LOCALIZED STATES IN A NONLINEAR MEDIUM CONTAINING A PLANE DEFECT LAYER WITH NONLINEAR PROPERTIES

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In the framework of the quasiclassical approach, the soliton states localized near a plane defect layer with nonlinear properties at different signs of nonlinearity and different characters of the interaction between elementary excitations in the system and the defect layer have been studied. The quantum-mechanical interpretation of corresponding nonlinear localized modes is proposed in terms of bound states of a large number of elementary excitations. The existence domains for such states are determined. The properties of those states and their dependence on the character of the interaction of elementary excitations with one another and with the defect are studied.

1. Introduction

The research carried out in this work is immediately connected with a rather new actual direction in modern theoretical physics, the theory of nonlinear waves and solitons in solid-state physics. The recent studies in this domain were directed toward studying the solitons in actual physical systems with regard for their discreteness, structure imperfection, internal microstructure, and other features. From the viewpoint of technological applications, the largest attention is focused on multilayered structures of various types. For instance, these are multilayered magnetic systems, the application of which can be of interest owing to their magnetooptical properties and the phenomenon of giant magnetoresistance in them, as well as multilayered crystals with multiatomic unit cells, high-temperature superconducting compounds, and others. In nonlinear optics, layered and modulated media are used in fiber systems, optical delay lines, and so forth (see, e.g., works [1, 2]).

In nonlinear optics, nonlinear media (e.g., a magnet, an elastic crystal, or an optically transparent insulator) are considered, as a rule. Such a medium contains narrow layers, which differ from the medium itself by their properties. In the case of waves with a stationary profile, the problem is equivalent to a study of nonlinear excitations in a one-dimensional system with point defects (nonlinear local vibrations). For a single isolated defect, this problem has been studied thoroughly in works [3, 4] for various signs of the nonlinearity of a medium and for various interactions between elementary excitations and the defect. In particular, the system with a defect characterized by linear properties was analyzed. Systems with nonlinear defects in a linear environment were studied, e.g., in works [5, 6].

In this work, in the framework of a nonlinear Schrödinger equation with an arbitrary sign of the *nonlinearity* of a medium, we have considered the excitations, which are localized near a plane *nonlinear* defect layer, in two cases: mutual attraction or repulsion between elementary excitations and the defect layer. For gaining a better understanding of the physical nature of considered nonlinear localized states, their quasiclassical quantization is carried out, and a relation between the total energy of the system and the number of bound elementary excitations in it is deduced.

2. Soliton States Localized Near the Defect Layer

While studying the soliton excitations, the effective size of which, depending on the soliton frequency, can vary in a wide interval, let us consider a local defect as a perturbation of parameters of the nonlinear medium. Let this perturbation be concentrated in a region with a size much smaller than the soliton width. In the presence of such a defect with nonlinear properties, the nonlinear Schrödinger equation for the field variable $u(z,t)$ looks like

$$i\,\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial z^2}+2\sigma\left|u\right|^2 u=-\lambda\delta(z)\left|u\right|^2 u, \tag{1}$$

where $\sigma=\pm 1$ characterizes the interaction between elementary excitations ($\sigma=+1$ corresponds to their mutual attraction and $\sigma=-1$ to repulsion), and λ is the characteristic magnitude of the defect (its "capacity"). Elementary excitations are effectively attracted to the defect, if $\lambda>0$, and repulsed from it, if $\lambda<0$.

In a defect-free environment ($\lambda=0$) and in the linear limit ($\sigma=0$), the dispersion law for linear waves $u\left(z,t\right)\sim\exp\left\{i\left(kz-\omega t\right)\right\}$ has the form $\omega=k^2$, and the spectrum of linear perturbations is extended over the semiaxis $\omega\geq 0$.

Note that, in the case of a defect in a linear medium, there also exist vibrations localized at the defect. For a linear defect with "capacity" λ, i.e. when the right-hand side of Eq. (1) equals $-\lambda\delta(z)u$, such localized states are possible only in the case of attracting defect ($\lambda>0$) [4]. Their frequency equals $\omega_l=\lambda^2/4$ and lies below the lower edge of the continuous spectrum. At $\lambda<0$, such localized vibrations are absent.

For a nonlinear defect considered in this work and a linear medium, the corresponding equation for $u(z,t)$ – an analog of Eq. (1) – has the following solution for a stationary localized state:

$$u=u_0\exp\left\{-\varepsilon\left|z\right|-i\omega t\right\}. \tag{2}$$

Here, $\varepsilon=\sqrt{-\omega}$ and $u_0=\sqrt{\frac{2}{\lambda}}\sqrt{\varepsilon}$. Hence, we obtain the same relation between the localized state frequency ω and the field amplitude at the defect site as that between the frequency of an anharmonic oscillator and the amplitude of its oscillations,

$$\omega=-\frac{\lambda^2}{4}u_0^4. \tag{3}$$

Introducing the total number of elementary excitations in the system,

$$N=\int\limits_{-\infty}^{+\infty}\left|u\right|^2 dz, \tag{4}$$

and the total system energy,

$$W=\int\limits_{-\infty}^{+\infty}\left\{\left|\frac{\partial u}{\partial z}\right|^2-\frac{\lambda}{2}\delta(z)\left|\,u\,\right|^4\right\}dz, \tag{5}$$

one can see that, in the framework of the model concerned, those parameters of the system do not depend on the frequency,

$$N=\frac{2}{\lambda},\quad W=0. \tag{6}$$

However, this property is not universal. Making allowance for the medium nonlinearity in a vicinity of the defect, when the left-hand side of Eq. (1) includes the term $2\sigma\left|u\right|^2 u$, and taking only linear properties of the defect into account, we obtain the following dependences [4]:

$$N=2\sigma\left(\varepsilon-\frac{\lambda}{2}\right),\quad W=-\frac{\sigma}{3}\left(2\varepsilon^3-\frac{\lambda^3}{4}\right). \tag{7}$$

Equation (1) of motion is the Euler equation for a Lagrangian with the following density:

$$L=\frac{i}{2}\left(u^*\frac{\partial u}{\partial t}-u\frac{\partial u^*}{\partial t}\right)-\left|\frac{\partial u}{\partial z}\right|^2+\sigma\left|u\right|^4+\frac{\lambda}{2}\delta(z)\left|u\right|^4. \tag{8}$$

Let us seek the stationary solutions of the nonlinear Schrödinger equation (1) in the form

$$u\left(z,t\right)=u\left(z\right)\exp(-i\omega t), \tag{9}$$

where $u(z)\to 0$ as $z\to\pm\infty$. Then Eq. (1) for the function $u(z)$ reads

$$\frac{\partial^2 u}{\partial z^2}+\omega u+2\sigma u^3=-\lambda\delta(z)u^3. \tag{10}$$

The solution of Eq. (10) is reduced to the solution of the homogeneous equation

$$\frac{\partial^2 u}{\partial z^2}+\omega u+2\sigma u^3=0 \tag{11}$$

in the regions $z>0$ and $z<0$ with the following boundary conditions at the point $z=0$:

$$u\left|_{+0}\right.=u\left|_{-0}\right., \tag{12}$$

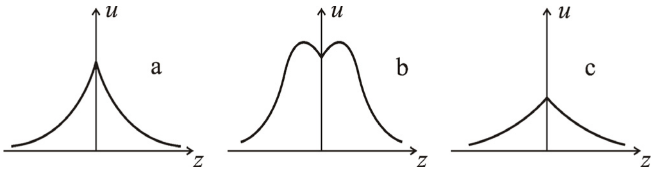

Fig. 1. Field distributions in a nonlinear localized state in the cases $\sigma = +1$ and $\lambda > 0$ (a), $\sigma = +1$ and $\lambda < 0$ (b), and $\sigma = -1$ and $\lambda > 0$ (c)

$$\frac{\partial u}{\partial z}\bigg|_{+0} - \frac{\partial u}{\partial z}\bigg|_{-0} = -\lambda\ u^3\,\big|_0 . \tag{13}$$

In the case $\sigma = +1$, the solution that satisfies the boundary conditions looks like

$$u\,(z) = \frac{\varepsilon}{\mathrm{ch}\ [\varepsilon\ (\,|\,z| - z_0)]}, \tag{14}$$

where the parameter $\varepsilon \equiv \sqrt{-\omega}$ characterizes the amplitude and the localization region of the solution, as well as the excitation frequency, and the parameters ε and z_0 are connected by a relation that follows from the second boundary condition (13),

$$\mathrm{sh}\ (2\varepsilon z_0) = -\lambda\varepsilon. \tag{15}$$

Formula (15) shows that $\mathrm{sgn}\, z_0 = -\mathrm{sgn}\,\lambda$, the interval of allowable frequencies is unbounded from below at any λ-sign, and the highest possible frequency of the solution equals zero, coinciding with the lower edge of the linear wave spectrum.

In the case $\sigma = -1$, the solution that satisfies the boundary conditions looks like

$$u\,(z) = \frac{\varepsilon}{\mathrm{sh}\ [\varepsilon\ (\,|\,z| - z_0)]}. \tag{16}$$

The quantity z_0 can now acquire only negative values, because the localized state in the case $\sigma = -1$ is realized only provided that the defect is attractive ($\lambda > 0$). The relation between the parameters ε and z_0 is determined by the same relation (15), as in the case where $\sigma = +1$. From relation (15), it follows that, at positive λ, it must be $z_0 < 0$. Similarly to the previous case ($\sigma = +1$), the interval of allowable frequencies is unbounded from below, and the maximum value (zero) corresponds to the edge of the linear wave spectrum.

Hence, the nonlinear localized states exist, if the following relations between the parameters σ and λ hold: (a) $\sigma = +1$ and $\lambda > 0$, (b) $\sigma = +1$ and $\lambda < 0$, and (c) $\sigma = -1$ and $\lambda > 0$. The same localized states in the system were found in work [4]. Let us examine the structure of the solutions in those three cases (Fig. 1) in more details.

(a) $\sigma = +1$ and $\lambda > 0$. The vibration amplitude maximum is located at the impurity site (because $z_0 < 0$), and the solution has the form exhibited in Fig. 1,a. In this case, the elementary excitations attract one another and are attracted to the defect. Near the edge of the linear wave spectrum, where $\omega \to 0$ ($\varepsilon \to 0$), relation (15) implies that $z_0 \approx -\lambda/2$, and the amplitude of localized state depends on the frequency as follows:

$$u(z = 0)|_{\ \omega \to 0} \approx \sqrt{-\omega}. \tag{17}$$

At $\omega = 0$, the nonlinear localized mode transforms into ordinary vibrations of linear theory.

(b) $\sigma = +1$ and $\lambda < 0$. The elementary excitations attract one another. However, they are repulsed from the defect, and the vibration amplitude maximum does not coincide with the defect location. The localized state is a bound state of two solitons, which are symmetrically arranged on both sides of the defect, with their centers being located at the points $\pm z_0$ (Fig. 1,b). In the limit of the lowest possible frequency of the solution, $\omega \to 0$, the distance between coupled solitons tends to a constant value $2z_0 \approx -\lambda = |\lambda|$, whereas the amplitude of defect vibrations approaches zero according to formula (17). The amplitude of solitons also tends to zero,

$$A|_{\omega=0} = u(z = \pm z_0)\,|_{\omega=0} = \varepsilon|_{\omega=0} = 0, \tag{18}$$

so that this limit is a low-amplitude one.

(c) $\sigma = -1$ and $\lambda > 0$. The excitations repulse one another, but they are attracted to the defect. The profile of a localized excitation looks approximately as that in case (a), i.e. the vibration amplitude maximum is located at the point of defect localization (Fig. 1,c). The frequency interval, where the local mode exists, is also the same, $-\infty < \omega < 0$. In the linear limit $\omega \to 0$, the parameter $z_0 \approx -\lambda/2$, as it was in case (a), but the amplitude of impurity vibrations tends now to a finite value (cf. Eq. (17))

$$u(z = 0)|_{\ \omega \to 0} \approx \frac{2}{\lambda}. \tag{19}$$

The solution for the localized state (16) transforms into a function with power-law asymptotes at the infinity, i.e. into an algebraic soliton of the form

$$u\,(z) = \frac{1}{|\,z| + \lambda/2}. \tag{20}$$

Earlier in work [4], it was indicated that, as a rule, the possibility for power-law solitons to exist at the edge of a continuous linear wave spectrum is associated with the account for competing nonlinearities in the evolution equations or with the presence of many-particle interactions of various types [7]. For instance, an algebraic soliton can emerge at the edge of a continuous linear wave spectrum, when the pair repulsion between quasiparticles and their three-particle attraction are considered, i.e. when terms of the type $u\,|u|^4$ are taken into account in Eq. (1) with $\sigma = -1$. It was shown earlier that such solitons are unstable [8].

In the considered case with a defect, the situation is similar in many respects. Namely, there are the interactions of two types: the particle-to-particle pair interaction described by the term $2\sigma\,|u|^2\,u$ and one-particle interaction between elementary excitations and the inhomogeneity, whose effective intensity depends, however, on the field intensity $\lambda_{\text{eff}} = \lambda\,|u|^2$ (in work [4], the one-particle interaction intensity was determined by the constant λ). Power-law solitons exist in our case in the presence of a pair repulsion between quasiparticles ($\sigma = -1$) or their attraction to the defect ($\lambda > 0$).

As follows from the consideration of three possible localized states, all localized states in our system, in contrast to the system analyzed in work [4], exist in the same frequency interval $-\infty < \omega < 0$. For the better understanding of the physical nature of these localized states, let us execute their quasiclassical quantization. Equation (1) describes the dynamics of a conservative system and, consequently, it has an evident integral of motion, namely the total energy of the system

$$W = \int\limits_{-\infty}^{+\infty} dz\,\left\{\left|\frac{\partial u}{\partial z}\right|^2 - \sigma\,|\,u\,|^4\, - \frac{\lambda}{2}\,\delta(z)\,|\,u\,|^4\right\}, \qquad (21)$$

In addition, it also has the additional integral of motion (4), i.e. the total number of elementary excitations—field quanta—localized in the system [3]. Up to now, we characterized the localized solution by its frequency ω (or the parameter ε). To clarify the quantum-mechanical nature of the soliton state, it is convenient to pass from the consideration of the frequency as its dynamic characteristic to the number of excitations, N, which are bound in this localized state.

Let us first consider the case $\sigma = +1$ and express the integrals of motion N and W in terms of the frequency ω (or the related parameter ε). Substituting solution (14) in expression (4) and taking the dependence $z_0(\varepsilon)$ into account (see Eq. (15)), we obtain the following relation for the total number of excitations:

$$N = 2\varepsilon\,(1 + \text{th}(\varepsilon z_0)) = 2\varepsilon + \frac{2}{\lambda}\,\left(1 - \sqrt{1 + (\lambda\varepsilon)^2}\right). \qquad (22)$$

From this formula, it is easy to reveal that, if the parameter λ is positive (i.e. the defect has attractive character), the total number of excitations in the system is bounded from above,

$$0 \le N < 2/\lambda, \quad \lambda > 0, \qquad (23)$$

with $N \to 2/\lambda$ as $\varepsilon \to +\infty$ and $N \to 0$ as $\varepsilon \to 0$. Note that the critical value $N^* = 2/\lambda$ corresponds to the total number of excitations in the system with nonlinear defect in a linear medium (see Eq. (6)).

If $\lambda < 0$, the parameter N can acquire any positive value,

$$N \ge 0, \quad \lambda < 0. \qquad (24)$$

From relation (22), it is easy to obtain the inverse relation $\varepsilon = \varepsilon(N)$,

$$\varepsilon = \frac{N}{4}\,\frac{4 - \lambda N}{2 - \lambda N} \qquad (25)$$

and, accordingly, the dependence $\omega = \omega(N)$,

$$\omega = -\left(\frac{N}{4}\right)^2\left(\frac{4 - \lambda N}{2 - \lambda N}\right)^2 = -\left(\frac{N}{4}\right)^2\left(1 + \frac{2}{2 - \lambda N}\right)^2. \qquad (26)$$

In a similar way, by substituting the explicit form of solution (14) in expression (21) and taking relation (15) into account, we can obtain the explicit expression for the total energy W of the system as a function of the parameter ε, $W = W(\varepsilon)$. Then, excluding the parameter ε from the expression $W = W(\varepsilon)$ and the dependence $N = N(\varepsilon)$ (see Eq. (22)) with the use of formula (25), it is easy to obtain a relation between the total energy of a localized state and the total number of elementary excitations bound in this state,

$$W\,(N) = \frac{N^3}{6} - \frac{\lambda N^4}{32} - \frac{N^3}{32}\,\frac{(4 - \lambda N)^2}{2 - \lambda N}. \qquad (27)$$

We note once more that the total number of excitations, N, cannot exceed the maximum value $N^* = 2/\lambda$ (see Eq. (23)).

We now consider the case $\sigma = -1$. Let us substitute solution (16) into formula (4) for N. In view of the dependence $z_0(\varepsilon)$, relation (15) yields the following dependence $N = N(\varepsilon)$ different from expression (22) obtained for the case $\sigma = +1$:

$$N = -2\varepsilon\,(1 + \text{cth}(\varepsilon z_0)) = -2\varepsilon + \frac{2}{\lambda}\,\left(1 + \sqrt{1 + (\lambda\varepsilon)^2}\right). \qquad (28)$$

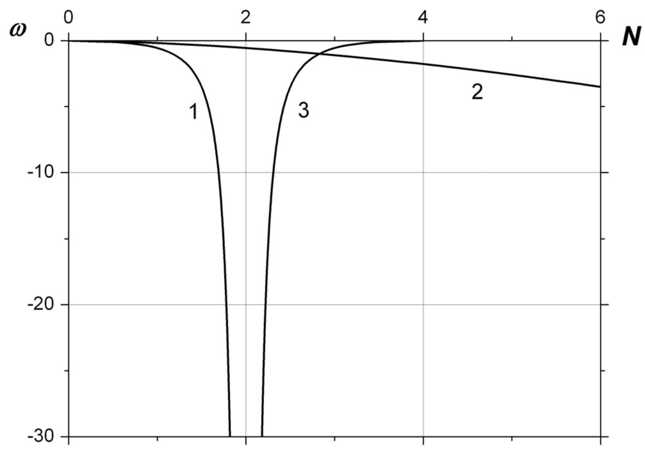

Fig. 2. Dependences $\omega = \omega(N)$ for nonlinear localized states of three possible types: $\sigma = +1$ and $\lambda = +1$ (*1*), $\sigma = +1$ and $\lambda = -1$ (*2*), and $\sigma = -1$ and $\lambda = +1$ (*3*)

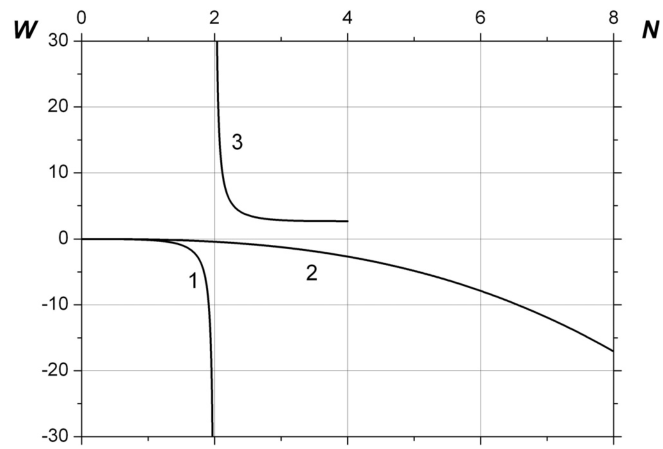

Fig. 3. Dependences $W = W(N)$ for nonlinear localized states of three possible types: $\sigma = +1$ and $\lambda = +1$ (*1*), $\sigma = +1$ and $\lambda = -1$ (*2*), and $\sigma = -1$ and $\lambda = +1$ (*3*)

It is easy to determine from this formula that the total number of excitations in the system, N, falls within the interval (the parameter λ is positive in the case $\sigma = -1$)

$$2/\lambda < N \leq 4/\lambda \tag{29}$$

with $N \to 2/\lambda$ as $\varepsilon \to +\infty$ and $N \to 4/\lambda$ as $\varepsilon \to 0$.

From expression (28), it is easy to determine the inverse relation $\varepsilon = \varepsilon(N)$ (cf. Eq. (25)),

$$\varepsilon = -\frac{N}{4}\frac{4 - \lambda N}{2 - \lambda N}. \tag{30}$$

In this case ($\sigma = -1$), the dependence $\omega = \omega(N)$ has the same analytic form (26) as that in the case $\sigma = +1$. This is also true for the dependence $W = W(N)$ of the total energy of a localized state on the number of elementary excitations bound in this state, i.e. formula (27) is valid in both cases $\sigma = +1$ and $\sigma = -1$.

At $\sigma = +1$, the positivity of the integral of motion N in Eq. (22) and relations (23) and (24) yield the conclusion that any positive ε-values are possible at any λ-values. The boundary condition (15) does not impose any additional restrictions on a domain, where the solution exists. This domain is determined by the simple relation

$$\varepsilon \geq 0. \tag{31}$$

At $\sigma = -1$ (when the solution exists only if $\lambda > 0$), the positivity of N-value in Eq. (28), relation (29), and boundary condition (15) imply that the domain, where the solution exists, is also determined by relation (31).

The value $\varepsilon = 0$ ($\omega = 0$) corresponds to the lower edge of a linear wave spectrum.

In Fig. 2, the dependences $\omega = \omega(N)$ are depicted for all three possible localized states and the values $\lambda = \pm 1$ ($N^* = 2$). The allowed regions for the parameter N – Eqs. (23), (24), and (29) – are taken into account.

Figure 3 illustrates the dependences $W = W(N)$ obtained for all possible localized states ($\lambda = \pm 1$ and $N^* = 2$).

By differentiating dependence (27) with respect to N and by using relations (22) and (28) for the dependence $N(\varepsilon)$, it is easy to verify that the relation

$$\frac{\partial W}{\partial N} = \omega, \tag{32}$$

which is usual for one-frequency solitons and is valid for the conservative nonlinear systems with the integral of motion N, is also obeyed. Therefore, the frequency of nonlinear local vibrations plays the role of a chemical potential for relevant bound elementary excitations.

3. Conclusions

In this work, all possible stationary states localized at a nonlinear defect (in a vicinity of the plane defect layer) with various properties have been studied with the use of a nonlinear Schrödinger equation with an arbitrary sign of the nonlinear term. The result obtained can be interpreted in terms of elementary excitations that interact with one another and with the defect.

It is found that the emergence of states localized at the defect layer with nonlinear properties surrounded by a nonlinear medium is possible at any anharmonicity sign (at any σ-sign) in the case where the elementary excitations are attracted to the defect layer ($\lambda > 0$). If the excitations attract one another ($\sigma = +1$), the localization of a nonlinear excitation near the defect layer is possible, even if the elementary excitations are repulsed from the defect ($\lambda < 0$). In the case where $\sigma = -1$ (repulsion of excitations from one another), the nonlinear localized excitations are possible only if $\lambda > 0$. The quasiclassical quantization of the determined localized modes is carried out, and the dependence of the total energy of the system on the total number of elementary excitations (quasiparticles) is derived.

The results obtained can be useful for studying the localized states in a system with two nonlinear defects and, as a further extension, in a periodic system of nonlinear defects (plane defect layers) in a nonlinear medium.

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ЛОКАЛІЗОВАНІ СТАНИ У НЕЛІНІЙНОМУ СЕРЕДОВИЩІ З ПЛОСКИМ ДЕФЕКТНИМ ШАРОМ, ЯКИЙ МАЄ НЕЛІНІЙНІ ВЛАСТИВОСТІ

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Р е з ю м е

У межах квазікласичного підходу вивчено солітонні стани, які локалізовані біля плоского дефектного шару, що має нелінійні властивості, при різних знаках нелінійності середовища та різному характері взаємодії елементарних збуджень системи з дефектним шаром. Надано квантову інтерпретацію цих нелінійних локалізованих мод на мові зв'язаних станів з великою кількістю елементарних збуджень. Визначено області існування та досліджено властивості таких станів залежно від характеру взаємодії елементарних збуджень між собою та з дефектом.