

CALCULATION OF CRITICAL TEMPERATURE FOR A BOSE GAS WITH LONG-RANGE FORCES

V.S. PASTUKHOV

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Ivan Franko National University of Lviv
(12, Drahomanov Str., Lviv 79005, Ukraine; e-mail: volodyapastukhov@gmail.com)

The critical temperature has been calculated for a Bose gas with the power-law dependence of the potential energy of interaction between particles on the interparticle distance. The result obtained satisfies the limiting cases known from the literature. The parameters of the collective excitation spectrum of the model have been analyzed in the random phase approximation (RPA). The long-wavelength asymptote has been derived for the structure factor of the system above the phase transition temperature.

1. Introduction

The story of studying the Bose systems with long-range forces started from the work by Foldy [1]. It was the first one dealing with the theory of charged Bose gas imbedded into a compensating field; the author studied the properties of the system's ground state in the Bogoliubov approximation. A few works, in which the critical behavior of the model was considered, appeared only in the first half of the 1970s [2–5], when a rapid development of the theory of phase transitions and critical phenomena started. The corresponding authors confined the consideration to a self-consistent static variant of random phase approximation (RPA), which enables the dominant contributions to thermodynamic functions of the system in a vicinity of the phase transition temperature to be found and the critical indices of the model to be calculated.

This paper aims at calculating the critical temperature of a Bose system in the model with the power-law dependence of the potential of pair interaction between particles on the distance r between them, $1/r^{1+\sigma}$, at least at large r 's. The ground state of this model in the limit $\sigma = 1$ and for an arbitrary space dimensionality d has been qualitatively analyzed in work [6] with the help of renormalization group methods. In the interesting case $d = 1$, a complete agreement with the exact solution was obtained. The critical behavior of our model has not been analyzed.

Hence, we consider a set of N spinless particles in the three-dimensional volume V and at the temperature

T . It is more convenient to deal with a grand canonical ensemble. Therefore, let us introduce the chemical potential μ into consideration. The Fourier transform of the pair interaction potential is chosen in the form

$$\nu_\sigma(k) = \frac{\lambda_\sigma}{k^{2-\sigma}}, \quad 0 \leq \sigma \leq 2,$$

where $\lambda_\sigma > 0$. Absolutely analogously to the model of charged Bose gas in a compensating field, where the stability of the system in the thermodynamic limit ($N \rightarrow \infty$ and $V \rightarrow \infty$ holds, but the density $\rho = N/V \rightarrow \text{const}$) is ensured by a uniform oppositely charged background, so that the system as a whole is electrically neutral, we impose the requirement $\nu_\sigma(0) = 0$ ($\sigma \neq 2$) on the zeroth component of the potential Fourier transform. The case $\sigma = 2$ ($\lambda_2 = 4\pi\hbar^2 a/m$, where a is the s -scattering length), which corresponds to a model with δ -repulsion, should be considered separately. The explicit form for the potential of pair interaction between particles reads

$$\Phi_\sigma(r) = \frac{1}{2^{2-\sigma}\pi^{3/2}} \frac{\Gamma(1/2 + \sigma/2)}{\Gamma(1 - \sigma/2)} \frac{\lambda_\sigma}{r^{1+\sigma}}, \quad \sigma \leq 2,$$

where $\Gamma(x)$ is the gamma-function. In the framework of our model, three parameters of the length dimension can be constructed. The first one, which is the simplest, is associated with the equilibrium density of the system and is proportional to $\rho^{1/3}$. The second one is typical of the ideal Bose gas at finite temperatures, $k_0 = p_0 = \sqrt{2mT}/\hbar$. At last, the third parameter is $k_\sigma = (\rho\lambda_\sigma/T)^{1/(2-\sigma)}$ ($\sigma \neq 2$), and, in the case of Coulomb potential ($\lambda_0 = 4\pi e^2$), it is nothing else but the reciprocal Debye radius $k_{\sigma=0} \equiv k_D = \sqrt{4\pi e^2 \rho/T}$. Those quantities can be used to construct two dimensionless parameters. At finite temperatures, the perturbation theory is developed on the basis of a series expansion in the ratio k_σ/k_0 , which is proportional to $[\lambda_\sigma m \rho^{(\sigma-1)/3} / \hbar^2]^{1/(2-\sigma)}$ in a vicinity of the Bose condensation temperature T_0 of the ideal gas. In the limiting case of the Coulomb gas, $\sigma = 0$, this expansion is equivalent to that in the so-called Brueckner parameter $r_s = (3/4\pi\rho)^{1/3} \frac{me^2}{\hbar^2}$, the ratio between the average

distance between particles and the Bohr radius. In the opposite limit, $\sigma = 2$, the substitution $k_\sigma^{2-\sigma} \rightarrow \rho\lambda_2/T$ should be made, and the expansion in the framework of perturbation theory in a vicinity of T_c is carried out in the gas parameter $a\rho^{1/3}$.

The ground state of the model for a space dimension larger than two is a state with a smeared, owing to interaction, Bose condensate. For its properties to be described at small values of the parameter of nonideality, it is sufficient to take advantage of the Bogoliubov method of approximate secondary quantization. It is clear that, for the space dimension $d = 3$, the results of work [1] are reproduced in the limiting case of a Bose systems with Coulomb interaction, and the known results of Lee and Yang [7] are obtained for the model of weakly nonideal Bose gas. The long-wave asymptote for the spectrum of elementary excitations over the ground state has a characteristic behavior $E_\sigma(k) \sim k^{\sigma/2}$.

2. Structure Factor and Collective Excitation Spectrum

Let us analyze firstly the structure factor of the model

$$S_\sigma(k) = \int_{-\infty}^{\infty} d\omega S_\sigma(\omega, k), \quad (1)$$

where the notation

$$S_\sigma(\omega, k) = \frac{1}{\pi\rho} \frac{1}{1 - e^{-\beta\omega}} \times \frac{I(\omega, k)}{(1 + \nu_\sigma(k)R(\omega, k))^2 + (\nu_\sigma(k)I(\omega, k))^2}. \quad (2)$$

was introduced for the dynamic structure factor in RPA; $\beta = 1/T$; $R(\omega, k)$ and $I(\omega, k)$ are the real and imaginary, respectively, parts of the polarization operator

$$\Pi(\omega_n, k) = \frac{1}{V} \sum_{\mathbf{p}} n(\beta\xi_p) \times \left\{ \frac{1}{\xi_{|\mathbf{p}-\mathbf{k}|} - \xi_p - i\omega_n} + (\omega_n \rightarrow -\omega_n) \right\} \quad (3)$$

after its analytical continuation $i\omega_n \rightarrow \omega + i0$ (see Appendix), $n(x) = \{e^x - 1\}^{-1}$, $\xi_p = \varepsilon_p - \mu$, $\varepsilon_p = \hbar^2 p^2/2m$ is the spectrum of free particles, and $\omega_n = 2\pi nT$ ($n = 0, \pm 1, \dots$) are the Matsubara frequencies. Taking the arguments of work [8] into account, let us write down the

structure factor in the long-wave region ($k \ll k_\sigma$) in the form

$$S_\sigma(k) = \frac{1}{\rho\nu_\sigma^2(k)} [\partial R(\omega, k)/\partial\omega]_{\omega \rightarrow \omega_\sigma(k)}^{-1} \times \coth(\beta\omega_\sigma(k)/2), \quad (4)$$

where $\omega_\sigma(k)$ is the spectrum of collective excitations in the system, which is determined by the equation

$$1 + \nu_\sigma(k)R(\omega_\sigma(k), k) = 0. \quad (5)$$

By differentiating this equation formally with respect to $\nu_\sigma(k)$, we obtain a useful relation

$$[\partial R(\omega, k)/\partial\omega]_{\omega \rightarrow \omega_\sigma(k)}^{-1} = \nu_\sigma^2(k) \frac{\partial\omega_\sigma(k)}{\partial\nu_\sigma(k)}. \quad (6)$$

The first moment of the dynamic structure factor has to satisfy the condition

$$\int_{-\infty}^{\infty} d\omega \omega S_\sigma(\omega, k) = \frac{\hbar^2 k^2}{2m},$$

which, in our case of long-wave asymptotic range, gives rise to the relation

$$\frac{1}{\rho\nu_\sigma^2(k)} [\partial R(\omega, k)/\partial\omega]_{\omega \rightarrow \omega_\sigma(k)}^{-1} = \frac{\varepsilon_k}{\omega_\sigma(k)}, \quad (7)$$

and, with regard for equality (6), to the dependence

$$\omega_\sigma^2(k) = 2\rho\nu_\sigma(k)\varepsilon_k + \dots, \quad (8)$$

where the notation “...” means the part of the spectrum that is independent of the interaction. Note that, in the case of low interaction, equalities (4)–(8) are valid only in a very narrow interval of wave vectors in a vicinity of the zero-value point. Let us try to expand the applicability region for those formulas. Let a condition be imposed that formula (8) should correctly reproduce the long-wave asymptotics of the structure factor for the ideal Bose gas, when the interaction is switched off, $\nu_\sigma(k) \rightarrow 0$. It is clear that, in this case, we have to replace the real spectrum of collective excitations in the system by a certain effective one, for which the notation $\omega_\sigma(k)$ is preserved. Then, for the long-wave asymptote of the “spectrum” in the absence of interaction, we obtain

$$\omega_\sigma^2(k)|_{\nu_\sigma(k) \rightarrow 0} = 2T\varepsilon_k/S_0(k), \quad (9)$$

where $S_0(k)$ is the structure factor for the ideal Bose gas. At last, the condition $S_\sigma(k \rightarrow \infty) \rightarrow 1$ unambiguously

determines the behavior of the $\omega_\sigma(k)$ -function at large argument values, $\omega_\sigma(k \rightarrow \infty) \rightarrow \varepsilon_k$. Combining all the aforesaid together, we obtain the following formula for the effective spectrum of the system:

$$\omega_\sigma^2(k) = 2\rho\nu_\sigma(k)\varepsilon_k + 2T\varepsilon_k/S_0(k) + \varepsilon_k^2. \quad (10)$$

It should be substituted into expression (4) for the structure factor of the system with interaction,

$$S_\sigma(k) = \frac{\varepsilon_k}{\omega_\sigma(k)} \coth(\beta\omega_\sigma(k)/2). \quad (11)$$

It is of interest that now this expression correctly reproduces the classical limit of the theory,

$$S_\sigma(k)|_{\hbar \rightarrow 0} = \frac{1}{1 + \beta\rho\nu_\sigma(k)}. \quad (12)$$

Hence, we obtained an expression that correctly describes the structure factor of the system in the long-wave interval, $k \ll k_0$. Despite that relations (10) and (11) correctly reproduce certain limiting cases of the theory, it is worth emphasizing that the applicability scope of those formulas is confined to low interactions, in other words, to high densities of a Bose gas at $\sigma < 1$ and to low ones at $\sigma > 1$. The extension of the results obtained onto lower (larger) densities is not a trivial problem. On the one hand, it is connected with our procedure of ‘‘guessing’’, on the other hand, with the applicability of RPA.

Now, let us study the spectrum of ‘‘plasmons’’, i.e. the roots of Eq. (5). We rewrite this equation in the form (see Appendix)

$$1 - \left(\frac{k_\sigma}{k}\right)^{2-\sigma} \frac{k_0}{2k} \{f(\beta E_\sigma(k)/(2k/k_0) - k/2k_0, \beta\mu) - f(\beta E_\sigma(k)/(2k/k_0) + k/2k_0, \beta\mu)\} = 0, \quad (13)$$

where a new notation $E_\sigma(k)$ was introduced for the spectrum. The spectrum behavior is completely determined by the ratio k_σ/k_0 . First, if $E_\sigma(k)$ is a root of the equation, the quantity $-E_\sigma(k)$ is also a root. Bearing the positivity of the function $f(\varepsilon, y)$ in mind, we arrive at a conclusion that a condition for Eq. (13) to have real-valued roots is

$$\frac{\beta E_\sigma(k)}{k/k_0} > \varepsilon_0(\beta\mu),$$

where $\varepsilon_0(y)$ is the maximum point of the function $f(\varepsilon, y)$ with respect to its first argument, provided that the second argument is fixed. Second, for every value of parameter k_σ/k_0 ($\sigma < 2$), there exists a spectral branch

in the range $k \ll k_\sigma$ with a characteristic gap $\omega_0 = \sqrt{4\pi e^2 \hbar^2 \rho/m}$ in the case of a charged Bose gas. To verify this statement, it is enough to expand the function $f(\varepsilon, y)$ (see Appendix) at large values of its first argument into a series and to substitute the expansion into Eq. (13). As a result, for the spectrum in the limit $k_\sigma \gg k_0$, we obtain

$$E_\sigma^2(k) = 2\rho\nu_\sigma(k)\varepsilon_k + 2T\varepsilon_k \frac{g_{5/2}(e^{\beta\mu})}{g_{3/2}(e^{\beta\mu})} + \varepsilon_k^2, \quad (14)$$

which can be easily rewritten in the form [9]

$$E_\sigma^2(k) = 2\rho\nu_\sigma(k)\varepsilon_k + \frac{4}{3} \frac{K_0}{N} \varepsilon_k + \varepsilon_k^2, \quad (15)$$

where K_0/N is the average energy of the ideal Bose gas per one particle.

The third important feature in the spectrum of collective excitations in Bose systems with our model potential at $T > T_c$ is the existence of the end point k_f , i.e., the maximal value of wave vector, for which Eq. (5) still has a root. This peculiarity in the spectrum of Coulomb systems was not indicated by the authors of work [10], where the long-wave asymptotics of the plasmon spectrum in a charged Bose gas at all temperatures was studied. In the general case, the location of the end point is determined by the parameter k_σ/k_0 . Only the limiting cases of small and large values of this parameter can be analyzed analytically. In the case $k_\sigma/k_0 \ll 1$, taking Eq. (13) into account, it is easy to obtain that the quantity $E_\sigma(k_f)/(2k_f/k_0)$ tends to $\varepsilon'_0(\beta\mu)$, the minimum point of the first derivative of function $f'(\varepsilon, \beta\mu)$ with respect to the variable ε . Then, for the end point, we have

$$k_f/k_\sigma = (|f'(\varepsilon'_0(\beta\mu), \beta\mu)|/2)^{1/(2-\sigma)}, \quad k_\sigma \ll k_0. \quad (16)$$

In the opposite case, the difference $E_\sigma(k_f)/(2k_f/k_0) - k_f/2k_0$ tends to the maximum point $\varepsilon_0(\beta\mu)$ of the functions $f(\varepsilon, \beta\mu)$, and $E_\sigma(k_f)/(2k_f/k_0) + k_f/2k_0 \gg 1$. Then, we have the relation

$$\frac{k_f}{k_0} = \left[\frac{1}{2} \left(\frac{k_\sigma}{k_0}\right)^{2-\sigma} f(\varepsilon_0(\beta\mu), \beta\mu) \right]^{1/(3-\sigma)}, \quad k_\sigma \gg k_0. \quad (17)$$

One should bear in mind that it is not an adequate procedure to analyze the limit $k_\sigma \gg k_0$ in RPA.

Let us demonstrate now that the spectrum damping

$$\Gamma_\sigma(k) = I(E_\sigma(k), k)[\partial R(\omega, k)/\partial \omega]_{\omega \rightarrow E_\sigma(k)}^{-1}, \quad (18)$$

is low. Using the formula for the function $I(\omega, k)$ given in Appendix and the last relation, we write down that, in the long-wave limit,

$$\frac{\Gamma_\sigma(k)}{E_\sigma(k)} \Big|_{k \rightarrow 0} = \left(\frac{\pi}{8}\right)^{1/2} \left(\frac{k_\sigma}{k}\right)^{3(1-\sigma/2)} \times \\ \times n \left(\frac{1}{2} (k_\sigma/k)^{2-\sigma} - \beta\mu \right) / g_{3/2}(e^{\beta\mu}). \quad (19)$$

In the limit $\mu \rightarrow \infty$ and at $\sigma = 0$, this formula reproduces the well-known result of the classical plasma theory,

$$\Gamma(k)/\omega_0 = \left(\frac{\pi}{8}\right)^{1/2} \left(\frac{k_D}{k}\right)^3 e^{-k_D^2/2k^2}. \quad (20)$$

In the case $\sigma = 2$, the behavior of the spectrum is quite different. In work [8], it was shown that not all values of nonideality parameter $a\rho^{1/3}$ allow Eq. (13) to have solutions; namely, only if $4\pi\hbar^2 a\rho/(mT) > 2/|f'(\varepsilon'_0(\beta\mu), \beta\mu)|$, there emerge two sound branches in the spectrum at $T > T_c$, one of which is suppressed by a substantial damping. At the critical point, there exists only one branch in the phonon spectrum for $4\pi\hbar^2 a\rho/(mT_c) > -\zeta(3/2)/\zeta(1/2) = 1.789$.

3. One-Particle States

The self-energy part or the mass operator in RPA looks like [11]

$$\Sigma(\omega_n, p) = -\frac{1}{\beta V} \sum_{\omega'_n} \sum_{\mathbf{k}} \frac{\nu_\sigma(k)}{1 + \nu_\sigma(k)\Pi(\omega'_n, k)} \times \\ \times \frac{1}{i(\omega_n + \omega'_n) - \xi_{|\mathbf{p}+\mathbf{k}|}}, \quad (21)$$

where the equality $\nu_\sigma(0) = 0$ has already been taken into account. After the analytical continuation into the upper half-plane, we introduce the notation

$$\Sigma_R(\omega, p) = \text{Re } \Sigma(\omega_n, p) \Big|_{i\omega_n \rightarrow \omega + i0}, \quad (22)$$

$$\Sigma_I(\omega, p) = \text{Im } \Sigma(\omega_n, p) \Big|_{i\omega_n \rightarrow \omega + i0}. \quad (23)$$

Now, the equation for the renormalized spectrum [12]

$$\tilde{\xi}_p = \xi_p + \Sigma_R(\tilde{\xi}_p, p) \quad (24)$$

can be written down. It is evident that elementary excitations are stable, provided that the damping is infinitesimally small,

$$\gamma_p = Z(p)\Sigma_I(\tilde{\xi}_p, p), \quad (25)$$

$$Z^{-1}(p) = 1 - \frac{\partial \Sigma_R(\omega, p)}{\partial \omega} \Big|_{\omega=\tilde{\xi}_p}. \quad (26)$$

The equilibrium density of the system

$$\rho = -\frac{1}{V} \sum_{\mathbf{p}} \lim_{\tau \rightarrow +0} \frac{1}{\beta} \sum_{\omega_n} \frac{e^{i\omega_n \tau}}{i\omega_n - \xi_p - \Sigma(\omega_n, p)} = \\ = \frac{1}{V} \sum_{\mathbf{p}} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{n(\beta\omega)\Sigma_I(\omega, p)}{(\omega - \xi_p - \Sigma_R(\omega, p))^2 + \Sigma_I^2(\omega, p)}, \quad (27)$$

which actually is an equation for the determination of the chemical potential of the system, can be rewritten in the form

$$\rho = \frac{1}{V} \sum_{\mathbf{p}} Z(p)n(\beta\tilde{\xi}_p), \quad (28)$$

which we use to find the renormalized temperature of the Bose condensation, provided that $\tilde{\mu} = 0$. Further calculations for the mass operator were carried out only for temperatures in a vicinity of the critical one (strictly speaking, in the limiting case $T \rightarrow T_c + 0$), being conceptually close to the calculations carried out in work [13]. We extract the Hartree–Fock contribution

$$\Sigma(\omega_n, p) = \frac{1}{V} \sum_{\mathbf{k}} \nu_\sigma(k)n(\beta\xi_{|\mathbf{k}-\mathbf{p}|}) + \\ + \frac{1}{\beta V} \sum_{\omega'_n} \sum_{\mathbf{k}} \nu_\sigma(k) \frac{\nu_\sigma(k)\Pi(\omega'_n, k)}{1 + \nu_\sigma(k)\Pi(\omega'_n, k)} \times \\ \times \frac{1}{i(\omega_n + \omega'_n) - \xi_{|\mathbf{p}+\mathbf{k}|}} \quad (29)$$

from formula (21) and rewrite the fraction in the second sum using the spectral relations

$$\frac{\nu_\sigma(k)\Pi(\omega'_n, k)}{1 + \nu_\sigma(k)\Pi(\omega'_n, k)} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega - i\omega'_n} \times \\ \times \frac{\nu_\sigma(k)I(\omega, k)}{(1 + \nu_\sigma(k)R(\omega, k))^2 + (\nu_\sigma(k)I(\omega, k))^2}.$$

After that, the sum over ω'_n can be easily calculated,

$$\begin{aligned} \Sigma(\omega_n, p) &= \frac{1}{V} \sum_{\mathbf{k}} \nu_{\sigma}(k) n(\beta \xi_{|\mathbf{k}-\mathbf{p}|}) + \\ &+ \frac{1}{V} \sum_{\mathbf{k}} \nu_{\sigma}(k) \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{n(\beta \omega) - n(\beta \xi_{|\mathbf{k}-\mathbf{p}|})}{\omega - \xi_{|\mathbf{k}-\mathbf{p}|} + i\omega_n} \times \\ &\times \frac{\nu_{\sigma}(k) I(\omega, k)}{(1 + \nu_{\sigma}(k) R(\omega, k))^2 + (\nu_{\sigma}(k) I(\omega, k))^2}. \end{aligned} \quad (30)$$

In view of the complicated form of the expression, it is evident that, in order to carry out subsequent calculations, further simplifications are to be made. Since the region concerned (here, the wave vector amplitudes are smaller than k_f) contains a well-defined branch in the spectrum of collective excitations of the system, expression (30) can be rewritten by splitting each \mathbf{k} -sum into two sums, with the magnitudes of wave vectors \mathbf{k} 's in each of them being, respectively, smaller or larger than k_f . Then, in the sums over $k \leq k_f$, the integrals over the variable ω can be approximately calculated taking advantage of the relation

$$\begin{aligned} &\frac{1}{\pi \rho} \frac{I(\omega, k)}{(1 + \nu_{\sigma}(k) R(\omega, k))^2 + (\nu_{\sigma}(k) I(\omega, k))^2} \rightarrow \\ &\rightarrow \frac{1}{\rho \nu_{\sigma}(k)} \text{sign}(I(\omega, k)) \delta(1 + \nu_{\sigma}(k) R(\omega, k)), \end{aligned}$$

and making allowance for a low damping of the spectrum of collective excitations in the system. Ultimately, we obtain

$$\begin{aligned} \Sigma(\omega_n, p) &= \frac{1}{V} \sum_{\mathbf{k}, k \leq k_f} \nu_{\sigma}(k) n(\beta \xi_{|\mathbf{k}-\mathbf{p}|}) - \\ &- \frac{1}{V} \sum_{\mathbf{k}, k \leq k_f} [\partial R(E_{\sigma}(k), k) / \partial E_{\sigma}(k)]^{-1} \times \\ &\times \left\{ \frac{1 + n(\beta E_{\sigma}(k)) + n(\beta \xi_{|\mathbf{k}-\mathbf{p}|})}{E_{\sigma}(k) + \xi_{|\mathbf{k}-\mathbf{p}|} - i\omega_n} - \right. \\ &\left. - \frac{n(\beta E_{\sigma}(k)) - n(\beta \xi_{|\mathbf{k}-\mathbf{p}|})}{E_{\sigma}(k) - \xi_{|\mathbf{k}-\mathbf{p}|} + i\omega_n} \right\} + \dots, \end{aligned} \quad (31)$$

where the notation “...” stands for expression (30), in which the sums over \mathbf{k} are confined from below by the condition $k \geq k_f$. Now, while considering the limit $k_{\sigma}/k_0 \rightarrow 0$ and taking into account that k_f is also small, it is easy to see that the first term in formula (31) gives an insignificant contribution to the mass operator. On the other hand, with regard for the properties of the functions $I(\omega, k)$ and $R(\omega, k)$, we see that practically the whole contribution to the integral over ω in formula (30) is provided by the vicinities of two points, for which $\beta \omega \sim \pm k^2/k_0^2$. Let us try to “catch” this region using a trick similar to that used by us when calculating the structure factor. We adopt that expression (31) with the corresponding modifications (the substitution of $E_{\sigma}(k)$ by $\omega_{\sigma}(k)$, the elimination of the restriction on the summation over the wave vector, and the neglect of terms hidden under the ellipsis sign) can correctly describe the mass operator

$$\begin{aligned} \Sigma(\omega_n, p) &\rightarrow \frac{1}{V} \sum_{\mathbf{k}} \nu_{\sigma}(k) n(\beta \xi_{|\mathbf{k}-\mathbf{p}|}) - \\ &- \frac{1}{V} \sum_{\mathbf{k}} \nu_{\sigma}(k) \frac{\rho \nu_{\sigma}(k) \varepsilon_k}{\omega_{\sigma}(k)} \left\{ \frac{1 + n(\beta \omega_{\sigma}(k)) + n(\beta \xi_{|\mathbf{k}-\mathbf{p}|})}{\omega_{\sigma}(k) + \xi_{|\mathbf{k}-\mathbf{p}|} - i\omega_n} - \right. \\ &\left. - \frac{n(\beta \omega_{\sigma}(k)) - n(\beta \xi_{|\mathbf{k}-\mathbf{p}|})}{\omega_{\sigma}(k) - \xi_{|\mathbf{k}-\mathbf{p}|} + i\omega_n} \right\}. \end{aligned} \quad (32)$$

In this expression, we used relation (7) for the derivative of the real part of the polarization operator. In such a way, we effectively consider the contribution made by the ideal gas similarly to what was done while calculating the structure factor. Concerning the validity of such a trick in calculations, it is demonstrated below that the corresponding shifts of the critical temperature obtained in the framework of our model for two limiting cases—the charged Bose gas of a high density and the model of weakly nonideal gas—completely coincide in the main approximation with the standard results [2, 14]. At the same time, expression (32) makes it possible to analyze the limiting case of large nonideality-parameter values as well.

4. The Limiting Case of Small Nonideality Parameter

Hence, in what follows, we use expression (32). Taking into account that small k make a dominant contribution to the integral over the wave vector in the limit

$k_\sigma/k_0 \rightarrow 0$, we may change the Bose distribution $n(x)$ by $1/x$. Then, after a simple regrouping of terms and taking equality (10) for the real part of the self-energy part into account, we obtain

$$\begin{aligned} \Sigma_R(\omega, p) = & \frac{1}{\beta V} \sum_{\mathbf{k}} \nu_\sigma(k) \frac{\omega_\sigma^2(k) - 2\rho\nu_\sigma(k)\varepsilon_k}{\omega_\sigma^2(k)} \frac{1}{\varepsilon_{|\mathbf{k}-\mathbf{p}|}} - \\ & - \frac{1}{\beta V} \sum_{\mathbf{k}} \nu_\sigma(k) \frac{\rho\nu_\sigma(k)\varepsilon_k}{\omega_\sigma^2(k)} \frac{\omega}{\varepsilon_{|\mathbf{k}-\mathbf{p}|}} \left\{ \frac{1}{\omega_\sigma(k) + \varepsilon_{|\mathbf{k}-\mathbf{p}|} - \omega} - \right. \\ & \left. - \frac{1}{\omega_\sigma(k) - \varepsilon_{|\mathbf{k}-\mathbf{p}|} + \omega} \right\}. \end{aligned} \quad (33)$$

In the framework of ordinary perturbation theory, we can approximately write down the renormalized spectrum in the form

$$\tilde{\varepsilon}_p = \varepsilon_p + \Sigma_R(\varepsilon_p, p) - \Sigma_R(0, 0), \quad (34)$$

where the zeroing of the renormalized chemical potential $\tilde{\mu}$ has already been taken into account. The further calculation is easy. It is enough to substitute only the long-wave asymptote of the ‘‘spectrum’’,

$$\omega_\sigma^2(k) = 2\rho\nu_\sigma(k)\varepsilon_k + \frac{2\zeta(3/2)}{\pi^{3/2}} \frac{k^3}{k_0^3} T^2, \quad (35)$$

into formula (33). The dominating contribution can be easily demonstrated to equal

$$\begin{aligned} \beta[\Sigma_R(\varepsilon_p, p) - \Sigma_R(0, 0)] & \equiv \sigma(p/p_0) = \\ & = (2a_\sigma/\pi)^2 I_\sigma(\gamma) + \dots, \end{aligned} \quad (36)$$

where

$$I_\sigma(\gamma) = \int_0^\infty \frac{dkk}{k^{3-\sigma} + 1} \left\{ \frac{k}{2\gamma} \ln \left| \frac{k + \gamma}{k - \gamma} \right| - 1 \right\},$$

the notations $a_\sigma^{3-\sigma} = \pi^{3/2}(k_\sigma/k_0)^{2-\sigma}/\zeta(3/2)$ and $\gamma = p/(p_0 a_\sigma)$ are introduced, and all the terms of the higher orders of the parameter a_σ in the expansion series are omitted. It is evident that the integral $I_\sigma(\gamma) = -\gamma^2 \ln(\gamma)/3$ is not analytic at small arguments, with the form of this non-analytic behavior being independent of the power exponent σ in the formula for the interaction potential. It is a hint at a universal power behavior $\tilde{\varepsilon}_p \sim p^{2-\eta}$ of the leading asymptotic term in

the one-particle spectrum for the interacting system. It is important that other terms, which were not taken into account in formula (36), are analytic in γ . The dominant contribution to the chemical potential of the system at the critical point is

$$\mu/T_c = \frac{\pi}{3 - \sigma} \frac{(2a_\sigma/\pi)^2}{\sin(\pi/(3 - \sigma))}, \quad \sigma < 1, \quad (37)$$

where the divergence at $\sigma > 1$ is fictitious, resulting from the substitution $n(x) \rightarrow 1/x$ while changing from formula (32) to expression (33).

In order to find the critical temperature in our model, we must calculate the quantity $Z(p)$ using formula (27). For this purpose, we calculate the derivative

$$\begin{aligned} [\partial \Sigma_R(\omega, p)/\partial \omega] \Big|_{\omega \rightarrow \varepsilon_p} & \equiv z(p/p_0) = \\ & = \frac{a_\sigma}{\sqrt{\pi}\zeta(3/2)} J_\sigma(\gamma) + \dots, \end{aligned} \quad (38)$$

where

$$J_\sigma(\gamma) = \frac{1}{\gamma} \int_0^\infty \frac{dkk}{(k^{3-\sigma} + 1)^2} \ln \left| \frac{k + \gamma}{k - \gamma} \right|,$$

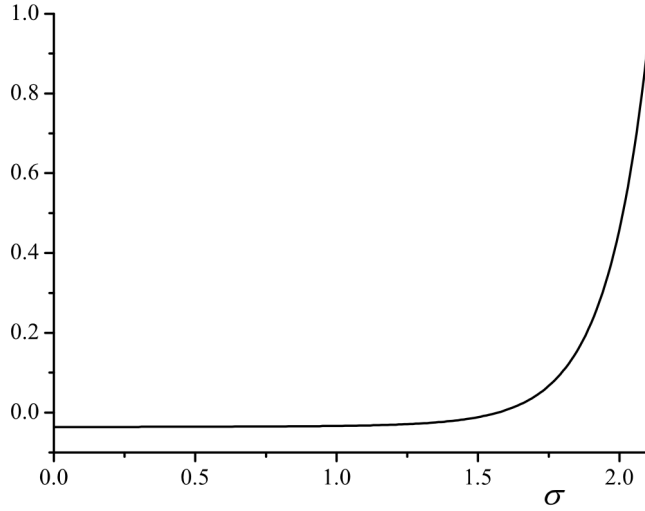
and only the leading term in the expansion series in a_σ is preserved again. The integral $J_\sigma(\gamma)$, in contrast to $I_\sigma(\gamma)$, is an analytic function of γ . At small argument values, it is constant and, at $\gamma \rightarrow \infty$, approaches the asymptote $1/\gamma^2$.

Now, we can proceed to the integration of formula (28) for the equilibrium density. Extracting the contribution for the density of the ideal gas, ρ_0 , we write down with the required accuracy that

$$\begin{aligned} \rho = \rho_0 + \frac{p_0^3}{2\pi^2} \int_0^\infty dp p^2 \{ n(p^2 + \sigma(p)) - n(p^2) \} + \\ + \frac{p_0^3}{2\pi^2} \int_0^\infty dp p^2 n(p^2 + \sigma(p)) z(p). \end{aligned}$$

After changing the integration variable, $p \rightarrow a_\sigma \gamma$, we obtain

$$\begin{aligned} \Delta\rho \equiv \rho - \rho_0 = \frac{p_0^3}{2\pi^2} a_\sigma^3 \times \\ \times \int_0^\infty d\gamma \gamma^2 \{ n(a_\sigma^2 \gamma^2 + (2a_\sigma/\pi)^2 I_\sigma(\gamma)) - n(a_\sigma^2 \gamma^2) \} + \end{aligned}$$



Dependence of the integral in formula (39) on the power exponent in the expression for the potential energy of the pairwise interaction between particles. The minimum values of -0.036 and 0.451 are attained in the Coulomb limit ($\sigma = 0$) and in the model of weakly nonideal gas ($\sigma = 2$), respectively

$$+ \frac{p_0^3}{2\pi^2} a_\sigma^3 \int_0^\infty d\gamma \gamma^2 n(a_\sigma^2 \gamma^2 + (2a_\sigma/\pi)^2 I_\sigma(\gamma)) \times \\ \times \frac{a_\sigma}{\sqrt{\pi} \zeta(3/2)} J_\sigma(\gamma).$$

At small a_σ , the dominant terms give the contribution

$$\Delta\rho = -\frac{p_0^3}{2\pi^2} a_\sigma \int_0^\infty d\gamma \frac{(2/\pi)^2 I_\sigma(\gamma)}{\gamma^2 + (2/\pi)^2 I_\sigma(\gamma)}. \quad (39)$$

There is no sense to make allowance for terms with higher powers of the parameter a_σ , because we use RPA, whereas “beyond-RPA” contributions are quadratic in this parameter.

To calculate the critical temperature T_c in this limit of the interaction, let us take advantages of the speculations of the authors of work [14], where a Bose system with the point-like interaction was considered. It is worth noticing that RPA is not absolutely correct for the gas model with short-range forces. We take into consideration that the equilibrium density does not depend on the interaction. Therefore,

$$\frac{d}{da_\sigma} \rho = \frac{\partial \rho}{\partial a_\sigma} + \frac{\partial \rho}{\partial T_c} \frac{\partial T_c}{\partial a_\sigma} = 0,$$

and, making the substitution

$$\frac{\partial \rho}{\partial T_c} \simeq \frac{\partial \rho_0}{\partial T_0} = \frac{3}{2} \frac{\rho_0}{T_0},$$

which has a required accuracy, we obtain

$$\frac{T_c(\sigma) - T_0}{T_0} = -\frac{2}{3} \frac{\Delta\rho}{\rho}. \quad (40)$$

Now, consider the limiting cases in brief. First, let us analyze the model of charged Bose gas in a compensating field. In this case, the quantity $a_0 = [\pi^{5/2}/6\zeta(3/2)]^{1/9} r_s^{1/3}$ can be expressed in terms of the Brueckner parameter. The numerical calculation of the quantity $\Delta\rho$ at $\sigma = 0$ making use of integral (39) and considering the equality for the critical temperature shift brings about the relation

$$\frac{T_c(0) - T_0}{T_0} = -0.026 r_s^{1/3}. \quad (41)$$

It is an exact result in the high-density limit, which was obtained for the first time in work [2] using a different calculation method. It is interesting that the critical temperature of a charged Bose gas is lower than the Bose condensation temperature in the ideal gas.

The other limiting value, $\sigma = 2$, corresponds to the model of weakly nonideal Bose gas. In this case, $a_2 = 2\pi^{3/2} \rho^{1/3} a / [\zeta(3/2)]^{1/3}$, and the function $I_2(\gamma)$ is convenient to be written down as follows:

$$I_2(\gamma) = -\frac{1}{\gamma} \int_0^\gamma \frac{dy y^2}{1-y^2} \ln |y|.$$

Then, by formally expanding the integrand in formula (39) in a series in $I_2(\gamma)$ and confining the expansion to the first-order term, we obtain

$$\Delta\rho = 2p_0^3 a_2 \int_0^\infty \frac{d\gamma}{\gamma^3} \int_0^\gamma \frac{dy y^2}{1-y^2} \ln |y|.$$

Now, changing the sequence of integration,

$$\Delta\rho = 2p_0^3 a_2 \int_0^1 dy y \int_0^\infty \frac{d\gamma}{1-\gamma^2} \ln |\gamma|,$$

and substituting the result into the expression for the critical temperature shift, we ultimately obtain [14]

$$\frac{T_c(2) - T_0}{T_0} = \frac{8\pi}{3[\zeta(3/2)]^{4/3}} \rho^{1/3} a = 2.328 \rho^{1/3} a. \quad (42)$$

In view of the result of numerical calculations of integral (39), we obtain

$$\frac{T_c(2) - T_0}{T_0} = 2.010 \rho^{1/3} a, \quad (43)$$

which is closer to the values obtained by Monte Carlo simulations (these are $(1.29 \pm 0.05)\rho^{1/3}a$ [15] and $(1.32 \pm 0.02)\rho^{1/3}a$ [16]) and the calculations using the renormalization group methods in the seven-loop approximation, $(1.27 \pm 0.11)\rho^{1/3}a$ [17].

5. Conclusions

In the framework of RPA and using the method proposed in this work to calculate the self-energy part of one-particle Green's functions, we managed to determine the dominant correction to the critical temperature in the model of a Bose gas with the long-range repulsive potential $1/r^{1+\sigma}$ in the high-density (for the case $\sigma < 1$) and low-density (for the case $\sigma > 1$) limits. It is important to emphasize that RPA is an exact approximation in the limit of a small nonideality parameter, only provided that $\sigma < 2$. It is of interest that the sign of the correction to the critical temperature changes from negative to positive at $\sigma = 1.57$. Since the experimental verification is most likely impossible, it would be interesting to compare our results with those obtained for this model with the use of the computer simulation. The method proposed in this work to calculate spectral parameters can be generalized to an arbitrary space dimension $d > 2$, where the behavior of the system would not differ qualitatively from the examined one. In the case $d \leq 2$, there is no phase transition into the Bose-condensate state at finite temperatures; and, as a consequence, the properties of the model change completely. The analysis of the spectrum of collective excitations in the system above the Bose condensation temperature revealed an interesting feature: the spectrum of the model has an end point, irrespective of the specific nonideality parameter value. This feature is absent for the condensate phase, considered in the same approximation, where the polarization operator (3) has an explicitly polar form. In this work, using an exact relation for the first moment of the dynamic structure factor, we managed to obtain the long-wave asymptote for the structure factor of the system.

We plan to continue the researches of this problem. In particular, we intend to find the critical temperatures for all k_σ/k_0 -values. This task, as well as a correct calculation of the next term in the expansion of formula (39) in a series in a_σ , evidently requires that the consideration should go beyond the RPA framework. We also intend to consider a possibility for an instability of the Wigner crystallization type to emerge in the charged Bose gas above the condensation temperature.

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APPENDIX

After the analytical continuation into the upper half-plane, the real and imaginary parts of the polarization operator (3) look like

$$\operatorname{Re} \Pi(\omega_n, k)|_{i\omega_n \rightarrow \omega + i0} = R(\omega, k),$$

$$\operatorname{Im} \Pi(\omega_n, k)|_{i\omega_n \rightarrow \omega + i0} = I(\omega, k),$$

where

$$I(\omega, k) = \frac{1}{16\pi} \beta k_0^3 \frac{k_0}{k} \ln \left| \frac{1 - \exp\{\beta\mu - [k/2k_0 + \beta\omega/(2k/k_0)]^2\}}{1 - \exp\{\beta\mu - [k/2k_0 - \beta\omega/(2k/k_0)]^2\}} \right|$$

and

$$R(\omega, k) = \beta \rho \frac{k_0}{2k} \{f(k/2k_0 + \beta\omega/(2k/k_0), \beta\mu) + f(k/2k_0 - \beta\omega/(2k/k_0), \beta\mu)\}.$$

The function

$$f(\varepsilon, y) = \varepsilon \int_0^1 \frac{dx}{\sqrt{1-x}} g_{1/2}(e^{y-x\varepsilon^2}) / g_{3/2}(e^y),$$

where

$$g_l(e^y) = \sum_{n \geq 1} \frac{e^{yn}}{n^l},$$

has the following asymptotes in the limiting cases:

$$f(\varepsilon \rightarrow 0, y) = 2\varepsilon \frac{g_{1/2}(e^y)}{g_{3/2}(e^y)},$$

$$f(\varepsilon \rightarrow \infty, y) = \frac{1}{\varepsilon} \left\{ 1 + \frac{1}{2\varepsilon^2} \frac{g_{5/2}(e^y)}{g_{3/2}(e^y)} + \dots \right\},$$

$$f(\varepsilon \rightarrow 0, 0) = [\pi^{3/2} \operatorname{sign}(\varepsilon) + 2\varepsilon \zeta(1/2)] / \zeta(3/2),$$

where $\zeta(x) = \sum_{n \geq 1} 1/n^x$ is the zeta-function.

1. L.L. Foldy, Phys. Rev. **124**, 649 (1961); Phys. Rev. **125**, 2208 (1962).
2. A.L. Fetter, Ann. Phys. **64**, 1 (1971).
3. S. Ma, Phys. Rev. Lett. **29**, 1311 (1972).
4. P.V. Panat, H.E. DeWitt, and J.C. Garrison, Phys. Rev. A **8**, 3154 (1973).
5. R.F. Bishop, J. Low Temp. Phys. **15**, 601 (1974).
6. E.B. Kolomeisky and J.P. Straley, Phys. Rev. B **46**, 13942 (1992).
7. T.D. Lee and C.N. Yang, Phys. Rev. **105**, 1119 (1957); T.D. Lee, K. Huang, and C.N. Yang, Phys. Rev. **106**, 1135 (1957).

8. I.O. Vakarchuk and V.S. Pastukhov, Zh. Fiz. Dosl. (to be published).
9. I.O. Vakarchuk, *Introduction to the Many-Body Problem* (Lviv Nat. Univ., Lviv, 1999) (in Ukrainian).
10. S.R. Hore and N.E. Frankel, Phys. Rev. B **12**, 2619 (1975).
11. A.L. Fetter and J.D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).
12. D.N. Zubarev, Usp. Fiz. Nauk **71**, 71 (1960).
13. V.S. Pastukhov, Ukr. Fiz. Zh. **56**, 556 (2011).
14. G. Baym, J.-P. Blaizot, and J. Zinn-Justin, Europhys. Lett. **49**, 150 (2000).
15. V. A. Kashurnikov, N. V. Prokof'ev, and B. V. Svistunov, Phys. Rev. Lett. **87**, 120402 (2001).
16. P. Arnold and G. Moore, Phys. Rev. Lett. **87**, 120401 (2001).
17. B. Kastening, Phys. Rev. A **69**, 043613 (2004).

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РОЗРАХУНОК КРИТИЧНОЇ ТЕМПЕРАТУРИ
БОЗЕ-ГАЗУ З ДАЛЕКОДІЙНИМИ СИЛАМИ*В.С. Пастухов*

Резюме

У статті обчислено критичну температуру моделі бозе-газу зі степеневим законом залежності потенціальної енергії попарної взаємодії від відстані між частинками. У граничних випадках результат відтворює відомі результати з літератури. У наближенні хаотичних фаз (RPA) проаналізовано параметри спектра колективних збуджень моделі та отримано довгохвильову асимптотику структурного фактора системи при температурах, вищих за температуру фазового переходу.