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B.YE. RYTSAR, Doctor Eng., Professor,
Institute of Telecommunications, Radioelectronics and Electronic Engineering,
L'viv polytechnic National University, Bandera str., 12, L'viv, 79013, Ukraine,
E-mail: bohdanrytsar@gmail.com

A NEW METHOD FOR SYMMETRY RECOGNITION IN BOOLEAN FUNCTIONS BASED ON THE SET-THEORETICAL LOGIC DIFFERENTIATION. I

The paper presents a new method for the recognition of the different types of total and partial symmetry in boolean functions based on the numeric set-theoretical differentiation. The proposed algorithm is based on the theorem on the recognition of different types of partial symmetry. This algorithm, compared to the known, has a relatively less computational complexity of realization due to a comparatively smaller number of operations and procedures necessary for the accomplishment of the given task. This is proved by the presented examples for the recognition of the proposed method of the different types of symmetry in complete and incomplete of Boolean functions, including given in the SOP format, taken for comparison reasons from publications of the well-known authors.

Keywords: *recognition of total and partial symmetry, Boolean function, numeric set-theoretical differentiation.*

Introduction

It is known [1–3] that an arbitrary boolean function is called symmetric if it does not change its value with any permutations of variables. Such property of symmetric functions, in comparison to others, gives a logical synthesis of digital functions that reveal various optimization problems in the process of designing digital devices. On the basis of symmetric functions it is much easier and cheaper (with a lower total number of components) to implement such devices as adders, code-converters, comparators, error-detecting devices, noise-resistant decoders etc. [3,4]. Symmetric functions are also effective in cryptography due to relatively less demanding storage requirements for large data sets [5–7]. Therefore, the searching for simple relatively realistic methods for recognizing of the different types of symmetry, as well as of functions of a partial symmertry, still remains actual.

The common feature of known methods and algorithms for identifying of the types of symmetric functions in boolean functions is the complexity of implementation. It is due to the fact that the basis of the known methods for recognizing of the types of symmetry is the analytical approach based on the Shannon expansion theorem [2,3,8] or a visual method based on the *K*-map [9,10]. Other known methods, such as the method of decomposition cloning on the basis of the ***q***-partition conjuncterms procedure, in particular the BRASh algorithm [11], or a method based on the analytical calculation of logical derivatives, in particular the BOOLE algorithm (available over the Internet) [12,13]. These methods are somewhat simpler than mentioned, but require a preconversion of a given function. However, this problem is much more complicated in the case of large-size functions and, especially, when such functions are not completely determined, but only partially [9,14].

In this paper we consider a new method for the recognition of symmetry in complete and incomplete functions on the basis of the author's proposed numerical set-theoretical definition of logical derivatives [15]. Compared with known methods, the proposed method algorithm differs by simpler practical implementation of the number of steps and the speed of detection and recognition of types of total and partial symmetries in complete and incomplete boolean functions and, as well as in the functions specified in SOP format, which is confirmed by numerous examples, borrowed from the publications of well-known authors.

The basic theoretical part

Types of symmetries in boolean functions are divided into two groups [1,2]: totally symmetric functions, if their values do not depend on any permutations of all n variables, and partially symmetric functions, if their values do not change due to permutations of some (but not all) variables. For example, $f(x_1,x_2,x_3)=\overline{x}_1\overline{x}_2\overline{x}_3 \vee x_1x_2x_3$ is a totally symmetric function, since any permutations (or re-indexing) of all its variables do not change its values. Instead, $f(x_1,x_2,x_3)=x_1x_2 \vee x_3$ it is a partially symmetric function relative to the variables x_1 and x_2 because the reindexing of these variables does not change the value of this function.

Among the totally symmetric functions distinguish the functions with a simple symmetry of n variables that are conventionally denoted as $x_1 \sim x_2 \sim \cdots \sim x_n$ and $\overline{x}_1 \sim \overline{x}_2 \sim \cdots \sim \overline{x}_n$ (where the tilde sign ~ is a symbol of symmetry), and functions with polysymmetry of n variables that are conventionally designated as $\overline{x}_1 \sim \overline{x}_2 \sim \cdots \sim \overline{x}_n, \tilde{x} \in \{x, \overline{x}\}$. For example, the above total symmetric function has a simple symmetry $x_1 \sim x_2 \sim x_3 / \overline{x}_1 \sim \overline{x}_2 \sim \overline{x}_3$, since inverting its variables does not affect the value of the function. In the case of polysymmetric functions, symmetry may be between inverse and non-inverse values of variables. These include linear functions: EXCLUSIVE OR $x_1 \oplus x_2 \oplus ... \oplus x_n$, where $\oplus$ is the symbol of the module 2 sum operation, and also EXCLUSIVE NOR $x_1 \approx x_2 \approx \cdots \approx x_n$, where is the symbol of the inversion module 2 sum operation. In particular, $x_1 \oplus x_2 \oplus x_3 = \overline{x}_1\overline{x}_2x_3 \vee \overline{x}_1x_2\overline{x}_3 \vee x_1\overline{x}_2\overline{x}_3 \vee x_1x_2x_3$ and $x_1 \approx x_2 \approx x_3 = \overline{x}_1\overline{x}_2\overline{x}_3 \vee \overline{x}_1x_2x_3 \vee x_1\overline{x}_2x_3 \vee x_1x_2\overline{x}_3$ are polysymmetric functions $\tilde{x}_1 \sim \tilde{x}_2 \sim \tilde{x}_3$.

The symmetric functions of n variables with a simple symmetry denote as $S_{k_1,k_2,...,k_p}(x_1,x_2,...,x_n)$ or $S_{k_1,k_2,...,k_p}(X)$ either S_K^n, where $K=\{k_1,k_2,...,k_p\}$, $k_i \in \{0,1,...,n\}$, $1 \le p \le n+1$ is the set of so-called $\boldsymbol{k}$-numbers, which are equal to the weights (by the number 1) of sets of variables, on which $S_K^n = 1$. In the set-theoretical form (STF) [16,17] the symmetric function S_K^n is sufficient to represent by the set of numerical (binary or decimal) conjuncterms. In the perfect STF Y^1 it is a set of minterms, on which $S_K^n = 1$. For example, the perfect SOP of the above symmetric function $S_{0,3}^3$ with simple symmetry $f=\overline{x}_1\overline{x}_2\overline{x}_3 \vee x_1x_2x_3$ has a perfect STF $Y^1=\{(000),(111)\}^1$, and the function given in SOP with partial symmetry has STF $Y^1=\{(11-),(--1)\}^1$. The polysymmetric function $S_{1,3}^3 = x_1 \oplus x_2 \oplus x_3$ has a perfect STF $Y^1=\{(001),(010),(100),(111)\}^1$, and a polysymmetric function $S_{0,2}^3=\overline{x}_1 \oplus x_2 \oplus x_3$ (here the inverse sign can be over any variable) has a perfect STF $Y^1=\{(000),(011),(101),(110)\}^1$. Among all 2^{n+1} symmetric functions $S_{k_1,k_2,...,k_p}^n$ elementary symmetric functions are simplest, since they have only one k-number and denote them as S_k^n, $0 \le k \le n$.

Partially symmetric functions $f=(x_1,...,x_i,...$ $...,x_j,...,x_n)$ relative to two variables (x_i,x_j) retain their values after permutation x_i and x_j, i.e.$(x_1,...$ $...,x_i,...,x_j,...,x_n)=(x_1,...,x_j,...,x_i,...,x_n)$.

Among functions with partial symmetry – simple $x_i \sim x_j / \overline{x}_i \sim \overline{x}_j$ or polysymmetry $\tilde{x}_i \sim \tilde{x}_j$, there may be other functions with antisymmetry $x_i \sim \overline{x}_j / \overline{x}_i \sim x_j$ [17]. For example, the function $f=x_1x_2\overline{x}_3 \vee \overline{x}_1x_2x_3$ has a partial polysymmetry $\tilde{x}_1 \sim \tilde{x}_3$, and the function $f=x_1\overline{x}_2\overline{x}_3 \vee \overline{x}_1x_2x_3$, for example, in addition to partial simple symmetry $x_2 \sim x_3 / \overline{x}_2 \sim \overline{x}_3$, it has also antisymmetry $x_1 \sim \overline{x}_2 / \overline{x}_1 \sim x_2$ and $x_1 \sim \overline{x}_3 / \overline{x}_1 \sim x_3$, that it has a mixed symmetry $x_1 \sim \overline{x}_2 \sim \overline{x}_3 / \overline{x}_1 \sim x_2 \sim x_3$.

An example of partial mixed symmetry may be the function $f=\overline{x}_3x_5 \vee (x_1 \oplus x_2 \oplus x_4)$ that has a partial polysymmetry $\tilde{x}_1 \sim \tilde{x}_2 \sim \tilde{x}_4$ and partial antisymmetry $x_3 \sim \overline{x}_5 / \overline{x}_3 \sim x_5$. Identification and recognition of mixed partial symmetries is quite difficult, if to use any analytical methods.

The described above types of symmetries are classified by cloning method based on the q-partition procedure of the binary minterms of function f using the so-called maximal clones $(n-2)$ – or 2-class [16,17]. It is shown that the type of symmetry in an arbitrary function f can be determined by using only numeric (binary or decimal) values of fixed subminterms of maximum clones, which are indicators of one or another type of symmetry in relation to the variables (x_i, x_j) of a given function. In this paper we compared the calculation results of the proposed method with the decomposition cloning method and a *BRASh* program [16]. Here we demonstrate that the above-mentioned types of symmetries in boolean functions are easier to detect with the aid of a vectorial ST derivative of the 2nd order [15].

The main part

It is known [15] that the vectorial ST-derivative of the 2-nd order with respect to arbitrary variables x_i and x_j of a function $f(x_1, x_2, ..., x_n)$, that is $\partial^2 Y^{\oplus} / \partial(x_i, x_j)$, is determined by overlaying on the given binary minterms $m_1, m_2, ..., m_z$ of the perfect STF Y^1 the function ***f*** of two masks of literals $\begin{Bmatrix} l_1 \cdots l_i \cdots l_j \cdots l_n \\ l_1 \cdots \overline{l}_i \cdots \overline{l}_j \cdots l_n \end{Bmatrix}, l \in \{0,1\}$:

$$Y^1 = \{m_1, m_2, ..., m_z\}^1 \overset{\partial^2/\partial(x_i,x_j)}{\Rightarrow} \begin{Bmatrix} l_1 \cdots l_i \cdots l_j \cdots l_n \\ l_1 \cdots \overline{l}_i \cdots \overline{l}_j \cdots l_n \end{Bmatrix} =$$

$$= \left\{ \begin{pmatrix} m_1 \\ m_1^* \end{pmatrix}, \begin{pmatrix} m_2 \\ m_2^* \end{pmatrix}, ..., \begin{pmatrix} m_z \\ m_z^* \end{pmatrix} \right\}^{\oplus} \Rightarrow \{\theta_1^{r_1}, \theta_2^{r_2}, ..., \theta_p^{r_s}\}, \quad (1)$$

where $\overset{\partial^2/\partial(x_i,x_j)}{\Rightarrow}$ is the operator of the vectorial ST-differentiation with respect to the variables x_i and x_j, as a result for each q-th (given) minterm $m_q = (\sigma_1 \cdots \sigma_i \cdots \sigma_j \cdots \sigma_n)$ formed by a minterm $m_q^* = (\sigma_1 \cdots \overline{\sigma}_i \cdots \overline{\sigma}_j \cdots \sigma_n)$, $\sigma \in \{0,1\}$. Their pairs form in the polynomial STF (PSTF $Y^{\oplus}$) a set $\left\{ \begin{pmatrix} m_1 \\ m_1^* \end{pmatrix}, \begin{pmatrix} m_2 \\ m_2^* \end{pmatrix}, ..., \begin{pmatrix} m_z \\ m_z^* \end{pmatrix} \right\}^{\oplus}$ that can be simplified by removing the same pairs of elements from it, and then, can be minimized in the polynomial format by the proposed method [18]. As a result, a set of conjuncterms of ranks $r_1, r_2, ..., r_s \in \{1, 2, ..., n\}$ is obtained, that is $\{\theta_1^{r_1}, \theta_2^{r_2}, ..., \theta_p^{r_s}\}$, $p \le 2z$, which represents the desired vectorial ST-derivative with respect to the variables (x_i, x_j). For example, the vectorial ST-derivative $\partial^2 Y^{\oplus} / \partial(x_1, x_2)$ of the 2nd order of the function $f = x_1\overline{x}_2\overline{x}_3 \vee \overline{x}_1 x_2 x_3$, having a perfect STF $Y^1 = \{(100), (011)\}^1$, is obtained as follows:

$$Y^1 = \{(100), (011)\}^1 \overset{\partial^2/\partial(x_1,x_2)}{\Rightarrow} \begin{Bmatrix} l_1 l_2 l_3 \\ \overline{l_1 l_2} l_3 \end{Bmatrix} =$$

$$= \left\{ \begin{pmatrix} 100 \\ 010 \end{pmatrix}, \begin{pmatrix} 011 \\ 101 \end{pmatrix} \right\}^{\oplus} = \{(100), (010), (011), (101)\}^1 =$$

$$= \{(10-), (01-)\}^1,$$

which corresponds to the analytical expression $\partial^2 f / \partial(x_1, x_2) = x_1\overline{x}_2 \vee \overline{x}_1 x_2$.

In the analytical version the vectorial derivative of the 2-nd order $\frac{\partial^2 f}{\partial(x_i, x_j)} = \frac{\partial f}{\partial x_i} \oplus \frac{\partial f}{\partial x_j} \oplus \frac{\partial^2 f}{\partial x_i \partial x_j}$, where

$$\frac{\partial f}{\partial x_i} = f(x_1, ..., x_{i-1}, 1_i, x_{i+1}, ..., x_n) \oplus$$

$$\oplus f(x_1, ..., x_{i-1}, 0_i, x_{i+1}, ..., x_n)$$

and

$$\frac{\partial f}{\partial x_j} = f(x_1, ..., x_{j-1}, 1_j, x_{j+1}, ..., x_n) \oplus$$

$$\oplus f(x_1, ..., x_{j-1}, 0_j, x_{j+1}, ..., x_n)$$

there are simple derivatives of the 1-nd order and $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$ it is the multiple derivative of the 2-nd order. For our function we have:

$$\frac{\partial^2}{\partial(x_1, x_2)}\left(x_1\overline{x}_2\overline{x}_3 \vee \overline{x}_1 x_2 x_3\right) = \overline{x}_2\overline{x}_3 \oplus x_2 x_3 \oplus \overline{x}_1 x_3 \oplus$$

$$\oplus x_1\overline{x}_3 \oplus 1 = x_1\overline{x}_2 \vee \overline{x}_1 x_2,$$

that corresponds to the previously obtained vectorial ST-derivarive of the 2-nd order $Y^1 = \{(10-), (01-)\}^1$.

Recognition of partial symmetries types in complete functions

Based on the above, we formulate a theorem on the recognition of different types of partial symmetries in complete boolean functions.

Theorem. The boolean function $f = (x_1, ..., x_i, ...$ $..., x_j, ..., x_n)$ given in the perfect STF $Y^1 = \{m_1, m_2, ...$ $..., m_z\}^1$, $2 < z < 2^n$, where the q-th binary minterm is $m_q = (\sigma_1 \sigma_2 \cdots \sigma_n)$, $\sigma \in \{0,1\}$, has a partial symmetry with respect to any pair of variables (x_i, x_j) of type

- polysymmetry, that is $\tilde{x}_i \sim \tilde{x}_j$, if it satisfies the condition

$$\frac{\partial^2 Y^{\oplus}}{\partial(x_i, x_j)} = \varnothing, \qquad (2)$$

where $\partial^2 Y^{\oplus} / \partial(x_i, x_j)$ is the vectorial ST-derivative of the 2-nd order with respect to (x_i, x_j) of the function *f*; moreover the condition (2) is fulfilled only for a pair of minterms of the complete function *f*;

- simple symmetry, that is $x_i \sim x_j$ or $\overline{x}_i \sim \overline{x}_j$, if it satisfies the condition

$$\frac{\partial^2 Y^{\oplus}}{\partial(x_i, x_j)} \cap \{(-_1 \cdots -_{i-1} 0_i -_{i+1} \cdots -_{j-1} 1_j -_{j+1} \cdots -_n), (-_1 \cdots -_{i-1} 1_i -_{i+1} \cdots -_{j-1} 0_j -_{j+1} \cdots -_n)\} = \varnothing, \qquad (3)$$

where $(-_1 \cdots -_{i-1} 0_i -_{i+1} \cdots -_{j-1} 1_j -_{j+1} \cdots -_n)$ and $(-_1 \cdots -_{i-1} 1_i -_{i+1} \cdots -_{j-1} 0_j -_{j+1} \cdots -_n)$ are ternary conjuncterms of the 2-nd rank;

- antisymmetry, that is $x_i \sim \overline{x}_j$ or $\overline{x}_i \sim x_j$, if it satisfies the condition

$$\frac{\partial^2 Y^{\oplus}}{\partial(x_i, x_j)} \cap \{(-_1 \cdots -_{i-1} 0_i -_{i+1} \cdots -_{j-1} 0_j -_{j+1} \cdots -_n), (-_1 \cdots -_{i-1} 1_i -_{i+1} \cdots -_{j-1} 1_j -_{j+1} \cdots -_n)\} = \varnothing \qquad (4)$$

where $(-_1 \cdots -_{i-1} 0_i -_{i+1} \cdots -_{j-1} 0_j -_{j+1} \cdots -_n)$ and $(-_1 \cdots -_{i-1} 1_i -_{i+1} \cdots -_{j-1} 1_j -_{j+1} \cdots -_n)$ are ternary conjuncterms of the 2-nd rank.

Proof. The theorem is based on the numerical set-theoretical interpretation of the orthogonality condition $Y^{\oplus} \cap Y^{\overline{\oplus}} = \varnothing$, corresponding to the analytic expression $(x_i \oplus x_j) \& \overline{(x_i \oplus x_j)} = 0$, where the logic operation of the conjunction (&) is reflected by the set-theoretical intersection operation ($\cap$), the expression $(x_i \oplus x_j)$ – by PSTF $Y^{\oplus}$, the expression $\overline{(x_i \oplus x_j)}$ – by PSTF $Y^{\overline{\oplus}}$, and zero (0) is empty set ($\varnothing$). In the analytic format for function $f = (x_1, ..., x_i, ..., x_j, ..., x_n)$ the conditions (2), (3), (4) look like this:

$$\frac{\partial^2 f}{\partial(x_i, x_j)} = 0, \qquad (5)$$

$$\frac{\partial^2 f}{\partial(x_i, x_j)} \& \overline{(x_i \oplus x_j)} = 0, \qquad (6)$$

$$\frac{\partial^2 f}{\partial(x_i, x_j)} \& (x_i \oplus x_j) = 0, \qquad (7)$$

where $\frac{\partial^2 f}{\partial(x_i, x_j)}$ is the vectorial derivative of the 2-nd order with respect to variables (x_i, x_j) of the function *f*.

Let it $f = x_i \oplus x_j$. Then, the vectorial derivative of the 2-nd order $\partial^2 f / \partial(x_i, x_j)$ will be defined as follows:

$$\frac{\partial^2 (x_i \oplus x_j)}{\partial(x_i, x_j)} = \frac{\partial(x_i \oplus x_j)}{\partial x_i} \oplus \frac{\partial(x_i \oplus x_j)}{\partial x_j} \oplus \frac{\partial^2 (x_i \oplus x_j)}{\partial x_i \partial x_j} = 1 \oplus 1 \oplus 0 = 0,$$

and, the vectorial ST-derivative $\partial^2 Y^{\oplus} / \partial(x_i, x_j)$ of the PSTF $Y^{\oplus} = \{(-1), (1-)\}^{\oplus} = \{(01), (10)\}^{\oplus}$ will be defined as:

$$Y^1 = \{(01), (10)\}^1 \overset{\partial^2/\partial(x_i, x_j)}{\Rightarrow} \left\{ \begin{matrix} l_i l_j \\ \overline{l_i l_j} \end{matrix} \right\} = \left\{ \begin{pmatrix} 01 \\ 10 \end{pmatrix}, \begin{pmatrix} 10 \\ 01 \end{pmatrix} \right\}^{\oplus} \Rightarrow \varnothing.$$

The same result is obtained for $f = \overline{x_i \oplus x_j}$ having perfect PSTF $Y^{\oplus} = \{(00), (11)\}^{\oplus}$. The partial polysymmetry $\tilde{x}_i \sim \tilde{x}_j$ is illustrated by the example of the function $f(x_1, x_2, x_3) = x_1 \overline{x}_2 \overline{x}_3 \vee \overline{x}_1 \overline{x}_2 x_3$ having perfect STF $Y^1 = \{(100), (001)\}^1$. Let us define vectorial ST-derivatives of the 2-order with respect to (x_1, x_2), (x_1, x_3) and (x_2, x_3) of this function and, then we verify the conditions (2), (3) and (4) of the theorem:

$$Y^1 \overset{\partial^2/\partial(x_1, x_2)}{\Rightarrow} \left\{ \begin{matrix} l_1 l_2 l_3 \\ \overline{l_1 l_2} l_3 \end{matrix} \right\} = \left\{ \begin{pmatrix} 100 \\ 010 \end{pmatrix}, \begin{pmatrix} 001 \\ 111 \end{pmatrix} \right\}^{\oplus} \cap$$

$$\cap \begin{cases} (01-), (10-) = \{(100), (010)\} \neq \varnothing \\ (00-), (11-) = \{(001), (111)\} \neq \varnothing \end{cases},$$

$$Y^1 \overset{\partial^2/\partial(x_1, x_3)}{\Rightarrow} \left\{ \begin{matrix} l_1 l_2 l_3 \\ \overline{l_1} l_2 \overline{l_3} \end{matrix} \right\} = \left\{ \begin{pmatrix} 100 \\ 001 \end{pmatrix}, \begin{pmatrix} 001 \\ 100 \end{pmatrix} \right\}^{\oplus} = \varnothing,$$

$$Y^1 \overset{\partial^2/\partial(x_2, x_3)}{\Rightarrow} \left\{ \begin{matrix} l_1 l_2 l_3 \\ l_1 \overline{l_2 l_3} \end{matrix} \right\} = \left\{ \begin{pmatrix} 100 \\ 111 \end{pmatrix}, \begin{pmatrix} 001 \\ 010 \end{pmatrix} \right\}^{\oplus} \cap$$

$$\cap \begin{cases} (-01), (-10) = \{(001), (010)\} \neq \varnothing \\ (-00), (-11) = \{(100), (111)\} \neq \varnothing \end{cases}.$$

Since only the vectorial ST-derivative $\partial^2 Y^{\oplus} / \partial(x_1, x_3) = \varnothing$, then according to the condition (2), the given function *f* has the partial polysymmetry $\tilde{x}_1 \sim \tilde{x}_3$. This is confirmed by the analytical method:

$$\frac{\partial^2 (x_1 \overline{x}_2 \overline{x}_3 \vee \overline{x}_1 \overline{x}_2 x_3)}{\partial(x_1, x_2)} = (\overline{x}_2 \overline{x}_3 \oplus \overline{x}_2 x_3) \oplus (x_1 \overline{x}_3 \oplus \overline{x}_1 x_3) \oplus \oplus (\overline{x}_3 \oplus x_3) = x_1 \oplus x_2 \oplus x_3,$$

$$\frac{\partial^2(x_1\overline{x}_2\overline{x}_3 \vee \overline{x}_1\overline{x}_2x_3)}{\partial(x_1,x_3)} = (\overline{x}_2\overline{x}_3 \oplus \overline{x}_2x_3) \oplus (\overline{x}_1\overline{x}_2 \oplus x_1\overline{x}_2) \oplus \oplus(\overline{x}_2 \oplus \overline{x}_2) = 0,$$

$$\frac{\partial^2(x_1\overline{x}_2\overline{x}_3 \vee \overline{x}_1\overline{x}_2x_3)}{\partial(x_2,x_3)} = (x_1\overline{x}_3 \oplus \overline{x}_1x_3) \oplus (\overline{x}_1\overline{x}_2 \oplus x_1\overline{x}_2) \oplus \oplus(\overline{x}_1 \oplus x_1) = x_1 \oplus x_2 \oplus x_3.$$

For example, we us verify the condition (6) for the obtained derivative $\partial^2 f / \partial(x_1,x_2) = x_1 \oplus x_2 \oplus x_3$:

$$\frac{\partial^2 f}{\partial(x_1,x_2)} \& (x_1 \oplus x_2) = (\overline{x}_1\overline{x}_2x_3 \vee \overline{x}_1x_2\overline{x}_3 \vee x_1\overline{x}_2\overline{x}_3 \vee \vee x_1x_2x_3)(x_1\overline{x}_2 \vee \overline{x}_1x_2) = x_1\overline{x}_2\overline{x}_3 \vee \overline{x}_1x_2\overline{x}_3 \neq 0.$$

Similarly, the condition (7) will not be satisfied, that indicates the absence of partial symmetry with respect to (x_1,x_2). This function does not have partial symmetry with respect to variables (x_2,x_3) because conditions (6) and (7) in this case are satisfied as well.

Note, that if condition (2) is satisfied with respect to all $C_n^2 = \frac{n!}{(n-2)!2!}$ possible pairs of its variables, then the function *f* is totally polysymmetric, that is $\tilde{x}_1 \sim \tilde{x}_2 \sim \cdots \sim \tilde{x}_n$, then it makes no sense to further consider conditions (4) and (5). Accordingly, a totally symmetric function with simple symmetry, i.e. $x_1 \sim x_2 \sim \cdots \sim x_n$ or $\overline{x}_1 \sim \overline{x}_2 \sim \cdots \sim \overline{x}_n$, either antisymmetry, i.e. $x_1 \sim \overline{x}_2 \sim \cdots \sim \tilde{x}_n$ or $\overline{x}_1 \sim x_2 \sim \cdots \sim \overline{\tilde{x}}_n$, will occur when the condition (3) or (4) holds for all possible pairs C_n^2 of its variables. If none of the conditions of *t*he theorem is satisfied, then the given function *f* is not characterized by mentioned types of symmetries with respect to variables (x_i,x_j).

The validity of the proved theorem is illustrated, by the following examples.

Based on an example of a previously considered function $f = x_1\overline{x}_2\overline{x}_3 \vee \overline{x}_1x_2x_3$ with a perfect STF $Y^1 = \{(100),(011)\}^1$ we consider the fulfillment of the conditions (2), (3) and (4) of the theorem:

$$Y^1 \overset{\partial^2/\partial(x_1,x_2)}{\Longrightarrow} \begin{Bmatrix} l_1l_2l_3 \\ \overline{l_1l_2}l_3 \end{Bmatrix} = \left\{\begin{pmatrix}100\\010\end{pmatrix},\begin{pmatrix}011\\101\end{pmatrix}\right\}^{\oplus} \cap \begin{cases}(01-),(10-) \neq \varnothing \\ (00-),(11-) = \varnothing\end{cases},$$

$$Y^1 \overset{\partial^2/\partial(x_1,x_3)}{\Longrightarrow} \begin{Bmatrix} l_1l_2l_3 \\ \overline{l}_1l_2\overline{l}_3 \end{Bmatrix} = \left\{\begin{pmatrix}100\\001\end{pmatrix},\begin{pmatrix}011\\110\end{pmatrix}\right\}^{\oplus} \cap \begin{cases}(0-1),(1-0) \neq \varnothing \\ (0-0),(1-1) = \varnothing\end{cases},$$

$$Y^1 \overset{\partial^2/\partial(x_2,x_3)}{\Longrightarrow} \begin{Bmatrix} l_1l_2l_3 \\ l_1\overline{l_2l_3} \end{Bmatrix} = \left\{\begin{pmatrix}100\\111\end{pmatrix},\begin{pmatrix}011\\000\end{pmatrix}\right\}^{\oplus} \cap \begin{cases}(-01),(-10) = \varnothing \\ (-00),(-11) \neq \varnothing\end{cases}$$

As one can see, the condition (2) of the theorem is not satisfied here (although we have even number of minterms), which indicates the absence of polysymmetry in this function. However, the conditions (3) and (4) of the theorem are satisfied with respect to certain pairs of variables. Based on that we can argue that the given function f is symmetric $x_1 \sim \overline{x}_2 / \overline{x}_1 \sim x_2$, $x_1 \sim \overline{x}_3 / \overline{x}_1 \sim x_3$ and $x_2 \sim x_3 / \overline{x}_2 \sim \overline{x}_3$ that corresponds to a mixed symmetry of type $x_1 \sim \overline{x}_2 \sim \overline{x}_3 / \overline{x}_1 \sim x_2 \sim x_3$.

An example of a totally polysymmetric function is $f = x_1 \oplus x_2 \oplus x_3$, i.e. $\tilde{x}_1 \sim \tilde{x}_2 \sim \tilde{x}_3$, having perfect STF $Y^1 = \{(001),(010),(100),(111)\}^1$. Thus, for that the condition (2) holds for all three pairs of variables (x_1,x_2), (x_1,x_3) and (x_2,x_3):

$$Y^1 \overset{\partial^2/\partial(x_1,x_2)}{\Longrightarrow} \left\{\cancel{\begin{pmatrix}001\\111\end{pmatrix}},\cancel{\begin{pmatrix}010\\100\end{pmatrix}},\cancel{\begin{pmatrix}100\\010\end{pmatrix}},\cancel{\begin{pmatrix}111\\001\end{pmatrix}}\right\}^{\oplus} = \varnothing,$$

$$Y^1 \overset{\partial^2/\partial(x_1,x_3)}{\Longrightarrow} \left\{\cancel{\begin{pmatrix}001\\100\end{pmatrix}},\cancel{\begin{pmatrix}010\\111\end{pmatrix}},\cancel{\begin{pmatrix}100\\001\end{pmatrix}},\cancel{\begin{pmatrix}111\\010\end{pmatrix}}\right\}^{\oplus} = \varnothing,$$

$$Y^1 \overset{\partial^2/\partial(x_2,x_3)}{\Longrightarrow} \left\{\cancel{\begin{pmatrix}001\\010\end{pmatrix}},\cancel{\begin{pmatrix}010\\001\end{pmatrix}},\cancel{\begin{pmatrix}100\\111\end{pmatrix}},\cancel{\begin{pmatrix}111\\100\end{pmatrix}}\right\}^{\oplus} = \varnothing .$$

Example 1 [12, p. 92, exercise 4.21]. Check whethere there are pairs of variables, for which the function $f(x_1,x_2,x_3,x_4) = (x_1 \oplus x_2 \oplus \overline{x}_3\overline{x}_4 \oplus x_2x_3\overline{x}_4) \vee x_1x_2$ is symmetric.

Solution. In the set-theoretical format the function *f* consists of PSTF $Y^{\overline{\oplus}}$ and STF Y^1: $\{Y^{\oplus},Y^1\} = = \left\{\{(1---),(-1--),(--00),(-110)\}^{\oplus},\{11--\}^1\right\}$. After transforming the PSTF $Y^{\overline{\oplus}}$ into STF Y^1 [19], we obtain the perfect STF Y^1 of the function f:

$$\left\{\{(8,\cancel{9},10,11,\cancel{12},\cancel{13},\cancel{14},\cancel{15}),(\cancel{4},\cancel{5},\cancel{6},\cancel{7},\cancel{12},\cancel{13},\cancel{14},\cancel{15}),(0,\cancel{4},\cancel{8},12),(\cancel{6},14)\}^{\oplus},\{(12,13,14,15)\}^1\right\} \Rightarrow$$
$$\Rightarrow \left\{\{9,10,11,5,7,0,12,14\}^{\oplus},\{(12,13,14,15)\}^1\right\} \Rightarrow$$
$$\Rightarrow \{0,5,7,9,10,11,12,13,14,15\}^1.$$

Note, that the condition (2) for all possible pairs $C_4^2 = 6$ of variables of the function *f* is not satisfied. Instead, the condition (3) will only be executed for the pair (x_2,x_4). It is easier to perform it for perfect STF $Y^0 = \{(0001),(0010),(0011),(0100),(0110),(1000)\}^0$ having lower power than perfect STF Y^1:

$$Y^0 \overset{\partial^2/\partial(x_2,x_4)}{\Longrightarrow} \left\{\cancel{\begin{pmatrix}0001\\0100\end{pmatrix}},\begin{pmatrix}0010\\0111\end{pmatrix},\cancel{\begin{pmatrix}0011\\0110\end{pmatrix}},\right.$$

$$\left.\cancel{\begin{pmatrix}0100\\0001\end{pmatrix}},\cancel{\begin{pmatrix}0110\\0011\end{pmatrix}},\begin{pmatrix}1000\\1101\end{pmatrix}\right\}^{\oplus} \Rightarrow \left\{\begin{pmatrix}0010\\0111\end{pmatrix},\begin{pmatrix}1000\\1101\end{pmatrix}\right\}\cap$$

$$\cap\{(-0-1),(-1-0)\}=\varnothing.$$

Therefore, the given function f has a partial simple symmetry $x_2 \sim x_4 / \overline{x}_2 \sim \overline{x}_4$ corresponding to [12].

Verification. In [12] the function is obtained $f(x_1,x_2,x_3,x_4)=x_2x_4 \vee (x_2 \oplus x_4)x_1\overline{x}_3 \vee \overline{(x_2 \vee x_4)}\,\overline{x}_1\overline{x}_3 \vee$ $\vee\, x_1x_3$, on the basis of which the symmetry is defined with respect to (x_2,x_4). After transformation of this expression we obtain STF $Y^1=\{(-1-1),(1100),(1001),(0000),(1-1-)\}^1 \equiv \{0,5,7,9,10,11,12,13,14,15\}^1$, corresponding to perfect STF Y^1 of the given function f.

Example 2. Determine which symmetric function is reflected in the test file xor5.pla.txt [20].

Solution. The given function $f(x_1,x_2,x_3,x_4,x_5)$ has perfect STF $Y^1=\{1,2,4,7,8,11,13,14,16,19,21,22,25,26,28,31\}^1$. We verify the condition (2) for a pair of variables (x_1,x_2):

$$Y^1 \overset{\partial^2/\partial(x_1,x_2)}{\Rightarrow} \left\{\cancel{\begin{pmatrix}00001\\11001\end{pmatrix}},\cancel{\begin{pmatrix}00010\\11010\end{pmatrix}},\cancel{\begin{pmatrix}00100\\11100\end{pmatrix}},\cancel{\begin{pmatrix}00111\\11111\end{pmatrix}},\right.$$

$$\cancel{\begin{pmatrix}01000\\10000\end{pmatrix}},\cancel{\begin{pmatrix}01011\\10011\end{pmatrix}},\cancel{\begin{pmatrix}01101\\10101\end{pmatrix}},\cancel{\begin{pmatrix}01110\\10110\end{pmatrix}},\cancel{\begin{pmatrix}10000\\01000\end{pmatrix}},$$

$$\cancel{\begin{pmatrix}10011\\01011\end{pmatrix}},\cancel{\begin{pmatrix}10101\\01101\end{pmatrix}},\cancel{\begin{pmatrix}10110\\01110\end{pmatrix}},\cancel{\begin{pmatrix}11001\\00001\end{pmatrix}},\cancel{\begin{pmatrix}11010\\00010\end{pmatrix}},$$

$$\left.\cancel{\begin{pmatrix}11100\\00100\end{pmatrix}},\cancel{\begin{pmatrix}11111\\00111\end{pmatrix}}\right\}^{\oplus}=\varnothing.$$

The condition (2) will also be satisfied for all the remaining nine (of $C_5^2=10$ possible) pairs of variables of function f. Therefore, it is a polysymmetric function $f=x_1\oplus x_2\oplus x_3\oplus x_4\oplus x_5$, and this is $S^5_{1,3,5}$.

Example 3. [5] Determine the type of symmetry in the function $f(x_1,x_2,x_3,x_4)$ given by perfect STF $Y^1=\{(0001),(0010),(0100),(0111),(1000),(1011),(1101),(1110),(1111)\}^1$.

Solution. The given function does not have polysymmetry, since the number of its minterms is odd. Instead, for all possible $C_4^2=6$ pairs of variables, the condition (3) will be fulfilled. Let illustrate it on an example of a pair of variables (x_1,x_2):

$$Y^1 \overset{\partial^2/\partial(x_1,x_2)}{\Rightarrow} \left\{\cancel{\begin{pmatrix}0001\\1101\end{pmatrix}},\cancel{\begin{pmatrix}0010\\1110\end{pmatrix}},\cancel{\begin{pmatrix}0100\\1011\end{pmatrix}},\cancel{\begin{pmatrix}0111\\1011\end{pmatrix}},\cancel{\begin{pmatrix}1000\\0100\end{pmatrix}},\right.$$

$$\left.\cancel{\begin{pmatrix}1011\\0111\end{pmatrix}},\cancel{\begin{pmatrix}1101\\0001\end{pmatrix}},\cancel{\begin{pmatrix}1110\\0010\end{pmatrix}},\begin{pmatrix}1111\\0011\end{pmatrix}\right\}^{\oplus} \Rightarrow \begin{pmatrix}1111\\0011\end{pmatrix}\cap$$

$$\cap\begin{cases}(00--),(11--)\neq\varnothing\\(01--),(10--)=\varnothing\end{cases}.$$

Since the simple symmetry holds for all possible pairs of variables, this indicates that the given function is a totally of the symmetric type $x_1\sim x_2\sim x_3\sim x_4 / \overline{x}_1\sim x_2\sim\overline{x}_3\sim\overline{x}_4$, and this is a symmetric function $S^4_{1,3,4}=x_1\oplus x_2\oplus x_3\oplus x_4\oplus x_1x_2x_3x_4$ corresponding to [5].

The partial mixed symmetry recognition is demonstrated on the above example of the function (*see Section* 2) $f=\overline{x}_3x_5\vee(x_1\oplus x_2\oplus x_4)$. In the set-theoretical format this function has this form $\{Y^1,Y^{\oplus}\}=\left\{\{(--0-1)\}^1,\{(1----),(-1---),(---1-)\}^{\oplus}\right\}$ and, after its transformation – as perfect STF $Y^1=\{1,2,6,7,8,11,12,13,16,19,20,21,25,26,30,31\}^1$. This function has a pair of minterms (17), so firstly we will verify the condition (2) of the theorem. Since the function is cumbersome, then, for the sake of simplicity, we will apply a special table. In the first line, we place the given minterms (highlighted by a bold font), and below, we place only the minterms of vectorial ST-derivatives obtained as a result of the fulfillment of the conditions of the theorem. For example, in the 2-nd row of the table there are placed the minterms of vectorial ST-derivative with respect to the variables (x_1,x_2) that are created by the mask of the literals $\overline{l}_1\overline{l}_2l_3l_4l_5$, etc. In the below table we outline the minterms of the derivatives equal to the given minterms.

As one can see Table, three lines in the table are crossed out for all the minterms created by masks $\overline{l}_1\overline{l}_2l_3l_4l_5$, $\overline{l}_1l_2l_3\overline{l}_4l_5$ and $l_1\overline{l}_2l_3\overline{l}_4l_5$. This indicates that the condition (2) is satisfied for three pairs of variables (x_1,x_2), (x_1,x_4) and (x_2,x_4) and therefore, the given function has a partial polysymmetry $\tilde{x}_1\sim\tilde{x}_2\sim\tilde{x}_4$. In addition, the given function is inherent partial antisymmetry $x_3\sim\overline{x}_5 / \overline{x}_3\sim x_5$, since the condition (4) for the vectorial ST-derivative $\partial^2Y^{\oplus}/\partial(x_3,x_5)$ is fulfilled, namely:

$$\{(00100),(00011),(01110),(01001),(10110),(10001),(11011)\}\cap\{(--0-0),(--1-1)\}=\varnothing.$$

Таблица 1. Theminterms created by masks

$l_1l_2l_3l_4l_5$	***00001***	***00010***	***00110***	***00111***	***01000***	***01011***	***01100***	***01101***	***10000***	***10011***	***10100***	***10101***	***11001***	***11010***	***11110***	***11111***
$\overline{l_1}\,\overline{l_2}l_3l_4l_5$	~~11001~~	~~11010~~	~~11110~~	~~11111~~	~~10000~~	~~10011~~	~~10100~~	~~10101~~	~~01000~~	~~01011~~	~~01100~~	~~01101~~	~~00001~~	~~00010~~	~~00110~~	~~00111~~
$\overline{l_1}l_2l_3\overline{l_4}l_5$	~~10011~~	~~10000~~	~~10100~~	~~10101~~	~~11010~~	~~11001~~	~~11110~~	~~11111~~	~~00010~~	~~00001~~	~~00110~~	~~00111~~	~~01011~~	~~01000~~	~~01100~~	~~01101~~
$l_1\overline{l_2}l_3\overline{l_4}l_5$	~~01011~~	~~01000~~	~~01100~~	~~01101~~	~~00010~~	~~00001~~	~~00110~~	~~00111~~	~~11010~~	~~11001~~	~~11110~~	~~11111~~	~~10011~~	~~10000~~	~~10100~~	~~10101~~
$l_1l_2\overline{l_3}l_4\overline{l_5}$	00100	~~00111~~	00011	~~00010~~	~~01101~~	01110	01001	~~01000~~	~~10101~~	10110	10001	~~10000~~	~~11110~~	~~11111~~	11011	~~11010~~

The group partial symmetry will be illustrated by next example.

Example 4 [21]. Determine the types of partial symmetries in the function $f(a,b,c,d,x,y)= = abxy \vee cdxy$.

Solution. The given function f has STF $Y^1=\{(11--11),(--1111)\}^1$ and its perfect STF $Y^1=\{(110011),(110111),(111011),(111111),(001111),(011111),(101111)\}^1$. This function does not have polysymmetry, because the number of minterms is odd and, consequently, the condition (2) will not be executed. Let us verify the conditions (3) and (4) only for those pairs of variables, where one of the conditions is satisfied:

$$Y^1 \overset{\partial^2/\partial(x_1,x_2)}{\Rightarrow} \begin{Bmatrix} l_1l_2l_3l_4l_5l_6 \\ \overline{l_1}\,\overline{l_2}l_3l_4l_5l_6 \end{Bmatrix} = \left\{ \begin{pmatrix}110011\\000011\end{pmatrix}, \begin{pmatrix}110111\\000111\end{pmatrix}, \begin{pmatrix}111011\\001011\end{pmatrix}, \begin{pmatrix}111111\\001111\end{pmatrix}, \begin{pmatrix}001111\\111111\end{pmatrix}, \begin{pmatrix}\cancel{011111}\\\cancel{101111}\end{pmatrix}, \begin{pmatrix}\cancel{101111}\\\cancel{011111}\end{pmatrix} \right\}^{\oplus} \cap$$

$$\cap \begin{cases} (01----),(10----)=\varnothing \\ (00----),(11----)\neq\varnothing \end{cases};$$

$$Y^1 \overset{\partial^2/\partial(x_3,x_4)}{\Rightarrow} \begin{Bmatrix} l_1l_2l_3l_4l_5l_6 \\ l_1l_2\overline{l_3}\,\overline{l_4}l_5l_6 \end{Bmatrix} = \left\{ \begin{pmatrix}\cancel{110011}\\\cancel{111111}\end{pmatrix}, \begin{pmatrix}\cancel{110111}\\\cancel{111011}\end{pmatrix}, \begin{pmatrix}\cancel{111011}\\\cancel{110111}\end{pmatrix}, \begin{pmatrix}\cancel{111111}\\\cancel{110011}\end{pmatrix}, \begin{pmatrix}001111\\000011\end{pmatrix}, \begin{pmatrix}011111\\010011\end{pmatrix}, \begin{pmatrix}101111\\100011\end{pmatrix} \right\}^{\oplus} \cap$$

$$\cap \begin{cases} (--01--),(--10--)=\varnothing \\ (--00--),(--11--)\neq\varnothing \end{cases};$$

$$Y^1 \overset{\partial^2/\partial(x_5,x_6)}{\Rightarrow} \begin{Bmatrix} l_1l_2l_3l_4l_5l_6 \\ l_1l_2l_3l_4\overline{l_5}\,\overline{l_6} \end{Bmatrix} = \left\{ \begin{pmatrix}110011\\110000\end{pmatrix}, \begin{pmatrix}110111\\110100\end{pmatrix}, \begin{pmatrix}111011\\111000\end{pmatrix}, \begin{pmatrix}111111\\111100\end{pmatrix}, \begin{pmatrix}001111\\001100\end{pmatrix}, \begin{pmatrix}011111\\011100\end{pmatrix}, \begin{pmatrix}101111\\101100\end{pmatrix} \right\}^{\oplus} \cap$$

$$\cap \begin{cases} (----01),(----10)=\varnothing \\ (----00),(----11)\neq\varnothing \end{cases}.$$

As we can see, only the condition (3) is satisfied for three pairs of variables (x_1,x_2), (x_3,x_4) and (x_5,x_6). This indicates a group of partial symmetries in a given function:

$$\underline{(x_1 \sim x_2)} \not\leftarrow \underline{(x_3 \sim x_4)} \not\leftarrow \underline{(x_5 \sim x_6)} \,/$$
$$/\,(x_1 \sim x_2) \not\leftarrow (x_3 \sim x_4) \not\leftarrow (x_5 \sim x_6).$$

The obtained result corresponds to [21].

If the function f given in SOP (or STF) contains nonorthogonal conjuncterms, it must be orthogonalized [16] or transformed into perfect STF Y^1. As an example of such a function it may be $f = x_1x_2 \vee x_3$ having STF $Y^1=\{(11-),(--1)\}^1$. After its orthogonalization we obtain:

$$Y^1=\{(11-),(--1)\}^1 \overset{ort}{\Rightarrow} \left\{(111),(11-)\cap(--0),\ (--1)\cap\begin{pmatrix}0--\\10-\end{pmatrix}\right\}^1 = \{(111),(110),(0-1),(101)\}^1.$$

We define now a vectorial ST-derivative of the 2-nd order, for example, with respect to a pair of variable (x_1,x_2):

$$Y^1=\{(111),(110),(0-1),(101)\}^1 \overset{\partial^2/\partial(x_1,x_2)}{\Rightarrow} \begin{Bmatrix} l_1l_2l_3 \\ \overline{l_1}\,\overline{l_2}l_3 \end{Bmatrix} = \left\{ \begin{pmatrix}111\\001\end{pmatrix}, \begin{pmatrix}110\\000\end{pmatrix}, \begin{pmatrix}0-1\\1-1\end{pmatrix}, \begin{pmatrix}101\\011\end{pmatrix} \right\}^{\oplus} \Rightarrow \{(110),(000)\}.$$

The identical result will be obtained for perfect STF $Y^1=\{(001),(011),(101),(110),(111)\}^1$.

Example 5. Determine the types of partial symmetries in the SOP function $f = x_2 \vee \overline{x}_1x_3$.

Solution. The given function f has STF $Y^1=\{(-1-),(0-1)\}^1$ and, after the orthogonalization procedure, it has STF $Y^1=\{(011),(010),(11-),(001)\}^1$. This function does not have polysymmetry because its number of minterms is odd. Let us verify the conditions (3) and (4):

$$Y^1 \overset{\partial^2/\partial(x_1,x_2)}{\Rightarrow} \begin{Bmatrix} l_1 l_2 l_3 \\ \overline{l_1}\,\overline{l_2} l_3 \end{Bmatrix} = \left\{ \begin{pmatrix}\cancel{001}\\ \cancel{111}\end{pmatrix}, \begin{pmatrix}010\\100\end{pmatrix}, \begin{pmatrix}011\\101\end{pmatrix}, \begin{pmatrix}11-\\00-\end{pmatrix}\right\}^{\oplus} \neq \varnothing;$$

$$Y^1 \overset{\partial^2/\partial(x_1,x_3)}{\Rightarrow} \begin{Bmatrix} l_1 l_2 l_3 \\ \overline{l_1} l_2 \overline{l_3} \end{Bmatrix} = \left\{ \begin{pmatrix}001\\100\end{pmatrix}, \begin{pmatrix}\cancel{010}\\ \cancel{111}\end{pmatrix}, \begin{pmatrix}011\\110\end{pmatrix}, \begin{pmatrix}11-\\01-\end{pmatrix}\right\}^{\oplus} \Rightarrow$$

$$\Rightarrow \left\{ \begin{pmatrix}001\\100\end{pmatrix}, \begin{pmatrix}\cancel{011}\\ \cancel{110}\end{pmatrix}, \begin{pmatrix}\cancel{110}\\ \cancel{011}\end{pmatrix}\right\} \Rightarrow \{(001),(100)\} \cap$$

$$\cap \begin{cases} \{(0-1),(1-0)\} \neq \varnothing \\ \{(0-0),(1-1)\} = \varnothing \end{cases};$$

$$Y^1 \overset{\partial^2/\partial(x_2,x_3)}{\Rightarrow} \begin{Bmatrix} l_1 l_2 l_3 \\ l_1 \overline{l_2}\,\overline{l_3} \end{Bmatrix} = \left\{ \begin{pmatrix}001\\010\end{pmatrix}, \begin{pmatrix}\cancel{010}\\ \cancel{001}\end{pmatrix}, \begin{pmatrix}\cancel{011}\\ \cancel{000}\end{pmatrix}, \begin{pmatrix}11-\\10-\end{pmatrix}\right\}^{\oplus} \neq \varnothing.$$

Since the condition (4) is fulfilled only for a pair (x_1, x_3), we have antisymmetry $x_1 \sim \overline{x}_3 / \overline{x}_1 \sim x_3$.

The identical result is obtained using the analytical method:

$$\frac{\partial^2 (x_2 \vee \overline{x}_1 x_3)}{\partial(x_1, x_3)} = \left(x_2 \oplus (x_2 \vee x_3)\right) \oplus \left(x_2 \oplus (x_2 \vee \overline{x}_1)\right) \oplus \overline{x}_2 =$$

$$= (x_2 \vee x_3) \oplus (x_2 \vee \overline{x}_1) \oplus \overline{x}_2 = (x_2 \vee x_3) x_1 \overline{x}_2 \oplus$$

$$\oplus\, \overline{x}_2 \overline{x}_3 (x_2 \vee \overline{x}_1) \oplus \overline{x}_2 = \overline{x}_2 (\overline{x_1 \oplus x_3}).$$

Since we have the condition (7) $\overline{x}_2(\overline{x_1 \oplus x_3})$ $(x_1 \oplus x_3) = 0$, this is confirmed by the presence of type antisymmetry $x_1 \sim \overline{x}_3 / \overline{x}_1 \sim x_3$ in the given function *f*.

Example 6 [22]. Determine the type of symmetry in a monotone increasing function $c = xy \vee xz \vee yz$.

Solution. The given function f has STF $Y^1 = \{(11-),(1-1),(-11)\}^1$ and, after the orthogonalization procedure, it has STF $Y^1 = \{(110),(101),(-11)\}^1$. Let us verify the conditions (2), (3) and (4):

$$Y^1 \overset{\partial^2/\partial(x_1,x_2)}{\Rightarrow} \left\{ \begin{pmatrix}110\\000\end{pmatrix}, \begin{pmatrix}\cancel{101}\\ \cancel{011}\end{pmatrix}, \begin{pmatrix}-11\\-01\end{pmatrix}\right\}^{\oplus} \Rightarrow$$

$$\Rightarrow \left\{ \begin{pmatrix}110\\000\end{pmatrix}, \begin{pmatrix}111\\001\end{pmatrix}\right\} \cap \begin{cases} (00-),(11-) \neq \varnothing \\ (01-),(10-) = \varnothing \end{cases};$$

$$Y^1 \overset{\partial^2/\partial(x_1,x_3)}{\Rightarrow} \left\{ \begin{pmatrix}\cancel{110}\\ \cancel{011}\end{pmatrix}, \begin{pmatrix}101\\000\end{pmatrix}, \begin{pmatrix}-11\\-10\end{pmatrix}\right\}^{\oplus} \Rightarrow$$

$$\Rightarrow \left\{ \begin{pmatrix}101\\000\end{pmatrix}, \begin{pmatrix}111\\010\end{pmatrix}\right\} \cap \begin{cases} (0-0),(1-1) \neq \varnothing \\ (0-1),(1-0) = \varnothing \end{cases};$$

$$Y^1 \overset{\partial^2/\partial(x_2,x_3)}{\Rightarrow} \left\{ \begin{pmatrix}\cancel{110}\\ \cancel{101}\end{pmatrix}, \begin{pmatrix}\cancel{101}\\ \cancel{110}\end{pmatrix}, \begin{pmatrix}-11\\-00\end{pmatrix}\right\}^{\oplus} \Rightarrow$$

$$\Rightarrow \left\{ \begin{pmatrix}-11\\-00\end{pmatrix}\right\} \cap \begin{cases} (-00),(-11) \neq \varnothing \\ (-01),(-10) = \varnothing \end{cases}.$$

Since the condition (3) holds for all possible pairs of variables, we have a total symmetric function with simple symmetry of the type $x_1 \sim x_2 \sim x_3 / \overline{x}_1 \sim \overline{x}_2 \sim \overline{x}_3$, namely $S^3_{2,3}$. The algorithm of the proposed method for the recognition of symmetry types in boolean functions is considered in Part II of the article.

Conclusion

Part II of the article describes the main theoretical statements of the new method for the recognition of symmetry in complete and incomplete boolean functions of *n* variables, based on numerical set-theoretical logical differentiation. The basis of the proposed method ëis the theorem for the recognition of different types of partial symmetry (polysymmetry, simple symmetry and antisymmetry).

The presented above examples confirm the validity of the proved theorem, and also illustrate the advantages of this method, compared with the known ones, in terms of the simplicity of its practical implementation.

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Б.Є. Рицар, доктор технічних наук, професор,
Національний університет «Львівська політехніка», вул. С. Бандери, 12, Львів, 79013, Україна,
E-mail: bohdanrytsar@gmail.com

НОВИЙ МЕТОД РОЗПІЗНАВАННЯ СИМЕТРІЇ У БУЛОВИХ ФУНКЦІЯХ НА ОСНОВІ ТЕОРЕТИКО-МНОЖИННОГО ЛОГІКОВОГО ДИФЕРЕНЦІЮВАННЯ. I

Вступ. Симетричні булові функції завдяки своїм специфічним властивостям мають широке застосування у проектуванні цифрових пристроїв, телекомунікаціях, криптографії тощо. Оскільки булові функції можуть мати різні типи симетрії з властивими їм особливостями, важливо вміти їх розпізнавати якомога простішими засобами. Проте проблема ускладнюється тим, що, з одного боку, функції можуть бути як одного типу, так і змішаного, а також як повністю симетричними, так і частково симетричними, а з другого боку, сама функція може бути не повністю визначена, тобто задана частково, або задана ДНФ. Сучасні методи розпізнавання типів симетрії ґрунтуються переважно на аналітичному підході (розкладі Шеннона), візуальному методі, аналітичному обчисленні логікових похідних і т.ін., надто складні щодо реалізації та мало ефективні для функцій великих розмірів і особливо, коли вони задані частково.

Мета статті — розробити простий для реалізації метод розпізнавання різних типів повних і частинних симетрій як у повних, так і частково заданих булових функціях.

Методи. У статті запропоновано новий метод розпізнавання різних типів повних і частинних симетрій, таких як полісиметрія, проста симетрія та антисиметрія, як у повністю, так і частинно заданих функціях на основі числового теоретико-множинного логікового диференціювання. Алгоритм методу ґрунтується на теоремі про розпізнавання різних типів частинних симетрій, який, порівняно з відомими, має відносно меншу обчислювальну складність за рахунок порівняно меншої кількості операцій і процедур, потрібних для виконання поставленої задачі.

Результат. Справедливість доведеної теореми засвідчують приклади розпізнавання різних типів повних і частинних симетрій як у повністю заданих функціях (частина I), так і частково заданих функціях (частина II), у тому числі заданих у ДНФ, які з метою порівняння ефективності запропонованого алгоритму запозичено з публікацій відомих авторів.

Висновки. Запропонований новий метод розпізнавання різних типів повних і частинних симетрій (полісиметрії, прості симетрії та антисиметрії) як у повністю, так і частково заданих булових функціях на основі числового теоретико-множинного логікового диференціювання відрізняється від відомих відносно простішою практичною реалізацією.

Ключові слова: *розпізнавання повних і часткових симетрій, булова функція, числове теоретико-множинне логікове диференціювання.*

Б.Е. Рыцар, доктор технических наук, профессор,
Национальный университет «Львівська політехніка», ул. С. Бандеры, 12, Львов, 79013, Украина,
E-mail: bohdanrytsar@gmail.com

НОВЫЙ МЕТОД РАСПОЗНАВАНИЯ СИММЕТРИИ В БУЛЕВЫХ ФУНКЦИЯХ НА ОСНОВЕ ТЕОРЕТИКО-МНОЖЕСТВЕННОГО ЛОГИЧЕСКОГО ДИФФЕРЕНЦИРОВАНИЯ. I

Введение. Симметричные булевые функции благодаря своим специфическим свойствам широко используются в проектировании цифровых устройств, телекоммуникациях, криптографии и т.п. Поскольку булевые функции могут иметь разные типы симметрии с присущими им особенностями, важно уметь их распознавать как можно простейшими способами. Но проблема усложняется тем, что, с одной стороны, функции могут быть как одного типа, так и смешанного, а также как полностью симметричными, так и частично симметричными, а с другой стороны, сама функция может быть не полностью определена, т.е. задана частично, или задана ДНФ. Современные методы распознавания типов симметрии основаны преимущественно на аналитическом подходе (разложении Шеннона), визуальном методе, аналитическом вычислении логических производных и др., слишком сложны в реализации и мало эффективны для функций больших размеров и особенно, когда они заданы частично.

Цель статьи — разработать простой в реализации метод распознавания разных типов полных и частичных симметрий, как в полных, так и частично заданных булевых функциях.

Методы. В статье предложен новый метод распознавания разных типов полных и частичных симметрий, таких как полисимметрия, простая симметрия и антисимметрия, как в полностью, так и частично заданных функциях на основе численного теоретико-множественного логического дифференцирования. Алгоритм метода основан на теореме распознавания разных типов частичных симметрий, который, в сравнении с известными, имеет относительно меньшую вычислительную сложность благодаря сравнительно меньшему количеству операций и процедур, необходимых для выполнения поставленной задачи.

Результат. Справедливость доказанной теоремы показывают примеры распознавания разных типов полных и частичных симметрий как в полностью заданных функциях (часть I), так и частично заданных функциях (часть II), в том числе заданных в ДНФ, которые с целью сравнения эффективности предложенного алгоритма взяты из публикаций известных авторов.

Выводы. Предложенный новый метод распознавания разных типов полных и частичных симметрий (полисимметрии, простые симметрии и антисимметрии) как в полностью, так и частично заданных булевых функциях на основе числового теоретико-множественного логического дифференцирования отличается от известных относительно простейшей практической реализацией.

Ключевые слова: *распознавание полных и частичных симметрий, булевая функция, числовое теоретико-множественное логическое дифференцирование.*