}^{\oplus}\}$ contains only one common minterm (1100). So, for minimization of the function f_1 we do the splitting procedure of the minterms of the perfect PSTF $\{Y_1^{\oplus}\}$ with the help of the matrix M_4^2 and covering of the last one which is highlighted in bold font:

$$Y_1^{\oplus} = \{(0011),(0101),(0110),(1000),(1001),(1100),(1111)\}^{\oplus}\overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow}\begin{bmatrix} ll-- \\ l-l- \\ l--l \\ -ll- \\ -l-l \\ --ll \end{bmatrix} = \begin{bmatrix} 00-- & \underline{01--} & \underline{01--} & \underline{10--} & \underline{10--} & \underline{11--} & \underline{11--} \\ \underline{\mathbf{0-1-}} & 0-0- & \underline{\mathbf{0-1-}} & \underline{\mathbf{1-0-}} & \underline{\mathbf{1-0-}} & \underline{\mathbf{1-0-}} & 1-1- \\ \underline{0--1} & \underline{0--1} & 0--0 & \underline{1--0} & \underline{1--1} & \underline{1--0} & \underline{1--1} \\ -01- & \underline{-10-} & \underline{-11-} & \underline{-00-} & \underline{-00-} & \underline{-10-} & \underline{-11-} \\ \underline{-0-1} & \underline{\mathbf{-1-1}} & \underline{-1-0} & -0-0 & \underline{-0-1} & \underline{-1-0} & \underline{\mathbf{-1-1}} \\ \underline{--11} & \underline{--01} & --10 & \underline{--00} & \underline{--01} & \underline{--00} & \underline{--11} \end{bmatrix} \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow}\{((0-1-),(0010),(0111)),((1-0-),(1101)),((-1-1),(0111),(1101))\}^{\oplus} \Rightarrow$$
$$\Rightarrow \{(\underline{0-1-}),(\underline{1-0-}),(-1-1),(0010)\}^{\oplus}.$$

Having applied to the underlined elements the rule (3) of theorem 1, we get the PSTF $Y_1^{\oplus} = \{(1---),(--1-),(-1-1),(0010)\}^{\oplus}$.

We apply the analogical transformation procedures to minimization of the function f_2:

$$Y_2^{\oplus} = \{(0001),(0010),(0100),(0111),(1010),(1011),(1100),(1101)\}^{\oplus} \overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow}\begin{bmatrix} ll-- \\ l-l- \\ l--l \\ -ll- \\ -l-l \\ --ll \end{bmatrix} = \begin{bmatrix} \underline{00--} & \underline{00--} & \underline{01--} & \underline{01--} & \underline{10--} & \underline{10--} & \underline{11--} & \underline{11--} \\ \underline{0-0-} & \underline{0-1-} & \underline{0-0-} & \underline{0-1-} & \underline{1-1-} & \underline{1-1-} & \underline{1-0-} & \underline{1-0-} \\ \underline{\mathbf{0--1}} & \underline{0--0} & \underline{0--0} & \underline{\mathbf{0--1}} & \underline{1--0} & \underline{1--1} & \underline{1--0} & \underline{1--1} \\ -00- & \underline{\mathbf{-01-}} & \underline{\mathbf{-10-}} & -11- & \underline{\mathbf{-01-}} & \underline{\mathbf{-01-}} & \underline{\mathbf{-10-}} & \underline{\mathbf{-10-}} \\ \underline{-0-1} & \underline{-0-0} & \underline{-1-0} & \underline{-1-1} & \underline{-0-0} & \underline{-0-1} & \underline{-1-0} & \underline{-1-1} \\ \underline{--01} & \underline{--10} & \underline{--00} & \underline{--11} & \underline{--10} & \underline{--11} & \underline{--00} & \underline{--01} \end{bmatrix} \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow}\{((0--1),(0011),(0101)),((-01-),(0011)),((-10-),(0101))\}^{\oplus} \Rightarrow \{(0--1),(\underline{-01-}),(\underline{-10-})\}^{\oplus}.$$

So, the minimal PSTF $Y_2^{\oplus} = \{(0--1),(--1-),(-1--)\}^{\oplus}$.

Answer. Minimal system PSTF $\{Y_{1,2}^{\oplus}\}$ looks like:

$$\begin{cases} Y_1^{\oplus} = \{(1---),(--1-),(-1-1),(0010)\}^{\oplus} \equiv \{(8,9,10,11,12,13,14,15), \\ \qquad (2,3,6,7,10,11,14,15),(5,7,13,15),(2)\}^{\oplus} = \{3,5,6,8,9,12,15\}^{\oplus} \\ Y_2^{\oplus} = \{(0--1),(--1-),(-1--)\}^{\oplus} \equiv \{(1,3,5,7),(2,3,6,7,10,11,14,15), \\ \qquad (4,5,6,7,12,13,14,15)\}^{\oplus} = \{1,2,4,7,10,11,12,13\}^{\oplus} \end{cases}.$$

The cost of realization of the minimized system is equal to $k_\theta^* / k_l^* / k_{in}^*$ =5/11/1, which corresponds to [33].

3.2. Algorithm of minimization of incomplete functions system. The examples of minimization

In case of the system of incomplete (inpredeterminated and weakly determinated) functions (27), given by the system (28) of the perfect PSTF $\{Y_i^{\oplus}, Y_i^{\oplus *}\}$, the algorithm of minimization by the method of conjuncterms splitting in the polynomial set-theoretical format combines realization of two algorithms: of system minimization of a complete functions (p. 3.1) taking into account the form of the giving system of incomplete functions (28) and minimization of an incomplete function (p. 2.2). Respectively, in this case the set of system minterms will consist of two subsets which will be separated by the symbol $Y_I^{\oplus}$ and reflected as $\{Y_I^{\oplus} \vdots Y_I^{\oplus *}\}$, $I \in \{1,2,...,s\}$, where $Y_I^{\oplus}$ is the set of system minterms of the perfect PSTF $\{Y_I^{\oplus *}\}$, and $Y_I^{\oplus *}$ is the set of system minterms of the perfect PSTF $\{Y_I^{\oplus *}\}$, here the mark * in particular case will be replaced by the symbol ~ or 0 in an analogical way to (27). Further, the minterms system undergo the splitting procedure with the help of the splitting matrix M_n^r and its covering, as a result of which some set of system conjuncterms of different ranks $\{Y_I^{\oplus} \vdots Y_I^{\oplus *}\}$ is formed. We should note that in the course of covering the matrix M_n^r two procedures are realized at a time: making the matrix compatible as its elements are used to maximum extent with higher capacity of the set I, and making it more definite (predeterminate) of the system due to use of the elements of the submatrix $Y_I^{\oplus *}$. After distribution of the last ones in the functions of the system we get PSTF $\{Y_I^{\oplus}, Y_I^{\oplus *}\}$, the elements of which for every function further undergo simplification procedure according to the rules of the respective theorems (p. 1.2), selecting out of possible variants of transformation those which will secure the compatible minimization of the given system in the best way.

Given further examples illustrate the peculiarities of minimization of the system of incomplete functions by the suggested method.

Example 15 [39, p. 228, example 5.1]. To minimize the system $F(X)$ of incomplete functions $f_1(a,b,c)$ and $f_2(a,b,c)$, given by the truth table with the help of splitting method in the set-theoretical format.

I	a	b	c	f_1	f_2
0	0	0	0	1	0
1	0	0	1	~	~
2	0	1	0	0	1
3	0	1	1	1	~
4	1	0	0	~	~
5	1	0	1	~	0
6	1	1	0	1	0
7	1	1	1	0	1

Solution. The given system $F(X)$ has the perfect PSTF $\begin{cases} Y_1^{\oplus} = \{0,3,6\}^{\oplus};\ Y_1^{\tilde{\oplus}} = \{1,4,5\}^{\tilde{\oplus}} \\ Y_2^{\oplus} = \{2,7\}^{\oplus};\ Y_2^{\tilde{\oplus}} = \{1,3,4\}^{\tilde{\oplus}} \end{cases}$. We will form a set of system minterms of its minterms doing the splitting procedure with the help of the matrix $Y_{1,2}^{\oplus} \vdots Y_{1,2}^{\tilde{\oplus}}$, and the procedure of its covering the matrix M_3^1, for example, for the mask $\{-l-\}$:

$$Y_{1,2}^{\oplus} \vdots Y_{1,2}^{\tilde{\oplus}} = \{(000)_1,(010)_2,(011)_1,(110)_1,(111)_2 \vdots (001)_{1,2},(011)_2,(100)_{1,2},(101)_1\}^{\oplus} \overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow} \begin{bmatrix} l-- \\ -l- \\ --l \end{bmatrix} = \left[\begin{array}{ccccc:cccc} 0-- & 0-- & 0-- & 1-- & 1-- & 0-- & 0-- & 1-- & 1-- \\ -0-_1 & \underline{\mathbf{-1-}}_2 & \underline{\mathbf{-1-}}_1 & \underline{\mathbf{-1-}}_1 & \underline{\mathbf{-1-}}_2 & -0-_{1,2} & \underline{\mathbf{-1-}}_2 & -0-_{1,2} & -0-_1 \\ --0 & --0 & --1 & --0 & --1 & --1 & --1 & --0 & --1 \end{array}\right] \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow} \{-l-\} = \big\{(000)_1, \big((-1-)_2,(011)_2,(110)_2\big), \big((-1-)_1,(010)_1,(111)_1\big) \vdots (001)_{1,2},(011)_2,(100)_{1,2},(101)_1\big\}^{\oplus}.$$

After removal of the system minterm $(011)_2$ the set of covering will look like

$$Y_{1,2}^{\oplus} \vdots Y_{1,2}^{\tilde{\oplus}} = \big\{(000)_1, \big((-1-)_2,(110)_2\big), \big((-1-)_1,(010)_1,(111)_1\big) \vdots (001)_{1,2},(011)_2,(100)_{1,2},(101)_1\big\}^{\oplus}.$$

Having distributed the system conjuncterms in the functions, we get the system PSTF $\{Y_{1,2}^{\oplus} \vdots Y_{1,2}^{\tilde{\oplus}}\}$

$$\begin{cases} Y_1^{\oplus} \vdots Y_1^{\tilde{\oplus}} = \{(-1-),(000),(010),(111) \vdots (001),(100),(101)\}^{\oplus} \\ Y_2^{\oplus} \vdots Y_2^{\tilde{\oplus}} = \{(-1-),(110) \vdots (001),(011),(100)\}^{\oplus} \end{cases}.$$

We will do the splitting procedure with the minterms of the PSTF $\{Y_1^{\oplus} \vdots Y_1^{\tilde{\oplus}}\}$ of the function f_1 with the help of the matrix M_3^1 and the procedure of its covering:

$$\{(000),(010),(111) \vdots (001),(100),(101)\}^{\oplus} \overset{S}{\Rightarrow} \begin{bmatrix} l-- \\ -l- \\ --l \end{bmatrix} = \left[\begin{array}{ccc:ccc} \underline{0--} & \underline{0--} & 1-- & \underline{0--} & 1-- & 1-- \\ -0- & -1- & -1- & -0- & -0- & -0- \\ --0 & --0 & \underline{--1} & \underline{--1} & --0 & \underline{--1} \end{array}\right] \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow} \big\{\big((0--),(011)\big), \big((--1),(011)\big)\big\}^{\oplus} \Rightarrow \{(0--),(--1)\}^{\oplus}.$$

Having taken in to account (3) $\left\{\begin{pmatrix}0--\\--1\end{pmatrix}\right\}^{\oplus} \equiv \left\{\begin{pmatrix}1--\\--0\end{pmatrix}\right\}^{\oplus}$ we will get two solutions of the minimal PSTF $Y_1^{\oplus} = \left\{(-1-), \begin{Bmatrix}1.\,(0--),(--1)\\2.\,(1--),(--0)\end{Bmatrix}\right\}^{\oplus}$.

We will do the analogical procedures for the minterms of the PSTF $\{Y_2^{\oplus} \vdots Y_2^{\tilde{\oplus}}\}$ of the function f_2, having applied the matrix M_3^2 for their splitting:

$$\{(110)\vdots(001),(011),(100)\}^{\oplus} \overset{S}{\Rightarrow} \begin{bmatrix} ll- \\ l-l \\ -ll \end{bmatrix} = \left[\begin{array}{c|ccc} 11- & 00- & 01- & 10- \\ \underline{1-0} & \underline{\underline{0-1}} & \underline{\underline{0-1}} & \underline{1-0} \\ -10 & -01 & -11 & -00 \end{array}\right] \overset{C}{\Rightarrow} \{(1-0),(0-1)\}^{\oplus}.$$

After the transformation according to the rule (3) $\begin{pmatrix}1-0\\0-1\end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{\begin{pmatrix}1--\\--1\end{pmatrix},\begin{pmatrix}--0\\0--\end{pmatrix}\right\}$, we get the minimal PSTF $Y_2^{\oplus} = \left\{(-1-), \begin{Bmatrix}1.\,(1--),(--1)\\2.\,(--0),(0--)\end{Bmatrix}\right\}^{\oplus}$.

Answer. The given system of functions has two solutions of minimization which reflect the PSTF

$$1.\ \begin{cases} Y_1^{\oplus} = \{(-1-),(--1),(0--)\}^{\oplus} \equiv \{(2,3,6,7),(1,3,5,7),(0,1,2,3)\}^{\oplus} = \{0,3,\mathbf{5},6\}^{\oplus} \\ Y_2^{\oplus} = \{(-1-),(--1),(1--)\}^{\oplus} \equiv \{(2,3,6,7),(1,3,5,7),(4,5,6,7)\}^{\oplus} = \{\mathbf{1},2,\mathbf{4},7\}^{\oplus} \end{cases}$$

$$2.\ \begin{cases} Y_1^{\oplus} = \{(-1-),(--0),(1--)\}^{\oplus} \equiv \{(2,3,6,7),(0,2,4,6),(4,5,6,7)\}^{\oplus} = \{0,3,\mathbf{5},6\}^{\oplus} \\ Y_2^{\oplus} = \{(-1-),(--0),(0--)\}^{\oplus} \equiv \{(2,3,6,7),(0,2,4,6),(0,1,2,3)\}^{\oplus} = \{\mathbf{1},2,\mathbf{4},7\}^{\oplus} \end{cases},$$

where the minterms of the sets $Y_1^{\tilde{\oplus}}$ and $Y_2^{\tilde{\oplus}}$ introduced as a result of additional predetermination are in bold font. The analytical expressions correspond to these solutions:

$$1.\ \begin{cases} f_1(a,b,c) = b \oplus c \oplus \overline{a} \\ f_2(a,b,c) = b \oplus c \oplus a \end{cases}; \qquad 2.\ \begin{cases} f_1(a,b,c) = b \oplus \overline{c} \oplus a \\ f_2(a,b,c) = b \oplus \overline{c} \oplus \overline{a} \end{cases}.$$

Cost of realization of the system for the solution 1 is equal to $k_\theta^* / k_l^* / k_{in}^* =$ 4/4/1, and for the solution 2 is $k_\theta^* / k_l^* / k_{in}^* =$ 4/4/2.

Both solutions are a better result if compared with [39], where cost of realization of the system is equal to $k_\theta^* / k_l^* / k_{in}^* =$ 4/7/2, namely: $\begin{cases} f_1(a,b,c) = b \oplus \overline{c} \oplus ab \\ f_2(a,b,c) = b \oplus ab\overline{c} \end{cases}$.

Example 16. To minimize the system of incomplete functions given by the perfect STF $\begin{cases} Y_1^1 = \{1,4,6\}^1; Y_1^{\sim} = \{3,5,7\}^{\sim} \\ Y_2^1 = \{0,2,4,7\}^1; Y_2^{\sim} = \{1,6\}^{\sim} \end{cases}$ in the polynomial set-theoretical format with the help of splitting method (*borrowed from* [40, p.4, example 3]).

Solution. We form a set of the minterms system of the perfect PSTF $\begin{cases} Y_1^{\oplus} = \{(001),(100),(110)\}^{\oplus};\ Y_1^{\tilde{\oplus}} = \{(011),(101),(111)\}^{\tilde{\oplus}} \\ Y_2^{\oplus} = \{(000),(010),(100),(111)\}^{\oplus};\ Y_2^{\tilde{\oplus}} = \{(001),(110)\}^{\tilde{\oplus}} \end{cases}$ doing the splitting procedure with the help of the matrix M_3^2 and the procedure of its covering. For example for the mask $\{l-l\}$ we have:

$$Y_{1,2}^{\oplus} \vdots Y_{1,2}^{\tilde{\oplus}} = \{(000)_2,(001)_1,(010)_2,(100)_{1,2},(110)_1,(111)_2 \vdots (001)_2,(011)_1,(101)_1,(110)_2,(111)_1\}^{\oplus} \overset{S}{\Rightarrow}$$

$$\overset{S}{\Rightarrow}\begin{bmatrix} ll- \\ l-l \\ -ll \end{bmatrix} = \left[\begin{array}{cccccc|ccccc} 00-_2 & 00-_1 & 01-_2 & 10-_{1,2} & 11-_1 & 11-_2 & 00-_2 & 01-_1 & 10-_1 & 11-_2 & 11-_1 \\ \underline{\mathbf{0-0}}_2 & \underline{\mathbf{0-1}}_1 & \underline{\mathbf{0-0}}_2 & 1-0_{1,2} & 1-0_1 & 1-1_2 & 0-1_2 & \underline{\mathbf{0-1}}_1 & 1-1_1 & 1-0_2 & 1-1_1 \\ -00_2 & -01_1 & -10_2 & -00_{1,2} & -10_1 & -11_2 & -01_2 & -11_1 & -01_1 & -10_2 & -11_1 \end{array}\right] \overset{C}{\Rightarrow}$$

$$\overset{C}{\Rightarrow}\{l-l\} = \{(0-0)_2,(0-1)_1,(100)_{1,2},(110)_1,(111)_2\}^{\oplus}.$$

Having distributed the system conjuncterms in the functions, we get the system PSTF $Y_{1,2}^{\oplus}$, doing step by step simplification for $Y_1^{\oplus}$ according to the rules (2) and (3), and for $Y_2^{\oplus}$ (7) and (2):

$$\begin{cases} Y_1^{\oplus} = \{(0-1),(\underline{100}),(\underline{110})\}^{\oplus} \Rightarrow \{(0-1),(1-0)\}^{\oplus} \Rightarrow \begin{Bmatrix} 1.\ (0--),(--0) \\ 2.\ (--1),(1--) \end{Bmatrix}^{\oplus} \\ Y_2^{\oplus} = \{(\underline{0-0}),(\underline{100}),(111)\}^{\oplus} \Rightarrow \{(--0),(\underline{110}),(\underline{111})\}^{\oplus} \Rightarrow \{(--0),(11-)\}^{\oplus} \end{cases}.$$

So, having taken into account the realization of the compatible minimization, the given system has the minimal PSTF $\begin{cases} Y_1^{\oplus} = \{(0--),(--0)\}^{\oplus} \equiv \{(0,1,2,3),(0,2,4,6)\}^{\oplus} = \{1,\mathbf{3},4,6\}^{\oplus} \\ Y_2^{\oplus} = \{(--0),(11-)\}^{\oplus} \equiv \{(0,2,4,6),(6,7)\}^{\oplus} = \{0,2,4,7\}^{\oplus} \end{cases}$.

Answer. Cost of realization of the minimized system is equal to $k_{\theta}^{*}/k_{l}^{*}/k_{in}^{*} = 3/4/2$ and is better than all seven results of additional predetermination given in [40, table 4, p. 5].

Conclusion

This article that is a logical continuation of the previous ones (*see* 1. Generalized of Set-Theoretical Simplify Rules of Conjuncterms; 2. Minimization of Complete and Incomplete Functions) describes the algorithm of the system minimization method of the complete and incomplete functions with *n* variables on the basis of conjuncterms splitting in the polynomial set-theoretical format. Efficiency of the suggested method is illustrated on the examples of minimization in the polynomial format of functions system borrowed from the well-known publications, the authors of which demonstrate their own methods. On the basis of the results comparison one can note that the described in the mentioned above papers advantages of the suggested method can be proved also in the case of application to the systems minimization of the functions.

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