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SUPER FIBONACCI GRACEFUL GRAPHS AND FIBONACCI CUBES

The popularity of Fibonacci cubes is due to their wide range of uses. In mathematical chemistry, this concept is used in the study of hexagonal graphs. In computer science, Fibonacci cubes are interesting from an algorithmic point of view. V. Hsu introduced them in 1993 to simulate the connections of multiprocessor computer networks. He wanted to get graphs with hypercube properties, the order of which is not a power of two. Therefore, the problem of embedding other graphs in Fibonacci cubes is of interest.

Keywords: *graph, hypercube, Fibonacci cube, super Fibonacci graceful labeling of graph.*

Introduction

One of the topical trends in the development of graph theory is the study of problems associated with various labeling of graphs. A. Rossa is considered the founder of the labeling theory, who in 1967, in his work "On certain valuations of the vertices of graph", defined four types of labels as a tool for decomposing a complete graph into isomorphic subgraphs. Among them is the β-valuation. S. Golomb in 1972 coined the term "graceful labeling" for it. "A dynamic survey of graph labeling" made by D. Gallian, which is annually republished and updated with the results of new research, contains brief information on the achievements in various types of graph labeling, including graceful ones. A survey on graceful trees 2006 by M. Edwards and L.Howard provides an in-depth look at this topic.

The continuation of the idea of the gracefulness of a graph is the case when the labeling is a mapping from a set of edges into a set consisting of elements of an arbitrary sequence. Of interest are the structural properties of graphs for which a series of Fibonacci numbers serves as this sequence. This article discusses two types of such labeling: graceful super Fibonacci and Θ-graceful. The first one was offered by K.M. Kathiresan and S. Amutha in 2010. They used the first n Fibonacci numbers for the labels of the vertices and edges of the n-size graph. The analysis of publications on this topic showed that super Fibonacci graceful graphs have a specific structure. The question arises: how do the degrees of its vertices affect the size and structure of this graph?

Θ-graceful labeling was introduced by B. Bresar and S. Klavzar in 2006 for partial cubes — isometric subgraphs of a hypercube. Partial cube topologies are popular schemes for modeling the connections of multiprocessor networks. Also, hypercubes and their subgraphs are being used in coding theory, mathematical chemistry and play an important role in the theory of metric graphs [1—7]. One of the features of the hypercube, which is passed on to any of its isometric subgraphs, is the efficient calculation of the distance between two vertices. This feature plays an important role in studying the properties of hypercube subgraphs related to the Djokovic-Winkler Θ-relation.

It, in turn, is defined on the set of graph edges and is an equivalence relation for partial cubes. This leads to the partition of the set of edges into Θ-classes. When you label a partial cube Θ-gracefully, all edges of the same Θ-class get the same labels, while the labels of edges from different Θ-classes should not be identical. Trees are examples of partial cubes. For them, each Θ-class contains only one edge. Therefore, the concepts of Θ-graceful and graceful tree labeling coincide. B. Bresar and S. Klavzar in the work " Θ-graceful labelings of partial cubes" defined types of Θ-graceful partial cubes, other than trees.

Among their representatives is the Fibonacci cube Γ_n. It was introduced by V. Hsu to obtain graphs with hypercube properties, the order of which is not a power of two [1]. Structures closely related to them were studied earlier in [3, 4]. The first n numbers of the Fibonacci series act as the labels of the Γ_n edges in its Θ-graceful labeling. The problems of characterization of super Fibonacci graceful and Θ-graphs, as well as the construction of the corresponding labeling in general form, are still remain open.

Formulation of the Problem

By a graph we mean an undirected graph without loops and multiple edges. The order of a graph is the number of vertices in it, and its size is the number of edges. Through $V(G)$, $E(G)$, deg(u) we denote the set of vertices, the set of edges, the degree u of the vertex of the graph G, respectively. Unless otherwise indicated, the distance $d_G(u,v)$ (or $d(u,v)$ between the vertices u and v of the graph G is equal to the length of the shortest path connecting these vertices, i.e. to the number of edges in it. The diameter of a graph is understood as the largest of the distances between its vertices. If the distance between two vertices u and v of the graph is equal to its diameter, then the shortest path connecting these vertices is called diametrical, while u and v — diametrically opposite.

Let $B_n = \{b_n b_{n-1} \ldots b_2 b_1 \mid b_i \in \{0,1\},\ 1 \le i \le n\}$ the set of binary tuples (strings) be of length n. Recall that the Hamming distance between two tuples of the same length is equal to the number of positions at which these tuples are different. A hypercube (or a Boolean cube, or a n-cube) Q_n can be viewed as a graph with 2^n vertices from B_n. Moreover, two vertices are adjacent if and only if the Hamming distance between them is equal to one. Q_n is also defined recursively as follows: $Q_0 = K_1$, $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$.

The interval $I(u,v)$ between the vertices u and v of the graph G is made up of all its vertices lying on the shortest u,v-path. The median of a triplet of vertices u, v, w of a graph G is a vertex belonging to the set $I(u,v) \cap I(u,w) \cap I(v,w)$. A connected graph G is considered median if each triplet of its vertices has a single median. A subgraph H of a connected graph G is called an isometric subgraph if $d_H(u,v) = d_G(u,v)$ for any $u,v \in V(H)$. This subgraph will be convex if, for any $u,v \in V(H)$, each shortest u,v-path from G lies in H. Isometric subgraphs of hypercubes are called *partial cubes*. It is known that partial cubes are median graphs.

Let a graph G and an injection or bijection f from $V(G)$ into a finite set L consisting of natural numbers be given. A function f is called *a vertex labeling of graceful type* if it induces a function $f^*(uv) = |f(u) - f(v)|$ on the set of edges, where $uv \in E(G)$. In turn, the function value $f^*(uv)$ is often referred as the label or weight of edge uv. We denote W the set of edge weights.

Depending on the requirements for L and W, various graceful vertex labelings are obtained. Their representatives are super Fibonacci graceful and Θ-graceful labelings. The purpose of this article is to study new properties of graphs that these labelings allow.

Properties of Super Fibonacci Graceful Graphs

Let a graph G of size q and its vertex labeling f of a graceful type be given. The labeling f is called *super Fibonacci graceful* if the following conditions are met: $L = \{F_0, F_1, F_2, F_3, \ldots, F_q\}$;

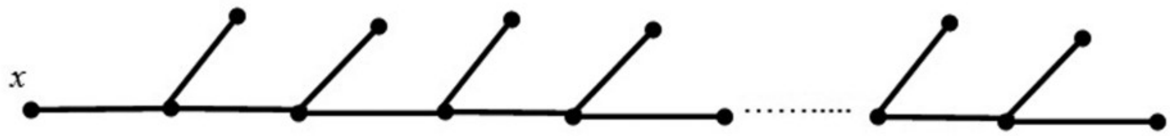
Fig. 1. Caterpillar T_x with selected vertex x

$W = \{F_1, F_2, F_3, \ldots, F_q\}$, where F_i — are the numbers of the Fibonacci series, $i = 1, 2, \ldots, q$, $F_0 = 0$; the function f is injective, while f^* — bijective. A graph that allows such a labeling is called *super Fibonacci graceful*. This concept K.M. Kathiresan, S. Amutha proposed for the Fibonacci series, consisting of numbers $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$, and M. Semeniuta — for the Fibonacci series, which are formed by numbers $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, \ldots$. In each case, the labels of the vertices and edges G remain unchanged, if we abstract from the values F_i, where $i = 1, 2, \ldots, q$. Consequently, all statements regarding the super Fibonacci graphs gracefulness for one of the indicated sets of values of the Fibonacci series numbers remain true for the second.

Further on, we will identify the vertices and edges of the graph with their labels. Theorems 1 and 2 describe the structural properties of super Fibonacci graceful graphs.

Theorem 1. Let G be a connected super Fibonacci graceful graph. Then $1 \le \deg(F_k) \le 4$ for any vertex $F_k \in V(G), F_k \ne F_0$.

Proof. Let G be a connected super Fibonacci graceful graph. Based on the properties of Fibonacci numbers, with the vertex $F_k \in V(G)$, $F_k \ne F_0$, in the graph G, only the vertices labeled F_{k-2}, F_{k-1}, F_{k+1}, F_{k+2} and F_0 can be adjacent. To avoid repeating edge labels, the number of vertices adjacent to F_k cannot exceed four. If $\deg(F_k) = 4$, then $F_k F_0 \in E(G)$. Hence, $1 \le \deg(F_k) \le 4$. The theorem is proved.

Consider a tree with a selected vertex x shown in Fig. 1. It belongs to a class of trees called caterpillars. A caterpillar is understood as trees that, after removing all vertices of degree one, are converted into apath — the base (or trunk) of the caterpillar. The caterpillar in Fig. 1 denote T_x, and its trunk — P.

Note that for the tree super Fibonacci graceful labeling is a bijective function.

Lemma 1. A caterpillar T_x of size n is a super Fibonacci graceful graph for any odd one n.

Proof. Let T_x has a size n. Since the number of edges for this caterpillar is odd, then $n = 2k - 1$, where $k = 1, 2, \ldots$. Let us enumerate the vertices T_x as follows. To the vertices of the trunk, starting with x, we assign the numbers: F_0, F_{2k-1}, F_{2k-3}, F_{2k-5}, ..., F_3, F_1. For the rest of the vertices, we use labels F_{2k-2}, F_{2k-4}, F_{2k-6}, ..., F_4, F_2, placing them in order of their remoteness from x.

The way of placing labels at the vertices T_x leads to the fact that all edges receive different labels, and these labels belong to the set $\{F_1, F_2, F_3, \ldots, F_{2k-1}\}$ for any $k = 1, 2, \ldots$. Therefore, T_x — is a super Fibonacci graceful graph. The lemma is proved.

Consider trees T_1, T_2, ..., T_m, each of which is isomorphic to T_x. The vertex x_i of the tree T_i, where $i = 1, 2, \ldots, m$ — is the isomorphic image of the vertex x. Let's construct a tree T_x^m by identifying all the vertices x_i. In this case, it is said that T_x^m is obtained as a result of a one-point connection of T_1, T_2, ..., T_m. For definiteness, trees T_i that are subgraphs of a tree T_x^m will be called its branches, and the common vertex x^* for them will be called the root.

Theorem 2. A tree T_x^m, each branch of which has a size n, is super Fibonacci graceful, if n — is odd and m — any natural number.

Proof. Let T_1, T_2, ..., T_m — are branches T_x^m and root x^* — is their common vertex. For a branch T_1 of size $n = 2k - 1$, where $k = 1, 2, \ldots$, we apply the labeling used in Lemma 1. To the vertices of the tree trunk T_2, starting from the vertex following x, we assign the numbers: F_{4k-2}, F_{4k-4}, F_{4k-6}, ..., F_{2k+2}, F_{2k}. For the rest of the vertices T_2, use the labels F_{4k-3}, F_{4k-5}, F_{4k-7}, ...

..., F_{2k+3}, F_{2k+1}, placing them in order of their remoteness from x. In a similar way, we label the vertices of the branches T_3, T_4,..., T_m. The performed labeling T_x^m is a bijective function from a set of vertices to a set $\{F_0, F_1, F_2, F_3, \ldots, F_{mn}\}$. It induces edge labeling. The set of edge labels will look like $\{F_1, F_2, F_3, \ldots, F_{mn}\}$. Hence, T_x^m — is super Fibonacci graceful tree for any m. The theorem is proved.

Consider a tree T_x^m and its branches T_i, where $i = 1, 2, \ldots, m$. Let T_i^* be a subgraph T_i that is a caterpillar with each non-terminal vertex of degree three. We form all T_i^* by removing k_i edges from T_i, where k_i is an even number, according to the following rule:

— at the first step, we remove from T_i a pair of vertices that are diametrically opposite to x^*, and denote the resulting tree as T^1; if the number of remote vertices is less then k, go to the second step;

— at the second step, we remove from the tree T^1 a pair of vertices that are diametrically opposite to x^*, the resulting tree is denoted as T^2;

— we repeat the action of the second step until the number of removed edges becomes equal k_i.

The operation of removing a vertex u of the graph $G = (V, E)$ consists in excluding the vertex u from V and all edges incident to u from E.

We denote T_x^m the tree obtained as a result of applying the described rule to T_x^m. It is a subgraph of T_x^m, in addition, $x^* \in V(T_x^{m*})$, T_i^* — are branches of T_x^{m*}.

Corollary. The tree T_x^{m*} is super Fibonacci graceful for every even k_i, where $i = 1, 2, \ldots, m$, and for any natural number m.

The validity of the assertion of the corollary is easy to obtain by applying arguments similar to those given in the proof of Theorem 2.

We call T_x^m a *complete super Fibonacci tree*, and T_x^{m*} — a super Fibonacci tree, if all branches in each of them have an odd size.

Θ-Gracious Labeling

We will use the Djokovic-Winkler Θ relation defined on the set of edges of the graph G. The edges xy and uv of the graph G are in a relationship Θ if $d(x,u) + d(y,v) \neq d(x,v) + d(y,u)$. P. Winkler proved that a connected graph G is a partial cube if and only if G is a bipartite and Θ is a transitive relation. Therefore, for a partial cube, Θ is an equivalence relation and partitions the set of its edges into equivalence classes, which are called Θ-classes.

If the set of edges of a graph G can be represented as a disjunct union $E(G) = F_1 \cup F_2 \cup \ldots \cup F_k$ of pairwise disjoint 1-factors (perfect matchings) of this graph, then this representation is called 1-factorization of G. A 1-factorization is called square if the pairwise union of its two 1-factors is a graph, all connected components of which are cycles of the length four.

Let G be a partial cube at the n vertices. B. Bresar and S. Klavzar called a bijection $f: V(G) \to \{0, 1, \ldots, n-1\}$ — a Θ-graceful labeling G if all edges in each Θ-class receive the same labels, and edges from different Θ-classes receive different labels and edge labels are determined according to the rule: $|f(x) - f(y)|$ for each $xy \in E(G)$. A graph that allows labeling f a is called Θ-graceful.

We use the decimal representation of the corresponding binary numbers $b_n b_{n-1} \ldots b_2 b_1 \in B_n$ as labels of the vertices of the hypercube. The relationship between 1-factorization and hypercube Θ-classes is presented in Theorem 3.

Theorem 3. Between the set of 1-factors of the square 1-factorization of the hypercube Q_n ($n \geq 2$) and the factor-set of the set $E(Q_n)$ with respect to equivalence, there is a one-to-one correspondence.

Proof. Q_n — is a Θ-graceful graph. The number of its parallel Θ-classes is n. A set of different edge labelsis of the form: $\{2^0, 2^1, \ldots, 2^{n-1}\}$. Each class contains 2^{n-1} edges.

For any n ($n \geq 2$) there is a square 1-factorization Q_n with 1-factors F_i, where $i = 1, 2, \ldots, n$. Moreover, F_i consists of 2^{n-1} edges incident to vertices that differ in position $i-1$ in their binary representation. Therefore, all edges of the factor F_i have the same labels, equal to 2^{i-1}. They are calculated as the modulus of the difference between the vertex labels. We assign to F_i a Θ-class with edge marks 2^{i-1}. The theorem is proved.

Consider the Fibonacci series: $F_1 = 1$, $F_2 = 2$, ..., $F_n = F_{n-2} + F_{n-1}$. For $n \geq 1$ of Fibonacci string,

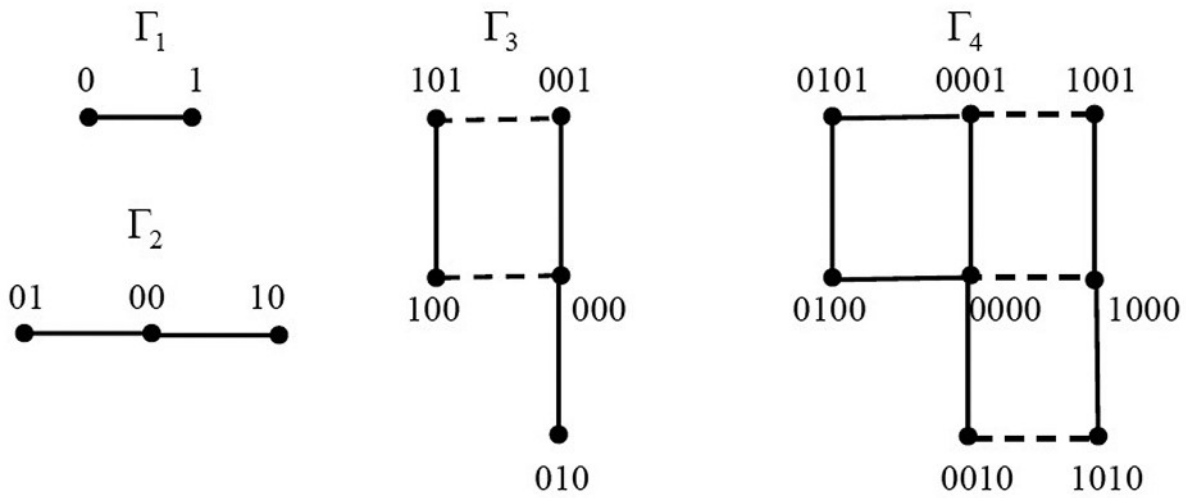

Fig. 2. Fibonacci cubes Γ_n at $n = 1, 2, 3, 4$

lengths n — are binary words (or strings) of the form:

$$F_n = \{b_n b_{n-1} \dots b_2 b_1 \in B_n \mid b_i \cdot b_{i+1} = 0, \; 1 \le i \le n-1\}$$

Note that the set F_n consists of all binary strings of length n not containing two consecutive ones. By Zeckendorf's theorem, any natural number can be uniquely written as a sum of inconsistent Fibonacci numbers. Let i — be a positive integer and $i \le F_n - 1$, denote $F(i) = b_n b_{n-1} \dots b_2 b_1 \in F_n$. Thus $i = \sum_{j=1}^{n} b_j F_j$, where F_j is the j-th term of the Fibonacci sequence. Each Fibonacci number written as a Fibonacci string contains one only in the i-th position.

The Fibonacci cube Γ_n of order n ($n = 0, 1, 2, \dots$) is understood as a graph with a set of vertices $V = F$ and the property: two of its vertices are adjacent if the Hamming distance between them is equal to one. The Fibonacci cube Γ_n is a subgraph of a hypercube Q_n. Γ_n is obtained from Q_n by removing all vertices containing at least two consecutive ones.

B. Bresard and S. Klavzar proved that Γ_n is Θ-graceful graph [8]. The factor-set of a set of $E(\Gamma_n)$ with respect to Θ consists of n Θ-classes of equivalence. Let's designate them Θ_1^n, Θ_1^n ,..., Θ_2^n . All edges from Θ_n^n have labels equal to F_i , where $i = 1, 2, \dots, n$.

As known, the Fibonacci cube Γ_n can be obtained using cubes of lower dimension Γ_{n-1} and Γ_{n-2} connected by F_{n-1} edges. This presentation is called fundamental decomposition of Γ_n . Fig. 2 presents cubes Γ_n for $n = 1, 2, 3, 4$. In Γ_3 and Γ_4 the edges connecting cubes of lower dimensionsare shown by dotted lines. The labels of the vertices in the form of Fibonacci strings are also indicated.

B. Kong, S. Zheng and S. Sharma described a construction that imitates the fundamental decomposition. Empty sequence $g_0 = \lambda$ and sequence $g_1 = 0, 1$ represent graphs Γ_0 and Γ_1, respectively. For $n \ge 2$, let $g_n = 0g_{n-1}, 10\overline{g}_{n-2}$, where $\overline{g}$ means the sequence reverse to g, and αg is the sequence obtained from g by adding a line α before each element from g. Thus, the first sequences g_i are:

$g_0 = \lambda$

$g_1 = 0, 1$

$g_2 = 01, 00, 10$

$g_3 = 010, 000, 001, 101, 100$

$g_4 = 0100, 0101, 0001, 0000, 0010, 1010, 1000, 1001$

The sequence g_n contains all the vertices Γ_n. Moreover, by induction, the consecutive terms in $0g_{n-1}$ as well as in $10\overline{g}_{n-2}$ differ in one position. In addition, the last term $0\overline{g}_{n-1}$ and the first term $10\overline{g}_{n-2}$ also differ in one position. Based on the described construction method Γ_n, the statement of Lemma 2 is obvious. It is also easy to obtain a proof of Lemma 3.

Lemma 2. For each Γ_n, we have $\Theta_i^n = \Theta_i^{n-1} \cup \Theta_i^{n-}$, where $i = 1, 2, \ldots, n-1$ and Θ_n^n consists of $\left|\Theta_n^n\right| = F_{n-1}$ edges connecting Γ_{n-1} and Γ_{n-2} .

Lemma 3. Γ_n contains only two diametrically opposite vertices˙

Proof. Let a graph Γ_n be given. Consider a vertex $x = b_n b_{n-1} \ldots b_2 b_1 \in V(\Gamma_n)$ at which $b_n = 0$, $b_{n-1} = \overline{b}_n = 1, b_{n-i} = \overline{b}_{n-(i-1)}$, where $i = 2, 3, \ldots, n-1$ and a vertex $y = \overline{b}_n \overline{b}_{n-1} \ldots \overline{b}_2 \overline{b}_1 \in V(\Gamma_n)$, where b_i and $\overline{b}_i$ take opposite values. The Hamming distance between *x* and *y* is the greatest and equals *n*. There are only two such vertices in Γ_n and they are diametrically opposite. The lemma is proved.

The edges of the diametrical chain Γ_n, which we denote D_n, belong to different Θ-classes. Since the diametrical chain D_n has a length *n*, it contains one representative of each class. Let us illustrate with an example Γ_7 and Γ_8 the construction of the corresponding diametrical chains, indicating the sequence of vertices through which they pass. With this method of setting D_n, we take into account that each pair of adjacent vertices of the sequence is connected by an edge at D_n.

For a graph Γ_7, the sequence of chain vertices is: 0101010, 0101000, 0100000, 0000000, 1000000, 1010000, 1010100, 1010101 .

For a graph Γ_8, the sequence of chain vertices D_8 is: 01010101, 01010100, 01010000, 01000000, . 00000000, 10000000, 10100000, 10101000, 101010.

Let us write down the sequence of edge labels for each case in the order of their sequence in the indicated chains. First for Γ_7: $F_2, F_4, F_6, F_7, F_5, F_3, F_1$ then for Γ_8: $F_1, F_3, F_5, F_7, F_8, F_6, F_4, F_2$. We denote D_n^F a sequence of this kind for Γ_n . There is not a single diametrical chain between the indicated vertices. However, the result is similar.

We conclude that:

Θ-classes: $F_2, F_4, F_6, \ldots, F_{2k-2}, F_{2k}, F_{2k+1}, F_{2k-1}, \ldots, F_5, F_3, F_1$ set D_n^F for any $n = 2k+1$;

Θ-classes:

$$F_1, F_3, F_5, \ldots, F_{2k-3}, F_{2k-1}, F_{2k}, F_{2k-2}, \ldots, F_6, F_4, F_2$$

set D_n^F for any $n = 2k$, where $k = 1, 2, \ldots$ and $F_i \in V(\Gamma_n)$, $i = 1, 2, \ldots, n$. Using D_n^F it is not difficult to construct D_n, since the values of the labels of the diametrically opposite vertices are known.

Conclusion

The popularity of Fibonacci cubes is due to their wide range of uses. In mathematical chemistry, this concept is used in the study of hexagonal graphs.

In computer science, Fibonacci cubes are interesting from an algorithmic point of view. The problem of embedding other graphs in Fibonacci cubes is of interest.

A new class of subgraphs of Fibonacci cubes is distinguished in the paper. Alternative definitions of the Fibonacci cube are considered. In one case, it is a subgraph of a hypercube. The next definition is related to median graphs. This made it possible to obtain versatile approaches to the study of properties of the super Fibonacci graceful graphs and to reveal the properties inherited from partial cubes.

This article discusses the structural properties of graphs that allow one to obtain new classes of super Fibonacci graceful graphs. The connection between the square 1-factorization of a hypercube and its - classes is demonstrated.

The fundamental decomposition of the Fibonacci cube is used to describe its new properties. The range of applications for the Fibonacci cube is expanding. Therefore, the continuation of the study of its properties is an urgent task.

These properties can be used to determine the structural and numerical invariants of such graphs. The results obtained can be useful to specialists in the field of coding theory, theory of parallel computing, theoretical chemistry, as well as in the study of interconnection topology.

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СУПЕР ФІБОНАЧЧІ ГРАЦІОЗНІ ГРАФИ І КУБИ ФІБОНАЧЧІ

Вступ. Популярність кубів Фібоначчі пов'язана з їх широким колом застосувань. У математичній хімії це поняття використовується при дослідженні гексагональних графів. В інформатиці куби Фібоначчі цікаві з алгоритмічної точки зору. В. Хсу їх ввів в 1993 році для моделювання з'єднань багатопроцесорних обчислювальних мереж. Він хотів отримати графи з властивостями гіперкуба, порядок яких не є ступенем двійки. Тому становить інтерес проблема вкладення інших графів в куби Фібоначчі.

Мета даної статті полягає в дослідженні взаємозв'язку між супер Фібоначчі граціозними графами і кубами Фібоначчі.

Методи. При доведенні теорем використані структурні властивості кубів Фібоначчі, методи теорії графів і теорії розміток.

Результати. Виділено новий клас підграфів кубів Фібоначчі. Розглянуто альтернативні визначення куба Фібоначчі. В одному випадку — це підграф гіперкуба. Наступне визначення пов'язане з медіанними графами. Це дозволило отримати різносторонні підходи до вивчення властивостей супер Фібоначчі граціозних графів і виявити властивості, успадковані від часткових кубів.

Висновок. Супер Фібоначчі граціозні графи, як підграфи кубів Фібоначчі, мають привабливу рекурсивну структуру і пов'язані з нею властивості. За допомогою цих властивостей можна визначити структурні та чисельні інваріанти таких графів. Отримані результати можуть бути корисними фахівцям в області теорії кодування, теорії паралельних обчислень, теоретичної хімія, а також при вивченні топології міжмережевих з'єднань.

Ключові слова: *гіперкуб, куб Фібоначчі, супер Фібоначчі граціозна розмітка граф.*