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A NEW METHOD OF THE LOGICAL FUNCTIONS MINIMIZATION IN THE POLYNOMIAL SET-THEORETICAL FORMAT. «HANDSHAKING» PROCEDURE

A new minimization method of logic functions of n variables in polynomial set-theoretical format has been considered. The method based on the so-called "handshaking" procedure. This procedure reflects the iterative polynomial extension of two conjuncterms of different ranks, the Hamming distance between which can be arbitrary. The advantages of the suggested method are illustrated by the examples.

Keywords: *minimization of the logical functions, conjuncterms, polynomial set-theoretical format, Hamming distance, «handshaking» procedure.*

Introduction

Despite the benefits, the implementation of procedures for minimization of logical functions in ESOP (EXOR Sum-Of-Product) is complicated in comparison to that of logical functions in SOP (Sum-Of-Product) [1–8]. As mentioned in parts 1–3 of the previously published articles[1] about the method for minimization of logical functions in ESOP, one of the significant reasons of the implementation complexity is that the simplification procedures [20–26] are not generalized regarding the Hamming distance d between two arbitrary conjuncterms with different ranks: thus the final minimization of the given function is not guaranteed.

In the mentioned above method, a generalized approach to the minimization of arbitrary set functions (*see* parts 1 and 2) and systems of such functions (*see* part 3) in ESOP for arbitrary Hamming distances between two conjuncterms with different ranks is proposed for the first time. This approach is based on a visual pattern method [28, 31], as a result of which a set of transformed ESOP conjuncterms, having lower rank than rank of two initial (given) conjuncterms, is formed. Transformed conjuncterms can be used for further to simplification of the given function to decrease simplificantly the implementation cost, reflected by the ratio k_0^* / k_l^*, where k_0^* is the amount of conjuncterms, and k_l^* is the amount of literals of the minimized function.

In parts 1 and 2, there are formulated and proved theorems for different possible initial transformation conditions that are determined by the ranks of the initial conjuncterms pairs. In particulary, Theorem 1 (T1) considers two minterms, Theorem

1 УСиМ, 2015, № 2 (part 1), № 4 (part 2), № 5 (part 3).

Table **4.1**

d	Theorem 1	Theorem 2	Theorem 3
1	$\begin{pmatrix}\bar\alpha\\ \alpha\end{pmatrix}\overset{\circ}{\Rightarrow}(-)$	$\begin{pmatrix}-\\ \tilde\alpha\end{pmatrix}\overset{\circ}{\Rightarrow}(\bar{\bar\alpha})$	-
2	$\begin{pmatrix}\bar\alpha\bar\beta\\ \alpha\beta\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha-\\ -\beta\end{pmatrix},\begin{pmatrix}\alpha-\\ -\bar\beta\end{pmatrix}\right\}$	$\begin{pmatrix}\bar\alpha-\\ \alpha\tilde\beta\end{pmatrix}\overset{\circ}{\Rightarrow}\begin{pmatrix}--\\ \alpha\bar{\bar\beta}\end{pmatrix}$ $\begin{pmatrix}-\bar\beta\\ \tilde\alpha\beta\end{pmatrix}\overset{\circ}{\Rightarrow}\begin{pmatrix}--\\ \bar{\bar\alpha}\beta\end{pmatrix}$	$\begin{pmatrix}\tilde\alpha-\\ -\tilde\beta\end{pmatrix}\overset{\circ}{\Rightarrow}\begin{pmatrix}\bar{\bar\alpha}-\\ -\bar{\bar\beta}\end{pmatrix}$
3	$\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma\\ \alpha\beta\gamma\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha\bar\beta-\\ \bar\alpha-\gamma\\ -\beta\gamma\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta-\\ -\bar\beta\gamma\\ \alpha-\gamma\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\\ -\beta\bar\gamma\\ \alpha\beta-\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\\ \bar\alpha\beta-\\ -\beta\gamma\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\\ \alpha\bar\beta-\\ \alpha-\gamma\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\\ \alpha-\bar\gamma\\ \alpha\beta-\end{pmatrix}\right\}$	$\begin{pmatrix}\bar\alpha\bar\beta-\\ \alpha\beta\tilde\gamma\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha--\\ -\beta-\\ \alpha\beta\bar{\bar\gamma}\end{pmatrix},\begin{pmatrix}-\bar\beta-\\ \alpha--\\ \alpha\beta\bar{\bar\gamma}\end{pmatrix}\right\}$ $\begin{pmatrix}\bar\alpha-\bar\gamma\\ \alpha\tilde\beta\gamma\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha--\\ --\gamma\\ \alpha\bar{\bar\beta}\gamma\end{pmatrix},\begin{pmatrix}--\bar\gamma\\ \alpha--\\ \alpha\bar{\bar\beta}\gamma\end{pmatrix}\right\}$ $\begin{pmatrix}-\bar\beta\bar\gamma\\ \tilde\alpha\beta\gamma\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}-\bar\beta-\\ --\gamma\\ \bar{\bar\alpha}\beta\gamma\end{pmatrix},\begin{pmatrix}--\bar\gamma\\ -\beta-\\ \bar{\bar\alpha}\beta\gamma\end{pmatrix}\right\}$	$\begin{pmatrix}\tilde\alpha\bar\beta-\\ -\beta\tilde\gamma\end{pmatrix}\overset{\circ}{\Rightarrow}\begin{pmatrix}---\\ \bar{\bar\alpha}\bar\beta-\\ -\beta\bar{\bar\gamma}\end{pmatrix}$ $\begin{pmatrix}\bar\alpha\tilde\beta-\\ \alpha-\tilde\gamma\end{pmatrix}\overset{\circ}{\Rightarrow}\begin{pmatrix}---\\ \bar\alpha\bar{\bar\beta}-\\ \alpha-\bar{\bar\gamma}\end{pmatrix}$ $\begin{pmatrix}\tilde\alpha-\bar\gamma\\ -\tilde\beta\gamma\end{pmatrix}\overset{\circ}{\Rightarrow}\begin{pmatrix}---\\ \bar{\bar\alpha}-\bar\gamma\\ -\bar{\bar\beta}\gamma\end{pmatrix}$
4	$\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma\bar\delta\\ \alpha\beta\gamma\delta\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha\bar\beta\gamma-\\ \bar\alpha\bar\beta-\delta\\ \bar\alpha-\gamma\delta\\ -\beta\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta\gamma-\\ \bar\alpha\bar\beta-\delta\\ -\beta\gamma\delta\\ \alpha-\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma-\\ \bar\alpha-\bar\gamma\delta\\ -\beta\bar\gamma\delta\\ \alpha\beta-\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma-\\ \bar\alpha-\bar\gamma\delta\\ \bar\alpha\beta-\delta\\ -\beta\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta\gamma-\\ -\bar\beta\bar\gamma\delta\\ \alpha\bar\beta-\delta\\ \alpha-\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta\gamma-\\ -\bar\beta\bar\gamma\delta\\ \alpha-\bar\gamma\delta\\ \alpha\beta-\delta\end{pmatrix},\right.$ $\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ \bar\alpha-\gamma\bar{\bar\delta}\\ -\beta\gamma\bar{\bar\delta}\\ \alpha\beta\gamma-\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ \bar\alpha-\gamma\bar{\bar\delta}\\ \bar\alpha\beta\gamma-\\ -\beta\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ -\bar\beta\gamma\bar\delta\\ \alpha\bar\beta\gamma-\\ \alpha-\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ -\bar\beta\gamma\bar\delta\\ \alpha-\gamma\bar{\bar\delta}\\ \alpha\beta\gamma-\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ \bar\alpha\bar\beta\gamma-\\ \bar\alpha-\gamma\delta\\ -\beta\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ \bar\alpha\bar\beta\gamma-\\ -\bar\beta\gamma\delta\\ \alpha-\gamma\delta\end{pmatrix},$ $\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ -\beta\bar\gamma\bar{\bar\delta}\\ \alpha\beta\bar\gamma-\\ \alpha\beta-\delta\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ -\beta\bar\gamma\bar{\bar\delta}\\ \alpha\beta-\bar\delta\\ \alpha\beta\gamma-\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ \bar\alpha\beta\bar\gamma-\\ \bar\alpha\beta-\delta\\ -\beta\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ \bar\alpha\beta\bar\gamma-\\ -\beta\bar\gamma\delta\\ \alpha\beta-\delta\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ \bar\alpha\beta-\bar\delta\\ -\beta\gamma\bar\delta\\ \alpha\beta\gamma-\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ \bar\alpha\beta-\bar{\bar\delta}\\ \bar\alpha\beta\gamma-\\ -\beta\gamma\delta\end{pmatrix},$ $\left.\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \alpha\bar\beta\bar\gamma-\\ \alpha\bar\beta-\delta\\ \alpha-\gamma\delta\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \alpha\bar\beta\bar\gamma-\\ \alpha-\bar\gamma\delta\\ \alpha\beta-\delta\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \alpha\bar\beta-\bar\delta\\ \alpha-\gamma\bar\delta\\ \alpha\beta\gamma-\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \alpha\bar\beta-\bar\delta\\ \alpha\bar\beta\gamma-\\ \alpha-\gamma\delta\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \alpha-\bar\gamma\bar\delta\\ \alpha\beta\bar\gamma-\\ \alpha\beta-\delta\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \alpha-\bar\gamma\bar{\bar\delta}\\ \alpha\beta-\bar\delta\\ \alpha\beta\gamma-\end{pmatrix}\right\}$	$\begin{pmatrix}\bar\alpha\bar\beta\gamma-\\ \alpha\beta\gamma\tilde\delta\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha\bar\beta--\\ \bar\alpha-\gamma-\\ -\beta\gamma-\\ \alpha\beta\gamma\bar{\bar\delta}\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta--\\ -\bar\beta\gamma-\\ \alpha-\gamma-\\ \alpha\beta\gamma\bar{\bar\delta}\end{pmatrix},\begin{pmatrix}\bar\alpha-\gamma-\\ \bar\alpha\beta--\\ -\beta\gamma-\\ \alpha\beta\gamma\bar{\bar\delta}\end{pmatrix},\begin{pmatrix}\bar\alpha-\gamma-\\ -\beta\bar\gamma-\\ \alpha\beta--\\ \alpha\beta\gamma\bar{\bar\delta}\end{pmatrix},\begin{pmatrix}-\bar\beta\gamma-\\ \alpha\bar\beta--\\ \alpha-\gamma-\\ \alpha\beta\gamma\bar{\bar\delta}\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma-\\ \alpha-\bar\gamma-\\ \alpha\beta--\\ \alpha\beta\gamma\bar{\bar\delta}\end{pmatrix}\right\}$ $\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ \alpha\beta\tilde\gamma\delta\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha\bar\beta--\\ \bar\alpha--\delta\\ -\beta-\delta\\ \alpha\beta\bar{\bar\gamma}\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta--\\ -\bar\beta-\delta\\ \alpha--\delta\\ \alpha\beta\bar{\bar\gamma}\delta\end{pmatrix},\begin{pmatrix}\bar\alpha--\bar\delta\\ \bar\alpha\beta--\\ -\beta-\delta\\ \alpha\beta\bar{\bar\gamma}\delta\end{pmatrix},\begin{pmatrix}\bar\alpha--\bar\delta\\ -\beta-\bar\delta\\ \alpha\beta--\\ \alpha\beta\bar{\bar\gamma}\delta\end{pmatrix},\begin{pmatrix}-\bar\beta-\bar\delta\\ \alpha\bar\beta--\\ \alpha--\delta\\ \alpha\beta\bar{\bar\gamma}\delta\end{pmatrix},\begin{pmatrix}-\bar\beta-\bar\delta\\ \alpha--\bar\delta\\ \alpha\beta--\\ \alpha\beta\bar{\bar\gamma}\delta\end{pmatrix}\right\}$ $\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ \alpha\tilde\beta\gamma\delta\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha-\bar\gamma-\\ \bar\alpha--\delta\\ --\gamma\delta\\ \alpha\bar{\bar\beta}\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma-\\ --\bar\gamma\delta\\ \alpha--\delta\\ \alpha\bar{\bar\beta}\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha--\delta\\ --\gamma\bar\delta\\ \alpha-\gamma-\\ \alpha\bar{\bar\beta}\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha--\bar\delta\\ \bar\alpha-\gamma-\\ --\gamma\delta\\ \alpha\bar{\bar\beta}\gamma\delta\end{pmatrix},\begin{pmatrix}--\bar\gamma\bar\delta\\ \alpha-\bar\gamma-\\ \alpha--\delta\\ \alpha\bar{\bar\beta}\gamma\delta\end{pmatrix},\begin{pmatrix}--\bar\gamma\delta\\ \alpha--\bar\delta\\ \alpha-\gamma-\\ \alpha\bar{\bar\beta}\gamma\delta\end{pmatrix}\right\}$ $\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \tilde\alpha\beta\gamma\delta\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}-\bar\beta\bar\gamma-\\ -\bar\beta-\bar\delta\\ --\bar\gamma\bar\delta\\ \bar{\bar\alpha}\beta\gamma\delta\end{pmatrix},\begin{pmatrix}-\bar\beta\gamma-\\ --\bar\gamma\delta\\ -\beta-\delta\\ \bar{\bar\alpha}\beta\gamma\delta\end{pmatrix},\begin{pmatrix}-\bar\beta-\bar\delta\\ -\bar\beta\gamma-\\ --\gamma\delta\\ \bar{\bar\alpha}\beta\gamma\delta\end{pmatrix},\begin{pmatrix}-\bar\beta-\bar\delta\\ --\gamma\bar\delta\\ -\beta\gamma-\\ \bar{\bar\alpha}\beta\gamma\delta\end{pmatrix},\begin{pmatrix}--\gamma\bar\delta\\ -\beta\bar\gamma-\\ -\beta-\delta\\ \bar{\bar\alpha}\beta\gamma\delta\end{pmatrix},\begin{pmatrix}--\bar\gamma\bar\delta\\ -\beta-\bar\delta\\ -\beta\gamma-\\ \bar{\bar\alpha}\beta\gamma\delta\end{pmatrix}\right\}$	$\begin{pmatrix}\tilde\alpha\bar\beta\bar\gamma-\\ -\beta\gamma\tilde\delta\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}-\bar\beta--\\ --\gamma-\\ \bar{\bar\alpha}\bar\beta\bar\gamma-\\ -\beta\gamma\bar{\bar\delta}\end{pmatrix},\begin{pmatrix}-\beta--\\ --\bar\gamma-\\ \bar{\bar\alpha}\beta\bar\gamma-\\ -\beta\gamma\bar{\bar\delta}\end{pmatrix}\right\}$ $\begin{pmatrix}\alpha\tilde\beta\bar\gamma-\\ \alpha-\gamma\tilde\delta\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha---\\ --\gamma-\\ \bar\alpha\bar{\bar\beta}\bar\gamma-\\ \alpha-\gamma\bar{\bar\delta}\end{pmatrix},\begin{pmatrix}\alpha---\\ --\bar\gamma-\\ \bar\alpha\bar{\bar\beta}\bar\gamma-\\ \alpha-\gamma\bar{\bar\delta}\end{pmatrix}\right\}$ $\begin{pmatrix}\bar\alpha\bar\beta\tilde\gamma-\\ \alpha\beta-\bar\delta\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha---\\ -\beta--\\ \bar\alpha\bar\beta\bar{\bar\gamma}-\\ \alpha\beta-\bar{\bar\delta}\end{pmatrix},\begin{pmatrix}\alpha---\\ -\bar\beta--\\ \bar\alpha\bar\beta\bar{\bar\gamma}-\\ \alpha\beta-\bar{\bar\delta}\end{pmatrix}\right\}$ $\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ -\beta\tilde\gamma\delta\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}-\bar\beta--\\ ---\delta\\ \bar{\bar\alpha}\bar\beta-\bar\delta\\ -\beta\bar{\bar\gamma}\delta\end{pmatrix},\begin{pmatrix}-\beta--\\ ---\bar\delta\\ \bar{\bar\alpha}\bar\beta-\bar\delta\\ -\beta\bar{\bar\gamma}\delta\end{pmatrix}\right\}$ $\begin{pmatrix}\bar\alpha\tilde\beta-\bar\delta\\ \alpha-\bar\gamma\delta\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha---\\ ---\delta\\ \bar\alpha\bar{\bar\beta}-\delta\\ \alpha-\bar{\bar\gamma}\delta\end{pmatrix},\begin{pmatrix}\alpha---\\ ---\bar\delta\\ \bar\alpha\bar{\bar\beta}-\bar\delta\\ \alpha-\bar{\bar\gamma}\delta\end{pmatrix}\right\}$ $\begin{pmatrix}\tilde\alpha-\bar\gamma\bar\delta\\ -\tilde\beta\gamma\delta\end{pmatrix}\overset{\circ}{\Rightarrow}\left\{\begin{pmatrix}--\bar\gamma-\\ ---\delta\\ \bar{\bar\alpha}-\bar\gamma\bar\delta\\ -\bar{\bar\beta}\gamma\delta\end{pmatrix},\begin{pmatrix}--\gamma-\\ ---\bar\delta\\ \bar{\bar\alpha}-\bar\gamma\bar\delta\\ -\bar{\bar\beta}\gamma\delta\end{pmatrix}\right\}$
...	...	...	...
d	$d\times d!$	$d(d\times(d-1)!)$	$d\times(d-2)!\sum_{i=1}^{d}(d-i)$

2(T2) — a minterm and a conjuncterm of (n-1)-rank and Theorem 3 (T3) — two conjuncterms, one of (n-1)-rank and the other of (n-2)-rank.

Table 4.1 demonstrates the generalized formulas of sets the PSTF $Y^{\oplus}$ of transformed conjuncterms, reflected in T1, T2 and T3 for $d = 1, 2, 3$ and 4. For arbitrary initial d (*see* the last row in Table 4.1) the general amount k_θ^* of transformed conjuncterms will be determined by the power set of each set: for T1 that is one set of power set $d!$, for T2 that is d sets, each of powerset $(d-1)!$, and for T3 that is $\sum_{i=1}^{d}(d-i)$ sets, each of powerset $(d-2)!$. For example, if d=4 we will get the following amount of transformed conjuncterms: for T1 that is $k_\theta^* = 96$, for T2 that is $k_\theta^* = 96$, for T3 that is $k_\theta^* = 48$.

Minimization of logical function in ESOP using the transformed conjuncterms can be realized by two approaches. In the given function the distance d between an arbitrary pair of conjuncterms (or minterms, if the function is in perfect STF Y^1) is determined. Then, this function, with chosen some pair of (initial) conjuncterms with certain d_i, can be replaced by a set of conjuncterms using:

- a pre-formed library (its full set depends on the computer memory capacity for $d = d_{\max}$), from which for a specific given d_i a set of transformed conjuncterms is determined; see in Table 4.1 an example of such library for $d_{\max} = 4$;
- an iterative polynomial extension procedure of two initial conjuncterms with arbitrary ranks, spaced at a distance d, which will be called the *"handshaking" procedure*; the proposed name in the simplest case (according to Theorem T1) reflects conditionally, on the pattern of a given function, the motion of one vertex (minterm) to a distant distance $d > 1$ of the second vertex (minterm) on the edges

of the pattern (conjuncterms of the (n-1)-rank), that creates some (required) set of transformed conjuncterms of the (n-1)-rank (*see* part 1.2, Fig. 1 and 2).

Unlike the first approach having the significant limitations regarding the computer position and requiring the value of d, the second approach does not have such restrictions. Let us consider the latter in more details.

"Handshaking" Procedure

The "handshaking" procedure is based on Theorem 2 for (6) (see part 1), i.e. $\begin{pmatrix} - \\ \tilde{\alpha}_i \end{pmatrix} \overset{\oplus}{\Rightarrow} \bar{\tilde{\alpha}}_i$, where $\alpha_i \in \{0,1\}$, $i \in \{0,1,...,(n-1)\}$ is the arbitrary binary position in conjuncterm of rank $r \in \{1,2,\ldots,n\}$. In particular, for minterm $(\alpha_1 \cdots \alpha_i \cdots \alpha_n)$ from the function $f(x_1,...,x_i,...,x_n)$ the «handshaking» procedure is performed to i-th position 2^i looks as follows:

$$(\alpha_1 \cdots \alpha_i \cdots \alpha_n) \overset{2^i}{\Rightarrow} \begin{pmatrix} (\alpha_1 \cdots (-)_i \cdots \alpha_n) \\ (\alpha_1 \cdots \bar{\alpha}_i \cdots \bar{\alpha}_n) \end{pmatrix}.$$

The procedure is similar if we have a pair of conjuncterms with their ranks differing by one. In such case, the complete transformed conjuncterms set formation will combinatorially depend on the weights order of binary positions to which this procedure is performed.

Let us consider the transformed conjuncterms set formation using the «handshaking» procedure for theorems T1, T2 and T3 (*see* part 1).

Implementation of the Transformed Conjuncterms Set Based on Theorem T1

In general for $d = 2$ we obtain a transformed conjuncterms set for two binary position sequences $2^0 \to 2^1$ and $2^1 \to 2^0$. Thus, for generative minterm $(\overline{\alpha}\,\overline{\beta})$ we get:

$$\begin{pmatrix} \bar\alpha\bar\beta \\ \alpha\beta \end{pmatrix} \overset{2^0}{\Rightarrow} \begin{pmatrix} \bar\alpha- \\ \bar\alpha\beta \\ \alpha\beta \end{pmatrix} \overset{2^1}{\Rightarrow} \begin{pmatrix} \bar\alpha- \\ -\beta \\ \alpha\beta \\ \alpha\beta \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} \bar\alpha- \\ -\beta \end{pmatrix};$$

$$\begin{pmatrix} \bar\alpha\bar\beta \\ \alpha\beta \end{pmatrix} \overset{2^1}{\Rightarrow} \begin{pmatrix} -\bar\beta \\ \alpha\bar\beta \\ \alpha\beta \end{pmatrix} \overset{2^0}{\Rightarrow} \begin{pmatrix} -\bar\beta \\ \alpha- \\ \alpha\beta \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} -\bar\beta \\ \alpha- \end{pmatrix}.$$

For generative minterm $(\alpha\,\beta)$ the result will be the same:

$$\begin{pmatrix} \bar\alpha\bar\beta \\ \alpha\beta \end{pmatrix} \overset{2^0}{\Rightarrow} \begin{pmatrix} \alpha- \\ \alpha\bar\beta \\ \bar\alpha\bar\beta \end{pmatrix} \overset{2^1}{\Rightarrow} \begin{pmatrix} \alpha- \\ -\bar\beta \\ \bar\alpha\bar\beta \\ \bar\alpha\bar\beta \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} \alpha- \\ -\bar\beta \end{pmatrix};$$

$$\begin{pmatrix} \bar\alpha\bar\beta \\ \alpha\beta \end{pmatrix} \overset{2^1}{\Rightarrow} \begin{pmatrix} -\beta \\ \bar\alpha\beta \\ \bar\alpha\bar\beta \end{pmatrix} \overset{2^0}{\Rightarrow} \begin{pmatrix} -\beta \\ \bar\alpha- \\ \bar\alpha\bar\beta \\ \bar\alpha\bar\beta \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} -\beta \\ \bar\alpha- \end{pmatrix}.$$

Therefore, $\begin{pmatrix} \bar\alpha\bar\beta \\ \alpha\beta \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} \bar\alpha- \\ -\beta \end{pmatrix}, \begin{pmatrix} -\bar\beta \\ \alpha- \end{pmatrix} \right\}$, that corre sponds to (3) in part 1.

For $d = 3$, considering all possible sequences, we will get the following set of transformed conjuncterms of (n-1)-rank:

$$\begin{pmatrix} \bar\alpha\bar\beta\bar\gamma \\ \alpha\beta\gamma \end{pmatrix} \overset{2^0}{\Rightarrow} \begin{pmatrix} \bar\alpha\bar\beta- \\ \bar\alpha\bar\beta\gamma \\ \alpha\beta\gamma \end{pmatrix} \overset{2^1}{\Rightarrow} \begin{pmatrix} \bar\alpha\bar\beta- \\ \bar\alpha-\gamma \\ \bar\alpha\beta\gamma \\ \alpha\beta\gamma \end{pmatrix} \overset{2^2}{\Rightarrow} \begin{pmatrix} \bar\alpha\bar\beta- \\ \bar\alpha-\gamma \\ -\beta\gamma \end{pmatrix};$$

$$\begin{pmatrix} \bar\alpha\bar\beta\bar\gamma \\ \alpha\beta\gamma \end{pmatrix} \overset{2^0}{\Rightarrow} \begin{pmatrix} \bar\alpha\bar\beta- \\ \bar\alpha\bar\beta\gamma \\ \alpha\beta\gamma \end{pmatrix} \overset{2^2}{\Rightarrow} \begin{pmatrix} \bar\alpha\bar\beta- \\ -\bar\beta\gamma \\ \alpha\bar\beta\gamma \\ \alpha\beta\gamma \end{pmatrix} \overset{2^1}{\Rightarrow} \begin{pmatrix} \bar\alpha\bar\beta- \\ -\bar\beta\gamma \\ \alpha-\gamma \end{pmatrix};$$

$$\begin{pmatrix} \bar\alpha\bar\beta\bar\gamma \\ \alpha\beta\gamma \end{pmatrix} \overset{2^1}{\Rightarrow} \begin{pmatrix} \bar\alpha-\bar\gamma \\ \alpha\beta\bar\gamma \\ \alpha\beta\gamma \end{pmatrix} \overset{2^2}{\Rightarrow} \begin{pmatrix} \bar\alpha-\bar\gamma \\ -\beta\bar\gamma \\ \alpha\beta\bar\gamma \\ \alpha\beta\gamma \end{pmatrix} \overset{2^0}{\Rightarrow} \begin{pmatrix} \bar\alpha-\bar\gamma \\ -\beta\bar\gamma \\ \alpha\beta- \end{pmatrix};$$

$$\begin{pmatrix} \bar\alpha\bar\beta\bar\gamma \\ \alpha\beta\gamma \end{pmatrix} \overset{2^1}{\Rightarrow} \begin{pmatrix} \bar\alpha-\bar\gamma \\ \bar\alpha\beta\bar\gamma \\ \alpha\beta\gamma \end{pmatrix} \overset{2^0}{\Rightarrow} \begin{pmatrix} \bar\alpha-\bar\gamma \\ \bar\alpha\beta- \\ \bar\alpha\beta\gamma \\ \alpha\beta\gamma \end{pmatrix} \overset{2^2}{\Rightarrow} \begin{pmatrix} \bar\alpha-\bar\gamma \\ \bar\alpha\beta- \\ -\beta\gamma \end{pmatrix};$$

$$\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma\\ \alpha\beta\gamma\end{pmatrix}\overset{2^2}{\Rightarrow}\begin{pmatrix}-\bar\beta\bar\gamma\\ \alpha\bar\beta\bar\gamma\\ \alpha\beta\gamma\end{pmatrix}\overset{2^0}{\Rightarrow}\begin{pmatrix}-\bar\beta\bar\gamma\\ \alpha\bar\beta-\\ \alpha\bar\beta\gamma\\ \alpha\beta\gamma\end{pmatrix}\overset{2^1}{\Rightarrow}\begin{pmatrix}-\bar\beta\bar\gamma\\ \alpha\bar\beta-\\ \alpha-\gamma\end{pmatrix};$$

$$\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma\\ \alpha\beta\gamma\end{pmatrix}\overset{2^2}{\Rightarrow}\begin{pmatrix}-\bar\beta\bar\gamma\\ \alpha\bar\beta\bar\gamma\\ \alpha\beta\gamma\end{pmatrix}\overset{2^1}{\Rightarrow}\begin{pmatrix}-\bar\beta\bar\gamma\\ \alpha-\bar\gamma\\ \alpha\beta\bar\gamma\\ \alpha\beta\gamma\end{pmatrix}\overset{2^0}{\Rightarrow}\begin{pmatrix}-\bar\beta\bar\gamma\\ \alpha-\bar\gamma\\ \alpha\beta-\end{pmatrix}.$$

Therefore, $\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma\\ \alpha\beta\gamma\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha\bar\beta-\\ \bar\alpha-\gamma\\ -\beta\gamma\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta-\\ -\bar\beta\gamma\\ \alpha-\gamma\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\\ -\beta\bar\gamma\\ \alpha\beta-\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma\\ \bar\alpha\beta-\\ -\beta\gamma\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\\ \alpha\bar\beta-\\ \alpha-\gamma\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma\\ \alpha-\bar\gamma\\ \alpha\beta-\end{pmatrix}\right\}$ that corresponds to (4).

For example, for minterms pair $\begin{pmatrix}000\\ 111\end{pmatrix}$ we will get the following:

$\begin{pmatrix}000\\ 111\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}00-\\ 0-1\\ -11\end{pmatrix},\begin{pmatrix}00-\\ -01\\ 1-1\end{pmatrix},\begin{pmatrix}0-0\\ -10\\ 11-\end{pmatrix},\begin{pmatrix}0-0\\ 01-\\ -11\end{pmatrix},\begin{pmatrix}-00\\ 10-\\ 1-1\end{pmatrix},\begin{pmatrix}-00\\ 1-0\\ 11-\end{pmatrix}\right\}$, where the first subset, for instance, is formed as follows:

$$\begin{pmatrix}000\\ 111\end{pmatrix}\overset{2^0}{\Rightarrow}\begin{pmatrix}00-\\ 001\\ 111\end{pmatrix}\overset{2^1}{\Rightarrow}\begin{pmatrix}00-\\ 0-1\\ 011\\ 111\end{pmatrix}\overset{2^2}{\Rightarrow}\begin{pmatrix}00-\\ 0-1\\ -11\end{pmatrix}.$$

The same can be realized for sets of transformed conjuncterms of rank (n-1) for $d>3$.

Implementation of the Ttransformed Conjuncterms Set Based on Theorem T2

For $d=2$ the minterm extension procedure is easier performed on position, where the conjuncterm of (n-1)-rank has a line (–), namely:

$$\begin{pmatrix}\bar\alpha-\\ \alpha\tilde\beta\end{pmatrix}\overset{2^0}{\Rightarrow}\begin{pmatrix}\bar\alpha-\\ \alpha-\\ \alpha\bar{\tilde\beta}\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}--\\ \alpha\bar{\tilde\beta}\end{pmatrix}\text{ and }\begin{pmatrix}-\bar\beta\\ \tilde\alpha\beta\end{pmatrix}\overset{2^1}{\Rightarrow}\begin{pmatrix}-\bar\beta\\ -\beta\\ \bar{\tilde\alpha}\beta\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}--\\ \bar{\tilde\alpha}\beta\end{pmatrix},$$

that corresponds to (7): $\begin{pmatrix}\bar\alpha-\\ \alpha\tilde\beta\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}--\\ \alpha\bar{\tilde\beta}\end{pmatrix}$ and $\begin{pmatrix}-\bar\beta\\ \tilde\alpha\beta\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}--\\ \bar{\tilde\alpha}\beta\end{pmatrix}$.

For $d=3$ considering T1 for $d=2$ (3) we obtain (8), (9) and (10), accordingly:

$$\begin{pmatrix}\bar\alpha\bar\beta-\\ \alpha\beta\tilde\gamma\end{pmatrix}\overset{2^0}{\Rightarrow}\begin{pmatrix}\bar\alpha\bar\beta-\\ \alpha\beta-\\ \alpha\beta\bar{\tilde\gamma}\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha--\\ -\beta-\\ \alpha\beta\bar{\tilde\gamma}\end{pmatrix},\begin{pmatrix}-\bar\beta-\\ \alpha--\\ \alpha\beta\bar{\tilde\gamma}\end{pmatrix}\right\},$$

$$\begin{pmatrix}\bar\alpha-\bar\gamma\\ \alpha\tilde\beta\gamma\end{pmatrix}\overset{2^1}{\Rightarrow}\begin{pmatrix}\bar\alpha-\bar\gamma\\ \alpha-\gamma\\ \alpha\bar{\tilde\beta}\gamma\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha--\\ --\gamma\\ \alpha\bar{\tilde\beta}\gamma\end{pmatrix},\begin{pmatrix}--\bar\gamma\\ \alpha--\\ \alpha\bar{\tilde\beta}\gamma\end{pmatrix}\right\},$$

$$\begin{pmatrix}-\bar\beta\bar\gamma\\ \tilde\alpha\beta\gamma\end{pmatrix}\overset{2^2}{\Rightarrow}\begin{pmatrix}-\bar\beta\bar\gamma\\ -\beta\gamma\\ \bar{\tilde\alpha}\beta\gamma\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}-\bar\beta-\\ --\gamma\\ \bar{\tilde\alpha}\beta\gamma\end{pmatrix},\begin{pmatrix}--\bar\gamma\\ -\beta-\\ \bar{\tilde\alpha}\beta\gamma\end{pmatrix}\right\}.$$

For example, $\begin{pmatrix}01-\\ 101\end{pmatrix}\overset{2^0}{\Rightarrow}\begin{pmatrix}01-\\ 10-\\ 100\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}0--\\ -0-\\ 100\end{pmatrix},\begin{pmatrix}-1-\\ 1--\\ 100\end{pmatrix}\right\},$

$$\begin{pmatrix}0-0\\ 101\end{pmatrix}\overset{2^1}{\Rightarrow}\begin{pmatrix}0-0\\ 1-1\\ 111\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}0--\\ --1\\ 111\end{pmatrix},\begin{pmatrix}--0\\ 1--\\ 111\end{pmatrix}\right\},$$

$$\begin{pmatrix}-10\\ 101\end{pmatrix}\overset{2^2}{\Rightarrow}\begin{pmatrix}-10\\ -01\\ 001\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}-1-\\ --1\\ 001\end{pmatrix},\begin{pmatrix}--0\\ -0-\\ 001\end{pmatrix}\right\}.$$

For $d=4$ considering T1 for $d=3$ we get (11–14). For demonstration purposes we consider only the case (11), for instance $\begin{pmatrix}101-\\ 0100\end{pmatrix}\overset{2^0}{\Rightarrow}\begin{pmatrix}101-\\ 010-\\ 0101\end{pmatrix}$, by applying the T1 transformation to pair $\begin{pmatrix}101-\\ 010-\end{pmatrix}$ for $d=3$.

For six different possible sequences of binary positions, the formation of transformed conjuncterms for example for $2^1 \to 2^2 \to 2^3$ is presented as follows:

$$\begin{pmatrix}101-\\010-\end{pmatrix}\overset{2^1}{\Rightarrow}\begin{pmatrix}10--\\100-\\010-\end{pmatrix}\overset{2^2}{\Rightarrow}\begin{pmatrix}10--\\1-0-\\110-\\010-\end{pmatrix}\overset{2^3}{\Rightarrow}\begin{pmatrix}10--\\1-0-\\-10-\end{pmatrix}, \text{ and}$$

then we have $\begin{pmatrix}101-\\0100\end{pmatrix}\overset{2^0}{\Longrightarrow}\begin{pmatrix}10--\\1-0-\\-10-\\0101\end{pmatrix}$. For the remaining variants of the binary position sequences we obtain the required set of transformed conjuncterms that corresponds to (11), that is:

$$\begin{pmatrix}001-\\010-\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}10--\\1-0-\\-10-\\0101\end{pmatrix},\begin{pmatrix}10--\\-00-\\0-0-\\0101\end{pmatrix},\begin{pmatrix}1-1-\\-11-\\01--\\0101\end{pmatrix},\begin{pmatrix}1-1-\\11--\\-10-\\0101\end{pmatrix},\begin{pmatrix}-01-\\00--\\0-0-\\0101\end{pmatrix},\begin{pmatrix}-01-\\0-1-\\01--\\0101\end{pmatrix}\right\}.$$

Sets of transformed conjuncterms for $d > 3$ are formed in a similar way.

Implementation of the Transformed Conjuncterms Set Based on Theorem T3

If $d = 2$, the result (15) is obvious. For $d = 3$ we have (16), (17) and (18), which are formed as follows:

$$\begin{pmatrix}\tilde\alpha\bar\beta-\\-\beta\tilde\gamma\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{cases}\overset{2^2}{\Rightarrow}\begin{pmatrix}-\bar\beta-\\\bar{\tilde\alpha}\bar\beta-\end{pmatrix}\\\overset{2^0}{\Rightarrow}\begin{pmatrix}-\beta-\\-\beta\bar{\tilde\gamma}\end{pmatrix}\end{cases}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---\\\bar{\tilde\alpha}\bar\beta-\\-\beta\bar{\tilde\gamma}\end{pmatrix};$$

$$\begin{pmatrix}\bar\alpha\tilde\beta-\\\alpha-\tilde\gamma\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{cases}\overset{2^1}{\Rightarrow}\begin{pmatrix}\bar\alpha--\\\bar\alpha\bar{\tilde\beta}-\end{pmatrix}\\\overset{2^0}{\Rightarrow}\begin{pmatrix}\alpha--\\\alpha-\bar{\tilde\gamma}\end{pmatrix}\end{cases}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---\\\bar\alpha\bar{\tilde\beta}-\\\alpha-\bar{\tilde\gamma}\end{pmatrix};$$

$$\begin{pmatrix}\tilde\alpha-\bar\gamma\\-\tilde\beta\gamma\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{cases}\overset{2^2}{\Rightarrow}\begin{pmatrix}--\bar\gamma\\\bar{\tilde\alpha}-\bar\gamma\end{pmatrix}\\\overset{2^1}{\Rightarrow}\begin{pmatrix}--\gamma\\-\bar{\tilde\beta}\gamma\end{pmatrix}\end{cases}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---\\\bar{\tilde\alpha}-\bar\gamma\\-\bar{\tilde\beta}\gamma\end{pmatrix}.$$

For instance,

$$\begin{pmatrix}01-\\-01\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{cases}\overset{2^2}{\Rightarrow}\begin{pmatrix}-1-\\11-\end{pmatrix}\\\overset{2^0}{\Rightarrow}\begin{pmatrix}-0-\\-00\end{pmatrix}\end{cases}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---\\11-\\-00\end{pmatrix};$$

$$\begin{pmatrix}01-\\1-0\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{cases}\overset{2^1}{\Rightarrow}\begin{pmatrix}0--\\00-\end{pmatrix}\\\overset{2^0}{\Rightarrow}\begin{pmatrix}1--\\1-1\end{pmatrix}\end{cases}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---\\00-\\1-1\end{pmatrix};$$

$$\begin{pmatrix}0-1\\-00\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{cases}\overset{2^2}{\Rightarrow}\begin{pmatrix}--1\\1-1\end{pmatrix}\\\overset{2^1}{\Rightarrow}\begin{pmatrix}--0\\-10\end{pmatrix}\end{cases}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---\\1-1\\-10\end{pmatrix}.$$

For $d = 4$ we have (19)–(24). This can be demonstrated on the example of the transformed conjuncterms formation for the case (20) considering Theorem T1 for $d = 2$ (3):

$$\begin{pmatrix}000-\\1-10\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{cases}\overset{2^2}{\Rightarrow}\begin{pmatrix}0-0-\\010-\end{pmatrix}\\\overset{2^0}{\Rightarrow}\begin{pmatrix}1-1-\\1-11\end{pmatrix}\end{cases}\overset{\oplus}{\Rightarrow}\begin{pmatrix}0-0-\\010-\\1-1-\\1-11\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}0---\\010-\\--1-\\1-11\end{pmatrix},\begin{pmatrix}--0-\\010-\\1---\\1-11\end{pmatrix}\right\}.$$

If one of the initial conjuncterms has lower rank than described in Theorem T3, the extension procedure should be performed to the conjuncterm with greater rank considering all the possible sequences as it is shown in the example below:

$$\begin{pmatrix}-00-\\1-10\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{array}{l}\overset{2^2}{\Rightarrow}\begin{pmatrix}--0-\\010-\end{pmatrix}\\ \overset{2^0}{\Rightarrow}\begin{pmatrix}1-1-\\1-11\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}--1-\\0-1-\\1-11\end{pmatrix}\end{array}\right.\overset{\oplus}{\Rightarrow}\begin{pmatrix}--0-\\-10-\\--1-\\0-1-\\1-11\end{pmatrix}\overset{\oplus}{\Rightarrow}$$

$$\overset{\oplus}{\Rightarrow}\begin{pmatrix}----\\-10-\\0-1-\\1-11\end{pmatrix};\quad \begin{pmatrix}-00-\\1-10\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{array}{l}\overset{2^2}{\Rightarrow}\begin{pmatrix}--0-\\010-\end{pmatrix}\\ \overset{2^3}{\Rightarrow}\begin{pmatrix}--10\\0-10\end{pmatrix}\overset{2^0}{\Rightarrow}\begin{pmatrix}--1-\\--11\\1-11\end{pmatrix}\end{array}\right.\overset{\oplus}{\Rightarrow}$$

$$\overset{\oplus}{\Rightarrow}\begin{pmatrix}--0-\\-10-\\--1-\\--11\\1-10\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}----\\-10-\\--11\\1-10\end{pmatrix}.$$

Therefore, $\begin{pmatrix}-00-\\1-10\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}----\\-10-\\0-1-\\1-11\end{pmatrix},\begin{pmatrix}----\\-10-\\--1\\0-10\end{pmatrix}\right\}.$

Sets of transformed conjuncterms for $d>4$ are formed in a similar way.

The following example demonstrates the case for $d=5$:

$$\begin{pmatrix}01-10\\--10-\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{array}{l}\overset{2^0}{\Rightarrow}\begin{pmatrix}01-1-\\01-11\end{pmatrix}\overset{2^3}{\Rightarrow}\begin{pmatrix}0--1-\\00-1-\\01-11\end{pmatrix}\overset{2^4}{\Rightarrow}\\ \overset{2^2}{\Rightarrow}\begin{pmatrix}---0-\\--00-\end{pmatrix}\end{array}\right.$$

$$\overset{2^4}{\Rightarrow}\begin{pmatrix}---1-\\1--1-\\00-1-\\01-11\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---1-\\1--1-\\01-1-\\01-11\\---0-\\--00-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}-----\\1--1-\\00-1-\\01-11\\--00-\end{pmatrix};$$

$$\begin{pmatrix}01-10\\--10-\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{array}{l}\overset{2^0}{\Rightarrow}\begin{pmatrix}01-1-\\01-11\end{pmatrix}\overset{2^4}{\Rightarrow}\begin{pmatrix}-1-1-\\11-1-\\01-11\end{pmatrix}\overset{2^3}{\Rightarrow}\\ \overset{2^2}{\Rightarrow}\begin{pmatrix}---0-\\--00-\end{pmatrix}\end{array}\right.$$

$$\overset{2^3}{\Rightarrow}\begin{pmatrix}---1-\\-0-1-\\11-1-\\01-11\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---1-\\-0-1-\\11-1-\\01-11\\---0-\\--00-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}-----\\-0-1-\\11-1-\\01-11\\--00-\end{pmatrix};$$

$$\begin{pmatrix}01-10\\--10-\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{array}{l}\overset{2^4}{\Rightarrow}\begin{pmatrix}-1-10\\11-10\end{pmatrix}\overset{2^0}{\Rightarrow}\begin{pmatrix}-1-1-\\-1-11\\11-10\end{pmatrix}\overset{2^3}{\Rightarrow}\\ \overset{2^2}{\Rightarrow}\begin{pmatrix}---0-\\--00-\end{pmatrix}\end{array}\right.$$

$$\overset{2^3}{\Rightarrow}\begin{pmatrix}---1-\\-0-1-\\-1-11\\11-10\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---1-\\-0-1-\\-1-11\\11-10\\---0-\\--00-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}-----\\-0-1-\\-1-11\\11-10\\--00-\end{pmatrix};$$

$$\begin{pmatrix}01-10\\--10-\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{array}{l}\overset{2^4}{\Rightarrow}\begin{pmatrix}-1-10\\11-10\end{pmatrix}\overset{2^3}{\Rightarrow}\begin{pmatrix}---10\\-0-10\\11-10\end{pmatrix}\overset{2^0}{\Rightarrow}\\ \overset{2^2}{\Rightarrow}\begin{pmatrix}---0-\\--00-\end{pmatrix}\end{array}\right.$$

$$\overset{2^0}{\Rightarrow}\begin{pmatrix}---1-\\---11\\-0-10\\11-10\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---1-\\---11\\-0-10\\11-10\\---0-\\--00-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}-----\\---11\\-0-10\\11-10\\--00-\end{pmatrix};$$

$$\begin{pmatrix}01-10\\--10-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{cases}\overset{2^3}{\Rightarrow}\begin{pmatrix}0--10\\00-10\end{pmatrix}\overset{2^4}{\Rightarrow}\begin{pmatrix}---10\\1--10\\00-10\end{pmatrix}\overset{2^0}{\Rightarrow}\begin{pmatrix}---1-\\---11\\1--10\\00-10\end{pmatrix}\overset{\oplus}{\Rightarrow}\\ \overset{2^2}{\Rightarrow}\begin{pmatrix}---0-\\--00-\end{pmatrix}\end{cases}$$

$$\overset{\oplus}{\Rightarrow}\begin{pmatrix}---1-\\---11\\1--10\\00-10\\---0-\\--00-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}-----\\---11\\1--10\\00-10\\--00-\end{pmatrix};$$

$$\begin{pmatrix}01-10\\--10-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{cases}\overset{2^3}{\Rightarrow}\begin{pmatrix}0--10\\00-10\end{pmatrix}\overset{2^0}{\Rightarrow}\begin{pmatrix}0--1-\\0--11\\00-10\end{pmatrix}\overset{2^4}{\Rightarrow}\begin{pmatrix}---1-\\1--1-\\0--11\\00-10\end{pmatrix}\overset{\oplus}{\Rightarrow}\\ \overset{2^2}{\Rightarrow}\begin{pmatrix}---0-\\--00-\end{pmatrix}\end{cases}$$

$$\overset{\oplus}{\Rightarrow}\begin{pmatrix}---1-\\1--1-\\0--11\\00-10\\---0-\\--00-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}-----\\1--1-\\0--11\\00-10\\--00-\end{pmatrix}.$$

Hence $\begin{pmatrix}01-10\\--10-\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}-----\\1--1-\\00-1-\\01-11\\--00-\end{pmatrix},\begin{pmatrix}-----\\-0-1-\\11-1-\\01-11\\--00-\end{pmatrix},\right.$

$$\left.\begin{pmatrix}-----\\-0-1-\\-1-11\\11-10\\--00-\end{pmatrix},\begin{pmatrix}-----\\---11\\-0-10\\11-10\\--00-\end{pmatrix},\begin{pmatrix}-----\\---11\\1--10\\00-10\\--00-\end{pmatrix},\begin{pmatrix}-----\\1--1-\\0--10\\00-10\\--00-\end{pmatrix}\right\}.$$

Application of "Handshaking" Procedure for Minimization of Logical Functions

As mentioned above, the proposed "handshaking" procedure has no restrictions on the Hamming distance d between two conjuncterms of different ranks. The method of minimization of logical functions using such a procedure will be advantageously distinguished not only from minimization in disjunctive format (where $d=1$), but also from minimization in polynomial format, as per known publications [20–26], where the search for the minimal form of a given function is considered for $d \le 3$. The principal advantage of the proposed minimization method, based on the "handshaking" procedure, is on expansion of the search for a polynomial format of the minimal amount of conjuncterms with minimal rank for a given function. The proposed approach is shown by examples of minimization of complete and incomplete functions.

To illustrate the complete function we will use the so-called cha in function [28], which is particular due to its cyclic core matrix, and both of its minimal SOPs form, that are equivalent regarding the function implementation cost, containing conjuncterms of $(n-1)$-rank only. In particular, the chain function $f(x_1, x_2, x_3, x_4)$ with the perfect STF $Y^1 = \{0,1,2,5,7,10,14,15\}^1$ wewill have two minimization solutions with the same implementation cost [28]:

$$Y^1 = \begin{cases}1.\{(0,1),(2,10),(5,7),(14,15)\}^1\\2.\{(0,2),(1,5),(7,15),(10,14)\}^1\end{cases}.$$

Hence the implementation cost in SOP we have $k_\theta^* / k_l^* = 4/12$.

In ESOP we will obtain the function minimization result as follows. First, in the given function we determine the distance $d = d_{\min}$ between all minterm pairs. In our function the four pairs have $d=1$, for which we will get conjuncterms of 3-rank in solution 1) as per Theorem T1:

$$\begin{pmatrix}000-\\-010\end{pmatrix}\overset{\oplus}{\Rightarrow}(000-),\begin{pmatrix}0010\\1010\end{pmatrix}\overset{\oplus}{\Rightarrow}(-010),\begin{pmatrix}0101\\0111\end{pmatrix}\overset{\oplus}{\Rightarrow}(01-1),\begin{pmatrix}1110\\1111\end{pmatrix}\overset{\oplus}{\Rightarrow}(111-).$$

In a similar way we will get conjuncterms of 3-rank in solution 2):

$$\begin{pmatrix}0000\\0010\end{pmatrix}\overset{\oplus}{\Rightarrow}(00-0),\begin{pmatrix}0001\\0101\end{pmatrix}\overset{\oplus}{\Rightarrow}(0-01),\begin{pmatrix}0111\\1111\end{pmatrix}\overset{\oplus}{\Rightarrow}(-111),\begin{pmatrix}1010\\1110\end{pmatrix}\overset{\oplus}{\Rightarrow}(1-1-).$$

Regarding the conjuncterms pairs in solution 1), there are no pairs with $d = 2$, but with $d = 3$ there are pairs $\begin{pmatrix}000-\\-010\end{pmatrix},\begin{pmatrix}000-\\01-0\end{pmatrix},\begin{pmatrix}-010\\111-\end{pmatrix},\begin{pmatrix}01-1\\111-\end{pmatrix}$. Taking any of them, for example the pair $\begin{pmatrix}000-\\-010\end{pmatrix}$, we can apply Theorem T3 to it:

$$\begin{pmatrix}000-\\-010\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{cases}\overset{2^3}{\Rightarrow}\begin{pmatrix}-00-\\100-\end{pmatrix}\\ \overset{2^0}{\Rightarrow}\begin{pmatrix}-01-\\-011\end{pmatrix}\end{cases}\overset{\oplus}{\Rightarrow}\begin{pmatrix}-0--\\100-\\0-11\end{pmatrix}.$$

By attaching the remaining conjuncterms of 3-rank to the last pair, we obtain the set, where we apply the similar procedures considering Theorems T1 and T2 (*see* the underlined elements):

$$\begin{pmatrix}-0--\\100-\\-011\\01-1\\111-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}-0--\\1-0-\\-011\\01-1\\11--\end{pmatrix}/\begin{pmatrix}-0--\\10--\\-011\\01-1\\1-1-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}----\\1-0-\\-011\\01-1\\01--\end{pmatrix}/\begin{pmatrix}00--\\-011\\01-1\\1-11-\end{pmatrix}\overset{\oplus}{\Rightarrow}$$

$$\overset{\oplus}{\Rightarrow}\begin{pmatrix}----\\1-1-\\-011\\01-0\end{pmatrix}/\begin{pmatrix}0---\\-011\\01-0\\1-1-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}----\\1-0-\\-011\\01-0\end{pmatrix}/\begin{pmatrix}----\\1-0-\\01-0\\-011\end{pmatrix}.$$

While performing the same procedures to the other pairs of conjuncterms of 3-rank for both solution 1) and solution 2), the result remains the same. This indicates that the given chain function minimized in ESOP form, has, unlike to SOP, only one solution, namely:

$$Y^1=\{(0000),(0001),(0010),(0101),(0111),(1010),$$

$$(1110),(1111)\}^1\overset{\oplus}{\Rightarrow}\begin{pmatrix}----\\1-0-\\-011\\01-0\end{pmatrix}.$$

Therefore, the implementation cost for the given function in ESOP is better than in SOP, due to $k_\theta^* / k_l^* = 4/8$.

Furthermore, there are such functions that cannot be minimized in SOP form, while in ESOP they can be minimized thanks to the proposed method. Let consider an example of such function that has four minimization solutions (for $d = 2$):

$$Y^1=\{(0000),(0011),(1111)\}^1\overset{\oplus}{\Rightarrow}$$

$$\overset{\oplus}{\Rightarrow}\begin{cases}1.\ \{(000-),(00-1),(1111)\}^\oplus\\2.\ \{(00-0),(001-),(1111)\}^\oplus\\3.\ \{(0000),(0-11),(-111)\}^\oplus\\4.\ \{(0000),(-011),(1-11)\}^\oplus\end{cases}.$$

By the way, the similar result will be obtaining using Theorem T1 to the minterms pair $\begin{pmatrix}0000\\1111\end{pmatrix}$.

In case of incomplete weakly determined function, which is mainly defined with two sets Y^1 and Y^0, there is appropriate to determine which is specific lower-rank conjuncterms will be formed as a result of in predetermined (from set $Y^{\sim}$) in set Y^1 and will participate in the «handshaking» procedure before performing it (procedure). In order to do that, every minterm of set Y^1 is split into a set of lower-rank conjuncterms, for instance $(n-1)$-rank. If the split set contains minterm(s) of the set Y^0, such conjuncterms is not futher considered. In such way, a set of conjuncterms is formed from all the minterms, and afterwards, the «handshaking» procedure is applied to this set by using the corresponding theorems. As a result, the needed minimal form of the given function is generated in ESOP.

Let us demonstrate that with an example of weakly determined function, specified by perfect STF $\begin{cases}Y^1=\{5,9,12\}^1\\Y^0=\{1,6,8\}^0\end{cases}$ (in [41, p.120] this function is minimized by the *K*-map method). For each minterm of set Y^1 we will determine the set of only those conjuncterms of $(n-1)$-rank, that will participate in the following «handshaking» procedure:

$$(0101)\Rightarrow\{(010-),(01-1),(-101)\},$$

$$(1001)\Rightarrow\{(10-1),(1-01)\},$$

$$(1100)\Rightarrow\{(110-),(11-0),(-100)\}.$$

The formed conjuncterms can be grouped by classes (with line at the same position), and afterwards, the «handshaking» procedure can be performed to the conjuncterms pairs:

Table 4.2

d	XOR5 $k_\theta^*/k_l^* = 16/80$		6Sym $k_\theta^*/k_l^* = 50/300$		9Sym $k_\theta^*/k_l^* = 87/522$		9Symml_91 $k_\theta^*/k_l^* = 87/522$		Z9Sym $k_\theta^*/k_l^* = 420/3780$	
	k_θ/k_l	t	k_θ/k_l	t	k_θ/k_l	t	k_θ/k_l	t	k_θ/k_l	t
4	5/5	≈ 0	13/54	≈ 0	76/395	3 *ms*	78/402	3 *ms*	80/391	13 *ms*
5	5/5	≈ 0	13/54	3 *s*	76/394	9 *ms*	75/385	11 *ms*	80/388	25 *ms*
6	-	-	13/54	3 *s*	75/387	30 *ms*	74/382	24 *ms*	77/382	1*h* 10 *ms*
7	-	-	-	-	74/384	1*h* 20*ms*	70/368	1*h* 38 *ms*	77/382	1*h* 31 *ms*
8	-	-	-	-	73/382	2*h* 30*ms*	70/368	2*h* 6 *ms*	77/382	2*h* 23 *ms*
9	-	-	-	-	73/382	3*h*	70/368	2*h* 40 *ms*	77/382	1*h* 36 *ms*

Table 4.3

d	XOR5	6Sym	9Sym	9symml_91	Z9sym
4	0.0195	0.0468	0.6610	0.6904	0.0197
5	0.0195	0.0468	0.6594	0.6358	0.0196
6	-	0.0468	0.6391	0.6225	0.0186
7	-	-	0.6257	0.5672	0.0185
8	-	-	0.6140	0.5672	0.0185
9	-	-	0.6140	0.5672	0.0185

$$\begin{pmatrix} -101 \\ -100 \end{pmatrix}^{\oplus} \Rightarrow (-10-); \quad \begin{pmatrix} 010- \\ 110- \end{pmatrix}^{\oplus} \Rightarrow (-10-);$$

$$\begin{pmatrix} 01-1 \\ 10-1 \\ 11-0 \end{pmatrix}^{\oplus} \Rightarrow \begin{pmatrix} \begin{pmatrix} 0--1 \\ -0-1 \end{pmatrix} / \begin{pmatrix} -1-1 \\ 1--1 \end{pmatrix} \\ 11-0 \end{pmatrix}^{\oplus} \Rightarrow \begin{pmatrix} -1-1 \\ 1--1 \\ 11-0 \end{pmatrix}^{\oplus} \Rightarrow$$

$$\overset{\oplus}{\Rightarrow} \begin{pmatrix} -1-- \\ 1--1 \\ 01-0 \end{pmatrix} / \begin{pmatrix} -1-1 \\ 1--- \\ 10-0 \end{pmatrix}^{\oplus} \Rightarrow (1--1)/(-1-1).$$

Therefore, minimal (P) STF $Y = \{(-10-), (1--1)\} \equiv \{(4,5,12,13),(9,11,13,15)\}$, that corresponds to [41].

The Experimental Part

Based on the proposed "handshaking" procedure, an algorithm and a program for minimization an arbitrarily given logical function from n variables in a polynomial format have been developed. The developed program was applied to experiment with some benchmarks [15, 20 39], namely: XOR5 (n = 5), 6Sym (n = 6) and 9Sym, 9symml_91 and Z9sym (n = 9). These functions were minimized by the program for different of "handshaking" depths d.

The obtained result are presented in Table 4.2, where k_θ^* / k_l^* is the ratio of the total amount of conjuncterms to the total amount of literals of a certain function before its minimization, respectively k_θ / k_l is the ratio after the minimization, t is the time spent on the minimization.

One can see, the result of minimization with benchmarks XOR5 and 6Sym does not depend on the depth d and is always the same. And for the benchmarks 9Sym, 9symml_91 and Z9sym, the obtained result demonstrates a decrease in the ratio k_θ / k_l, which illustrates the tendency to improve the minimization results by the increasing of depth d. Such feature of the "handshaking" procedure can be estimated by the *relative implementation* cost

$$k_r = \frac{k_\theta^* k_l^*}{k_\theta k_l}. \tag{4.1}$$

In the Table 4.3 we demonstrate how the value of the relative cost of implementation (4.1) changes from the "handshaking" depth d due to the minimization of benchmarks by the proposed method.

Therefore, the closer the value of the relative implementation cost k_r is to 0, the more effective the minimization is performed. Thus, if k_r>1, the result obtained relatively to the implementation cost

k_r of the minimized function will be worse than for non-minimized function. Based on that, one can use the parameter k_r to control the "handshaking" procedure while minimizing an arbitrarily given function from n variables.

Tables 4.2 and 4.3 also demonstrate that the best result for Z9Sym is obtained starting with $d = 6$, while it remains unchanged with increasing of d. A similar trend is observed for 9symml_91 and 9Sym, and for the first function, starting with $d = 6$, and for the second function, starting with $d = 8$. For all there cases, the minimization result improves with increasing of d, but is also aligned with increasing of the minimization time t. It is easy to assume that the optimal depth of the "handshaking" d depends on the function itself, that in some way affects the minimization time. Therefore, in practice it is important to record the optimal value of d, when the minimization result will improve or will not change.

Hence, the proposed “handshaking” minimization program is written in a high-level programming language (Python) and is not optimized for performance and multitasking. However, the time of execution the “handshaking” procedure only in the case of conditional 100 minterms with a Hamming distance $d = 10$ between them, can take at least 150 hours even for a conventional ideal modern processor core performing one operation with one bit per clock cycle (excluding auxiliary operations, counters, writing/reading, memory allocation, etc.).

The "handshake" procedure is quite demanding in combinatorial terms, but compared to known approaches allows to obtain one of the most accurate possible minimization results of logical functions in polynomial format.

Conclusions

The article proposes a new approach for the minimization of logical functions in a polynomial format and is based on the so-called "handshaking" procedure, which reflects the iterative polynomial extension of two conjuncterms of arbitrary ranks. The particularity of this procedure is that the Hamming distance d between these conjuncterms may be arbitrary. This significantly distinguishes the proposed approach from those known from publications [20–26], where the distance $d \leq 3$. An algorithm and a program for minimization of logical functions from n variables in a polynomial format were developed for this procedure, and the program was experimentally investigated/tested on different benchmarks. The obtained results illustrate the advantages of the proposed method due to the ability to minimize a given function in a polynomial format even when the Hamming distance d between two conjuncterms of different ranks is arbitrary.

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НОВИЙ МЕТОД МІНІМІЗАЦІЇ ЛОГІКОВИХ ФУНКЦІЙ
У ПОЛІНОМНОМУ ТЕОРЕТИКО-МНОЖИННОМУ ФОРМАТІ.
IV. ПРОЦЕДУРА «РУКОСТИСКАННЯ»

Вступ. Однією з причин складності мінімізації логікових функцій у поліномному форматі є те, що відомі процедури спрощення не мають узагальненого характеру щодо гемінґової відстані між довільними двома кон'юнктермами різних рангів заданої функції, що не гарантує її остаточної мінімізації. У відомих публікаціях на аналогічну тему згадана гемінґова відстань не перевищує число 3.

Мета. Метою цієї статті (яка є продовженням опублікованих статей в УСиМ у 2015 р. (№ 2, 4 і 5)) є розробка такої процедури над двома кон'юнктермами довільних рангів, гемінґова відстань між якими може бути довільною, а утворені внаслідок цього перетворені кон'юнктерми матимуть порівняно нижчі ранги і можуть бути використані для подальшого спрощення заданої функції за правилами, описаними в доведених теоремах (УСиМ № 2 за 2015 р.).

Методи. Запропоновано процедуру так званого «рукостискання» (яка умовно відображає на візерунку заданої функції рух одної вершини до віддаленої другої вершини), що ґрунтується на ітераційному розширенні двох числових кон'юнктермів різних рангів логікової функції, заданої у поліномному теоретико-множинному форматі. При цьому гемінґова відстань між цими кон'юнктермами може бути довільною. Перетворені кон'юнктерми утвореної множини використовуються для подальшого спрощення заданої функції на основі простих теоретико-множинних правил у поліномному форматі.

Результати. На основі процедури «рукостискання» розроблено алгоритм та програму мінімізації логікових функцій у поліномному теоретико-множинному форматі. Проведені на бенчмарках експериментальні дослідження програми ілюструють ефективність нового методу мінімізації логікових функцій у поліномному теоретико-множинному форматі.

Висновок. Запропоновано метод мінімізації логікових функцій у поліномному теоретико-множинному форматі, що ґрунтується на так званій процедурі «рукостискання», яка відображає ітераційне поліномне розширення двох початкових кон'юнктермів довільних рангів, унаслідок чого утворюється деяка множина перетворених кон'юнктермів, що мають нижчі ранги, ніж початкові. Особливість процедури в тому, що гемінґова відстань між цими двома кон'юнктермами може бути довільною, завдяки чому елементи утвореної множини можна використати для подальшої мінімізації заданої функції за певними правилами. Це принципово відрізняє запропонований метод від відомих щодо ефективності мінімізації, що підтверджують наведені приклади та експериментальні дослідження розробленої програми.

Ключові слова: *мінімізація логікової функції, кон'юнктерм, поліномний теоретико-множинний формат, гемінґова відстань, процедура «рукостискання».*