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SUPERINTEGRABLE SYSTEMS WITH ARBITRARY SPIN

Superintegrable quantum mechanical systems with spin are presented. These systems admit a generalized Laplace–Runge–Lenz vector and possess the dynamical symmetry with respect to group $SO(4)$ for bound states. Our discussion is restricted mainly to 2d and 3d systems, but a special class of arbitrary dimension systems is presented as well.

Keywords: superintegrability, supersymmetry, spin, anomalous interaction.

1. Introduction

Exactly solvable problems of quantum mechanics are interesting and popular subjects. They can be described fully and in a straightforward way free of complications caused by perturbation methods. Their solvability is usually connected with non-trivial symmetries, which are of particular interest by themselves. In addition, the exact solutions form complete sets of functions, which can be used to find solutions of other problems.

There are two properties of quantum mechanical systems that can make them exactly solvable: *supersymmetry* and *superintegrability*. There are very interesting relations between these properties [1]. Moreover, some of quantum mechanical systems, like a hydrogen atom or isotropic harmonic oscillator, are both superintegrable and supersymmetric, and exactly such systems are, as a rule, very interesting and important.

A quantum mechanical system with n degrees of freedom is called *integrable* if it admits $n-1$ commuting integrals of motion. If there exists at least one additional integral of motion, the system is called *superintegrable*.

The system is treated as supersymmetric in two cases: when some of its integrals of motion form a superalgebra, and when its Hamiltonian has a specific symmetry with respect to the Darboux transform, called the shape invariance.

The classification of superintegrable models of quantum mechanics was started with work [2], where 2d systems admitting second-order integrals of motion had been presented. We will not discuss the history of this interesting field (see survey [3], but restrict ourselves to contemporary results obtained for superintegrable systems with spin. Moreover, just the systems which admit symmetries analogous to the Fock symmetry of a hydrogen atom will be considered. In other words, we will present superintegrable systems with spin that admit analogues of the Laplace–Runge–Lenz (LRL) vector.

The LRL vector is a corner stone of celestial mechanics. It has a great value also in quantum mechanics. In particular, using this vector, Pauli found the spectrum of a hydrogen atom before the Schrödinger equation was discovered [4].

There are only few examples of 3d quantum mechanical systems with spin admitting the LRL vector. They are the MICZ–Kepler system [5], [6], a dyon with gyromagnetic ratio $g = 4$, interacting with a magnetic monopole field plus a Coulomb plus a fine-tuned inverse-square potential [7], and a neutral particle with a non-trivial dipole momentum [8]. In addition, a system with spin-orbit interaction and a special inverse-square potential with *fixed* coupling constant was presented in [9], but it is equivalent to the direct sum of two scalar hydrogen-like systems [8]. All the mentioned systems include particles with spin 1/2. One more example of a (reducible) system with spin is the supersymmetric extension of a 3d hydrogen atom discussed in [10], but it has some controversial points, see Appendix.

In the present paper, we classify the LRL vectors with arbitrary spin. It is done for 2d and 3d systems. In addition, we present some systems of arbitrary dimensions which admit this vector and so have a hidden symmetry of the Fock type.

Mathematically, the subject of this classification includes systems of coupled Schrödinger equations of the following form:

$$H\psi = E\psi, \quad \text{where} \quad H = \frac{p^2}{2m} + V(\mathbf{x}). \tag{1}$$

Here, ψ is a multicomponent wave function, and $V(\mathbf{x})$ is a *matrix potential*. The well-known example of Eq. (1) is the Schrödinger–Pauli equation for a neutral fermion.

Physically, just equations of the generic form (1) are requested to construct models of particles that have nontrivial dipole momenta. In particular, these particles can be neutral. A perfect example of such particle is a neutron. Its electrical charge is zero, but the magnetic moment (and, probably, the electric moment) is nontrivial.

We present a survey of the results concerning superintegrable systems (1). The results discussed in Subsection 3.3 are new; in Section 4, we present a short sketch of results published on the problem of a Runge–Lenz vector for systems with arbitrary spin.

2. Hydrogen Atom

This system is described by the following Hamiltonian:

$$H = \frac{p^2}{2m} + V(x), \tag{2}$$

where

$$p^2 = p_1^2 + p_2^2 + p_3^2, \quad p_1 = -i\frac{\partial}{\partial x_1}, \quad V = -\frac{\alpha}{x},$$
$$x = \sqrt{x_1^2 + x_2^2 + x_3^2}, \ \alpha = e^2 > 0.$$

Hamiltonian (2) commutes with the orbital momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$. An additional integral of motion is the LRL vector

$$\mathbf{K} = \frac{1}{2m}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) + \mathbf{x}V. \tag{3}$$

These integrals of motion satisfy the following commutation relations:

$$[L_a, H] = [K_a, H] = 0, \tag{4}$$

$$[L_a, L_b] = \mathrm{i}\varepsilon_{abc}L_c, \quad [K_a, L_b] = \mathrm{i}\varepsilon_{abc}K_c,$$
$$[K_a, K_b] = -\frac{2\mathrm{i}}{m}\varepsilon_{abc}L_cH. \tag{5}$$

On the set of eigenvectors of the Hamiltonian H corresponding to an eigenvalue E, relations (5) specify a Lie algebra, which is isomorphic to $so(4)$ for $E < 0$ and to $so(1,3)$ for $E > 0$. Using this symmetry and our knowledge of irreducible representations of algebra $so(4)$, it is possible to find the admissible eigenvalues E algebraically [4]. Let us present some details of this procedure for negative E.

Choosing a new basis

$$\mathbf{U}^+ = \frac{1}{2}(\mathbf{L} + \mathbf{K}'), \quad \mathbf{U}^- = \frac{1}{2}(\mathbf{L} - \mathbf{K}'), \tag{6}$$

where $\mathbf{K}' = \sqrt{-\frac{m}{2E}}\mathbf{K}$, we find that the vectors $\mathbf{U}^+$ and $\mathbf{U}^-$ commute with each other. Their components satisfy the commutation relations

$$[U_a^{\pm}, U_b^{\pm}] = i\varepsilon_{abc}U_c^{\pm}, \tag{7}$$

and so form a basis of the Lie algebra $so(4) \cong$ $\cong so(3) \bigoplus so(3)$. There are two Casimir operators for this algebra:

$$C_{\pm} = 4(\mathbf{U}^{\pm})^2 = \mathbf{L}^2 + \nu^2\mathbf{K}^2 \pm 2\nu\mathbf{L}\cdot\mathbf{K}, \tag{8}$$

where $\nu = \sqrt{-\frac{m}{2E}}$. Since $\mathbf{L}\cdot\mathbf{K} \equiv 0$, we have $C_+ = C_-$ in our case.

The irreducible representations of algebra $so(3)$, which is specified by the commutation relations (7), are labeled by integers or half-integers, and eigenvalues $c_{\pm}$ of the Casimir operator $C_{\pm}$ are:

$$c_- = 4q(q+1), \quad c_+ = 4g(g+1), \tag{9}$$

where q and g are non-negative integers or half-integers.

Since $\mathbf{K}^2 = \alpha^2 + (\mathbf{L}^2 + 1)\frac{2H}{m}$, Eq. (8) gives: $C_{\pm} =$ $= -1 - \frac{\alpha^2 m}{2E}$. Thus, it follows from (9) that

$$E = -\frac{m\alpha^2}{2n^2}, \quad n = 2q + 1 = 2g + 1 = 1, 2, \ldots. \tag{10}$$

We will see that, in an analogous manner, it is possible to find the energy spectrum of other models admitting the (generalized) LRL vector. The term "generalized LRL vector" denotes a vector integral of motion depending on the spin, which satisfy (together with the total orbital momentum) the same commutation relations (5) as those for the LRL vector for a hydrogen atom.

3. Systems with Spin $\frac{1}{2}$

3.1. 2d systems

The non-relativistic model of hydrogen atom ignores the spin of the electron. The first (historically) model with spin that admits an analogue of the LRL vector was discovered by Pron'ko and Stroganov as far back as in 1977 [12]. It is based on the Hamiltonian

$$\mathcal{H} = \frac{p_1^2 + p_2^2}{2m} + \lambda\frac{\sigma_1 x_2 - \sigma_2 x_1}{r^2}, \quad r^2 = x_1^2 + x_2^2 \qquad (11)$$

and describes a neutral spinor anomalously interacting with the magnetic field generated by a constant straight line current directed along the third coordinate axis. In (11), σ_1 and σ_2 are Pauli matrices, and λ is the integrated coupling constant.

Hamiltonian (11) commutes with the third component of the total orbital momentum

$$J_3 = x_1 p_2 - x_2 p_1 + S_3, \qquad (12)$$

where $S_3 = \frac{1}{2}\sigma_3$. There are also two more constants of motion for (11), namely:

$$\begin{aligned} K_1 &= \frac{1}{2}(J_3 p_1 + p_1 J_3) + \frac{m}{r}\mu(\mathbf{n})x_2, \\ K_2 &= \frac{1}{2}(J_3 p_2 + p_2 J_3) - \frac{m}{r}\mu(\mathbf{n})x_1, \end{aligned} \qquad (13)$$

where $\mu(\mathbf{n}) = \lambda(S_1 n_2 - S_2 n_1)$, $n_a = \frac{x_a}{r}$, and $\mathbf{n} = (n_1, n_2)$.

Operators (12) and (13) commute with $\mathcal{H}$ and satisfy the following relations:

$$\begin{aligned} &[J_3, K_1] = \mathrm{i}K_2, \quad [J_3, K_2] = -\mathrm{i}K_1, \\ &[K_1, K_2] = -2\mathrm{i}mJ_3\mathcal{H}. \end{aligned} \qquad (14)$$

The algebra spanned on the basis elements K_1, K_2, and J_3 can be defined on the sets of solutions of the eigenvalue problem:

$$\mathcal{H}\psi = E\psi. \qquad (15)$$

Changing Hamiltonian $\mathcal{H}$ by its eigenvalue E, we obtain the Lie algebra isomorphic to $so(3)$ provided $E < 0$ or to $so(1,2)$ if E is positive.

Using this symmetry, it is possible to find the eigenvalues E for coupled states algebraically. To do this, it is sufficient to rescale the operators K_1 and KA_2:

$$K_1 = \sqrt{-2mE}J_1, \quad K_2 = \sqrt{-2mE}J_2. \qquad (16)$$

Then J_1, J_2 and J_3 realize a representation of algebra $so(3)$. Supposing this representation be irreducible, we obtain the following constraint for eigenvectors:

$$C\psi \equiv (J_1^2 + J_2^2 + J_3^2)\psi = n(n+1)\psi. \qquad (17)$$

In addition, ψ is supposed to be an eigenvector of the operator J_3 commuting with $\mathcal{H}$ and C:

$$J_3\psi = k\psi, \quad k = \frac{1}{2}, \frac{3}{2}, \dots. \qquad (18)$$

Substituting (12), (13), and (16) into (17) and using (15) and (18), we obtain the energy spectrum in the form

$$E = -\frac{m\lambda^2}{(2n + 2k + 1)^2}, \qquad (19)$$

where $n = 0, 1, 2, \dots$.

The corresponding eigenvectors can be found using the supersymmetry and the shape invariance of Hamiltonian (11), as it was done in [13]. This supersymmetry is closely related to the integrals of motion (13), which are components of the 2d LRL vector. Moreover, the corresponding supercharges can be expressed via these components [14].

3.2. 3d system

To construct a 3d system with spin, it is necessary to change the angular momentum $\mathbf{L}$ by the total angular momentum

$$\mathbf{L} \to \mathbf{J} = \mathbf{L} + \mathbf{S}, \qquad (20)$$

where $\mathbf{S}$ is a spin vector, whose components satisfy the conditions

$$[S_a, S_b] = \mathrm{i}\varepsilon_{abc}S_c, \quad S_1^2 + S_2^2 + S_3^2 = s(s+1)I. \qquad (21)$$

For spin $\frac{1}{2}$, we have $\mathbf{S} = (\frac{1}{2}\sigma_1, \frac{1}{2}\sigma_2, \frac{1}{2}\sigma_3)$.

The LRL vector with spin is supposed to have the form

$$\hat{\mathbf{K}} = \frac{1}{2m}(\mathbf{p} \times \mathbf{J} - \mathbf{J} \times \mathbf{p}) + \mathbf{x}\hat{V}, \qquad (22)$$

where $\hat{V}$ is a potential that specifies the corresponding Hamiltonian (2). Now, it is a matrix of dimension $(2s+1)(2s+1)$ depending on $\mathbf{x}$.

Let us note that components of vector (22) can be rewritten in the form

$$K_a = \frac{1}{2}\left(p_b J_{ab} + J_{ab} p_b\right) + x_a\hat{V}, \qquad (23)$$

where $J_{ab} = \varepsilon_{abc}J_c$, $a, b, c = 1, 2, 3$, and ε_{abc} is the Levi-Civita symbol. This uniform representation is valid for both 2d and 3d cases, compare (23) for $a, b = 1, 2$ and (16).

By definition, vector (22) should commute with the Hamiltonian

$$\hat{H} = \frac{p^2}{2m} + \hat{V}. \tag{24}$$

The necessary and sufficient conditions for this commutativity are given by the following equations [11]:

$$[\hat{V}, J_{ab}] = 0, \tag{25}$$

$$x_a \nabla_a \hat{V} + \hat{V} = 0, \tag{26}$$

$$S_{ab} \nabla_b \hat{V} + \nabla_b \hat{V} Sab = 0, \tag{27}$$

where $\nabla_a = \frac{\partial}{\partial x_a}$ and $S_{ab} = \varepsilon_{abc} S_c$.

If conditions (25)–(27) are fulfilled, the operators $\mathbf{J}$, $\hat{\mathbf{K}}$, and $\hat{H}$ satisfy the same commutation relations (4) and (5) as the orbital momentum, LRL vector, and Hamiltonian of a hydrogen atom.

It can be proven that, up to a constant multiplier α, there is the unique choice for $\hat{V}$ [8]:

$$\hat{V} = \alpha \frac{\boldsymbol{\sigma} \cdot \mathbf{x}}{x^2}. \tag{28}$$

Moreover, this potential can be found without *a priori* restrictions on the generic form of vector integrals of motion given by equation (22) [15].

Thus, the total angular momentum and the LRL vector for spin $\frac{1}{2}$ have the form

$$\begin{aligned} \mathbf{J} &= \mathbf{L} + \frac{1}{2}\boldsymbol{\sigma}, \\ \hat{\mathbf{K}} &= \frac{1}{2m}(\mathbf{p} \times \mathbf{J} - \mathbf{J} \times \mathbf{p}) + \alpha \mathbf{x} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}}{x^2}, \end{aligned} \tag{29}$$

while the corresponding Hamiltonian is given by Eqs. (31) and (28).

Like in Section 2, we can find the eigenvalues of Hamiltonian (31), (28) algebraically. Using (29), we obtain $\hat{\mathbf{K}}^2 = \left(2\mathbf{J}^2 + \frac{3}{2}\right)\frac{H}{m} + \alpha^2$. Substituting this expression into (8) and setting $H = E$, we obtain

$$C_{\pm} = \left(\nu^2 \alpha^2 \pm \nu\alpha - \frac{3}{4}\right) I \tag{30}$$

. So, it follows from (9) that

$$(\nu\alpha - 1/2)^2 = (2q+1)^2, \quad (\nu\alpha + 1/2)^2 = (2g+1)^2$$

and

$$E = -\frac{m\alpha^2}{2N^2}, \tag{31}$$

where

$$N = n + 1/2, \quad n = 2q + 1 = 2g = 1, 2, \dots . \tag{32}$$

In contrast to the case of a hydrogen atom, the main quantum number N should be half-integer.

Let us note that Hamiltonian (31), (28) is shape-invariant, which makes it possible to find the energy spectrum (31), using tools of supersymmetric quantum mechanics. It has been done in work [8], where the corresponding eigenvectors are presented as well.

3.3. Systems of arbitrary dimension

It is well known that the LRL vector can be generalized to the case of a multidimensional Schrödinger equation with Coulomb potential [16]. Let us show that it is the case also for the case of multidimensional systems with spin.

Let us consider a multidimensional Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}, \tag{33}$$

where $\hat{p}^2 = p_1^2 + p_2^2 + \dots + p_d^2$, and V_d is a matrix potential. We suppose that this Hamiltonian is invariant with respect to the rotation group in d dimensions, whose generators have the standard form

$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \tag{34}$$

where $S_{\mu\nu}$ are matrices satisfying the familiar $so(d)$ commutation relations

$$[S_{\mu\nu}, S_{\lambda\sigma}] = \mathrm{i}(\delta_{\mu\lambda} S_{\nu\sigma} + \delta_{\nu\sigma} S_{\mu\lambda} - \delta_{\mu\sigma} S_{\nu\lambda} - \delta_{\nu\lambda} S_{\mu\sigma}), \tag{35}$$

where the subindices run from 1 to d. One more supposition is that, when reducing this matrix algebra to its subalgebra $so(3)$, one obtains a direct sum of irreducible representations $D\left(\frac{1}{2}\right)$. This means that the eigenvalues of the matrices $S_{\mu\nu}$ are equal to $\pm\frac{1}{2}$, and these matrices can be expressed via the basis elements γ_ν of the Clifford algebra

$$S_{\mu\nu} = \frac{1}{4}\left(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu\right), \tag{36}$$

which satisfy the relations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}. \qquad (37)$$

The dimension of the irreducible matrices γ_μ is equal to $2^{\left[\frac{d}{2}\right]}$, where $\left[\frac{d}{2}\right]$ is the entire part of $\frac{d}{2}$.

By definition, Hamiltonian (33) should commute with operators (34) and (23). It is the case iff the potential $\hat{V}$ satisfies relations (25)–(27).

The generic form of a potential satisfying (25) and (26) is given by the equations

$$\hat{V} = \frac{1}{r}\left(\lambda + \alpha\gamma_a n_a\right) \qquad (38)$$

for d odd and

$$\hat{V} = \frac{1}{r}\left(\lambda + \alpha\gamma_a n_a + \nu\gamma_{d+1}\gamma_a n_a\right) \qquad (39)$$

for d even. Here, λ, α, and ν are arbitrary parameters, $n_a = \frac{x_a}{r}$, $r = \sqrt{x_1^2 + x_2^2 + ... + x_d^2}$, and the summation from 1 to d is imposed over the repeating index a.

The remaining condition (27) reduces potentials (38) and (39) to the following unified form:

$$\hat{V} = \frac{\alpha}{r}\gamma_a n_a. \qquad (40)$$

Thus, we construct a d-dimensional quantum mechanical system with spin $\frac{1}{2}$ which is invariant with respect to the rotation group in d dimensions and admits the generalized LRL vector. This system is specified by Hamiltonian (33) with potential (40), i.e.,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{\alpha}{r^2}\gamma_a x_a, \qquad (41)$$

while the corresponding basis elements of algebra $so(d)$ and the additional vector integral of motion are given by Eqs. (34), (36) and (23), (36). These operators satisfy the commutation relations

$$[J_{ab}, \hat{H}] = [\hat{K}_a, \hat{H}] = 0, \qquad (42)$$

$$[\hat{K}_a, J_{bc}] = i(\delta_{ac}\hat{K}_b - \delta_{ab}\hat{K}_c), \quad [\hat{K}_a, \hat{K}_b] = -\frac{2\mathrm{i}}{m}J_{ab}\hat{H}, \qquad (43)$$

$$[J_{ab}, J_{cd}] = \mathrm{i}(\delta_{ac}J_{bd} + \delta_{bd}J_{ac} - \delta_{ad}J_{bc} - \delta_{bc}J_{ad}). \qquad (44)$$

Changing $\hat{H}$ by its eigenvalue $E < 0$ in (43) and (44), we obtain an algebra isomorphic to $so(d+1)$. More exactly, the operators J_{ab} and $J_{d+1a} = \sqrt{-\frac{m}{2E}}K_a$ satisfy the commutation relations (44), where the indices run from 1 to $d+1$.

Let us note that all Hamiltonians discussed in the previous sections are nothing but particular cases of (41). Like in Subsections 4.1 and 4.2, it is possible to use the hidden symmetry of Hamiltonian (41) w.r.t. algebra $so(d+1)$ to find its spectrum algebraically.

The second-order Casimir operator of group $SO(d+1)$ C_2 has the following form:

$$C_2 = J_{\mu\nu}J_{\mu\nu} = J_{ab}J_{ab} - \frac{1}{\nu}K_aK_a. \qquad (45)$$

Using the identity

$$K_aK_a = \alpha^2 + \left(J_{ab}J_{ab} + \frac{d(d-1)}{4}\right)\frac{\hat{H}}{m},$$

it is not difficult to express eigenvalues of Hamiltonian $\hat{H}$ via the eigenvalues c_2 of operator (45):

$$E = -\frac{2m\alpha^2}{d(d-1) + 4c_2}. \qquad (46)$$

Thus, analogues of the LRL vector exist for quantum mechanical multidimensional systems with spin. Moreover, like the case of a hydrogen atom, the energy spectrum of these systems can be found algebraically. The corresponding energy values are ultimately fixed and can be expressed via eigenvalues of the second order Casimir operator of group $SO(d+1)$ in accordance with Eq. (46).

4. Runge–Lenz Vector for Arbitrary Spin

In the previous section, we restrict ourselves to the discussion of LRL vectors with spin $\frac{1}{2}$. Let us show how the presented results can be extended to the case of systems with arbitrary spin.

4.1. Planar systems

Consider Hamiltonian (33), where $d = 2$, and $\hat{V}$ is a matrix of dimension $(2s+1)\times(2s+1)$. Let us suppose that it is rotationally invariant and describes a particle with spin s. Then it should commute with operator (12), where S_3 is a spin matrix which can be chosen in the diagonal form

$$S_3 = \mathrm{diag}(s, s-1, s-2, ..., -s). \qquad (47)$$

Let us show that the 2d LRL vector (16) can be generalized to the case of arbitrary s. To do it, we firstly represent the potential $\hat{V}$ in (33) as

$$\hat{V} = \frac{\alpha}{x}\mu(\mathbf{n}), \tag{48}$$

where $\mu(\mathbf{n})$ is the unknown matrix, and $\mathbf{n} = \frac{1}{x}(x_1, x_2)$. In other words, we represent Hamiltonian (33) in the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{\alpha}{x}\mu(\mathbf{n}). \tag{49}$$

Then operators (13) commute with $\hat{H}$ and satisfy Eqs. (14) if the matrix $\mu(\mathbf{n})$ satisfies the conditions [13]

$$[\mu(\mathbf{n}), J_3] = 0, \tag{50}$$

$$\mu_s(\mathbf{n})S_3 + S_3\mu(\mathbf{n}) = 0. \tag{51}$$

Solutions of Eqs. (51) were found in [17] and [8]. Since the results presented in [17] include superfluous arbitrary parameters, we will give solutions in the form obtained in [8]. We present no detailed calculations here, since they would be nothing but a reduced version of ones given in the following section.

The general solution of Eqs. (51) for $s = 1$ is given by the formulae

$$\mu(\mathbf{n}) = \nu(2(\mathbf{S}\times\mathbf{n})^2 - 1) + \lambda(2(\mathbf{S}\cdot\mathbf{n})^2 - 1). \tag{52}$$

Here, ν and λ are arbitrary real parameters, $\mathbf{S}\cdot\mathbf{n} = S_1n_1 + S_2n_2$ and $\mathbf{S}\times\mathbf{n} = S_1n_2 - S_2n_1$, S_1, and S_2 are matrices of spin 1:

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \quad S_2 = \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}, \quad S_3 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}. \tag{53}$$

For $s = \frac{3}{2}$, we obtain

$$\mu(\mathbf{n}) = \left(\nu + \mu S_3^2\right)\left(7\mathbf{S}\times\mathbf{n} - 4(\mathbf{S}\times\mathbf{n})^3\right). \tag{54}$$

Here, μ and ν are arbitrary parameters, S_1, S_2, and S_3 are the 4×4 matrices of spin $\frac{3}{2}$, which can be chosen in the form

$$S_1 = \frac{1}{2}\begin{pmatrix}0&\sqrt{3}&0&0\\\sqrt{3}&0&2&0\\0&2&0&\sqrt{3}\\0&0&\sqrt{3}&0\end{pmatrix}, \quad S_2 = \frac{\mathrm{i}}{2}\begin{pmatrix}0&-\sqrt{3}&0&0\\\sqrt{3}&0&-2&0\\0&2&0&-\sqrt{3}\\0&0&\sqrt{3}&0\end{pmatrix}, \quad S_3 = \frac{1}{2}\begin{pmatrix}3&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-3\end{pmatrix}. \tag{55}$$

Solutions for arbitrary spin include $s + \frac{1}{2}$ arbitrary paremeters for integer s and $s + 1$ parameters if s is half-integer. The corresponding matrix $\mu(\mathbf{n})$ can be represented in the form

$$\mu(\mathbf{n}) = \sum_{\nu\geq0}\lambda_\nu\tilde{B}_\nu\sum_\sigma(-1)^{[\sigma]}\Lambda_\sigma, \tag{56}$$

where the indices ν and σ take the values $s, s-1, ...$, $[\nu]$ is the entire part of ν, Λ_σ and B_ν are projector operators

$$\Lambda_\nu = \prod_{\nu'\neq\nu}\frac{\mathbf{S}\cdot\mathbf{n} - \nu'}{\nu - \nu'}, \quad \tilde{B}_\nu = \prod_{\nu'\neq\nu}\frac{S_3^2 - \nu'^2}{\nu^2 - \nu'^2}, \quad \tilde{C}_\nu = \frac{S_3}{\nu}\prod_{\nu'\neq\nu}\frac{S_3^2 - \nu'^2}{\nu^2 - \nu'^2}. \tag{57}$$

Solutions (52) and (54) are nothing but particular cases of (57).

4.2. Physical interpretation

Let us consider solution (52) for $s = 1$ and $\mu = 0$, $\lambda = \omega > 0$. The corresponding Hamiltonian (49) takes the form

$$\mathcal{H}_1 = \frac{p_1^2 + p_2^2}{2m} + \omega\left(\frac{2(\mathbf{S}\cdot\mathbf{x})^2}{r^3} - \frac{1}{r}\right) \tag{58}$$

and can be rewritten as

$$\mathcal{H}_1 = \frac{p_1^2 + p_2^2}{2m} + \omega Q_{ab}\frac{\partial E_a}{\partial x_b}, \tag{59}$$

where

$$Q_{ab} = S_aS_b + S_bS_a - \delta_{ab} \tag{60}$$

is the tensor of quadruple interaction, and

$$\mathbf{E} = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right) \qquad (61)$$

is an external vector field.

In accordance with (59), $\mathcal{H}_1$ can be interpreted as the Hamiltonian of a neutral spin-one particle with non-trivial quadrupole moment.

A more sophisticated problem is the physical interpretation of the vector field (61). It is difficult to interpret it as an electric field, since the corresponding charge density should be proportional to $\text{div}\mathbf{E} = \frac{1}{r}$ and is hardly realized experimentally. However, vector (61) solves the equations of axion electrodynamics with trivial current and charge [18].

Let us discuss one more interpretation of the spin-one Hamiltonian. Setting $\lambda = 0$ and $\mu = \omega > 0$ in (52), we can rewrite Hamiltonian (49) in the form

$$\hat{\mathcal{H}}_1 = \frac{p_1^2 + p_2^2}{2m} + \omega \frac{2(\mathbf{S}\cdot\mathbf{H})^2 - \mathbf{H}^2}{|\mathbf{H}|}, \qquad (62)$$

where $\mathbf{H}$ is the vector of a magnetic field, whose components are

$$H_1 = \frac{x_2}{r^2}, \quad H_2 = -\frac{x_1}{r^2}. \qquad (63)$$

Thus, the last term in (62) represents a nonlinear interaction with a well-defined magnetic field, which we have already met in Eq. (11).

Hamiltonians (58) and (62) are unitarily equivalent, namely,

$$\mathcal{H}_1 = U\hat{\mathcal{H}}_1 U^\dagger, \quad U = \exp\left(\frac{\mathrm{i}\pi}{2} S_3\right). \qquad (64)$$

However, Hamiltonian (49) with generic potential (52) with $\mu\lambda \neq 0$ is equivalent neither to (58) nor to (62) and includes two types of interaction terms discussed above.

Potential (54) for spin $\frac{3}{2}$ also can be represented in terms of external fields. In the case $\mu = 0$, $\nu = \frac{\lambda}{3} \neq 0$, we have the following form of Hamiltonian (49):

$$\mathcal{H} = \frac{p_1^2 + p_2^2}{2m} + \omega \left(\mathbf{S}\cdot\mathbf{H} - \frac{4}{7}\frac{(\mathbf{S}\cdot\mathbf{H})^3}{\mathbf{H}^2}\right), \qquad (65)$$

where $\omega = \frac{7\lambda}{3}$, $\mathbf{H}$ is the vector of a magnetic field, whose components are defined in Eq. (63), and $\mathbf{S}$ is the spin $\frac{3}{2}$ vector with components (55).

In addition to the standard Pauli term $\omega\mathbf{S}\cdot\mathbf{H}$, Hamiltonian (65) includes the interaction $\sim\frac{(\mathbf{S}\cdot\mathbf{H})^3}{\mathbf{H}^2}$, which is non-linear in the magnetic field.

Alternatively, for $\mu = -\frac{4}{3}\nu$, we obtain the representation

$$\mathcal{H} = \frac{p_1^2 + p_2^2}{2m} - \nu Q_{abc}\frac{\partial^2 H_a}{\partial x_b \partial x_c}, \qquad (66)$$

where

$$Q_{abc} = \sum_{P(a,b,c)} \left(S_a S_b S_c - \frac{7}{4} S_a \delta_{bc}\right) \qquad (67)$$

is the octuple interaction tensor,

$$\mathbf{H} = (x_2 \ln r, -x_1 \ln r, 0),$$

and the summation is imposed over all possible permutations of the indices a, b, and c.

The vector field $\mathbf{H}$ perfectly solves the equations of axion electrodynamics [18]. However, to treat it as a classical magnetic field, we need a current, whose density grows logarithmically with r.

4.3. Systems in three-dimensional space

To specify 3d systems with arbitrary spin, which admit the LRL vector, we are supposed to solve Eqs. (25)–(27).

It follows from (25) that $\hat{V}$ is a scalar matrix. All such matrices are polynomials in $\mathbf{S}\cdot\mathbf{n}$, where $\mathbf{S}$ is a spin vector, whose components S_a are expressed via S_{bc} as

$$S_a = \frac{1}{2}\varepsilon_{abc} S_{bc}.$$

Such polynomials can be expanded via projection operators:

$$\hat{V} = \sum_{\nu=-s}^{s} f_\nu \hat{\Lambda}_\nu. \qquad (68)$$

Here, f_ν are functions of x, and $\hat{\Lambda}_\nu$ are projectors onto the eigenvector space of the matrix $\mathbf{S}\cdot\mathbf{n}$, $\mathbf{n} = \frac{\mathbf{x}}{x}$, corresponding to the eigenvalue ν ($\nu, \nu' = s, s-1, ... -s$):

$$\hat{\Lambda}_\nu = \prod_{\nu' \neq \nu} \frac{\mathbf{S}\cdot\mathbf{n} - \nu'}{\nu - \nu'}. \qquad (69)$$

In contrast with the 2d projectors Λ_ν (57), operators (69) include the 3d scalar products $\mathbf{S}\cdot\mathbf{n} = S_1 n_1 + S_2 n_2 + + S_3 n_3$.

The matrices Λ_ν possess the standard projector properties

$$\Lambda_\nu\lambda_{\nu'} = \delta_{\nu\nu'}\Lambda_\nu, \quad \sum_{\nu=-s}^{s}\Lambda_\nu = I, \quad \mathbf{S}\cdot\mathbf{n} = \sum_{\nu=-s}^{s}\nu\Lambda_\nu.$$

Operators (68) satisfy condition (26) if

$$f_\nu(x) = \frac{c_\nu}{x}, \tag{70}$$

where c_ν are constants. The remaining equation (26) specifies these constants in the following manner [11]:

$$c_0 = \alpha, \quad c_\nu = 0, \quad \text{if} \quad \nu \neq 0 \tag{71}$$

if the spin is integer, and

$$c_\nu = \frac{\alpha}{\nu} \tag{72}$$

for a half-integer spin. Here, α is an arbitrary real parameter.

Substituting (70), (71), and (72) into (68), we obtain the explicit forms of potentials for arbitrary spins:

$$\hat{V} = \frac{\alpha}{x}\hat{\Lambda} = \frac{\alpha}{x}\Lambda_0 \tag{73}$$

for integer spins,

$$\hat{V} = \frac{\alpha}{x}\hat{\Lambda} = \frac{\alpha}{x}\sum_{\nu=-s}^{s}\frac{1}{\nu}\Lambda_\nu \tag{74}$$

for half-integer spins.

Thus, we find 3d Hamiltonians for arbitrary spin in the form (31), (73), and (74). All these Hamiltonians commute with the LRL vectors generalized by the presence of spin. The obtained list of Hamiltonians is complete in the case where the spin matrices realize irreducible representations of algebra $so(3)$.

4.4. Spins 1 and $\frac{3}{2}$

Potential (28) for spin $\frac{1}{2}$ is a particular case of the generic solution (74). Let us consider two more important cases, i.e., spin 1 and $\frac{3}{2}$.

For spin 1, relation (73) yields

$$\hat{V} = \frac{\alpha}{x}(1 - (\mathbf{S}\cdot\mathbf{n})^2) \tag{75}$$

that corresponds to the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{\alpha}{x}(1 - (\mathbf{S}\cdot\mathbf{n})^2). \tag{76}$$

Potential (75) qualitatively differs from the 2d potential (48), (52). Indeed, it satisfies

$$\hat{V}^2 = \frac{\alpha}{x}\hat{V}. \tag{77}$$

So, it is a projector multiplied by $\frac{\alpha}{x}$. On the other hand, the matrix $\mu(\mathbf{n})$ for the 2d potential (48) does not have this property.

An alternative representation of potential (75) is given by the formula

$$\hat{V} = \hat{Q}_{ab}\frac{\partial \hat{F}_a}{\partial x_b}, \tag{78}$$

where Q_{ab} is tensor (60) with $a, b = 1, 2, 3$, and

$$\hat{F}_a = -\frac{\alpha x_a}{x}. \tag{79}$$

For spin $\frac{3}{2}$, formula (74) reads

$$\hat{V} = \frac{2\alpha}{9}Q_{abc}\frac{\partial^2 \tilde{F}_a}{\partial x_b \partial x_c}, \tag{80}$$

where

$$\tilde{F}_a = x_a \ln x, \tag{81}$$

and Q_{abc} is tensor (67) with $a, b, c = 1, 2, 3$.

Potentials (75) and (80) represent the quadrupole and octupole interactions, respectively. The external fields (79) and (81) solve the equations of axion electrodynamics [18].

5. Discussion

We present an entire collection of superintegrable systems with spin, which admit a generalized LRL vector. This collection includes 2d and 3d systems with arbitrary spins and an arbitrary dimensional system with spin $\frac{1}{2}$. All these systems possess the hidden symmetry with respect to group $SO(4)$ for coupled states and group $SO(1, 3)$ for positive energy states. In other words, the Fock symmetry for the hydrogen atom is extended to a large class of other quantum mechanical systems.

The presented list of superintegrable systems is complete for the case of spin matrices realizing irreducible representations of the rotation group. More exactly, it includes all quantum mechanical systems for neutral particles, whose Hamiltonians (1) are invariant with respect to the rotation group and commute with a generalized LRL vector.

The extended number of integrals of motion makes the discussed systems superintegrable. This property makes many of them exactly solvable. One more attractive property of these systems is their supersymmetry and shape invariance, which present additional tools for finding their energy spectra and exact solutions. For the exact solutions of the described systems, see [8, 11–13, 15, 17].

The presented superintegrable systems describe neutral particles with non-trivial multipole moments. More exactly, for systems with spin $\frac{1}{2}$, only the dipole interaction is present. While, for spins 1 and $\frac{3}{2}$, we involve the quadrupole and octupole interactions, respectively.

We do not discuss recent results obtained in the classification of superintegrable systems (1), which do not admit a generalized LRL vector, see works [8, 19] and [20]. In contrast with scalar systems, the classification of equations with matrix potentials is in fact at its beginning and includes a number of open problems. In particular, the integrals of motion, which include discrete symmetries, were not discussed yet, though such integrals of motion in the scalar case are well known (see, e.g., [21–23]).

APPENDIX
Reducible systems with generalized LRL vector

In work [10], supersymmetric extensions for a d-dimensional hydrogen atom were discussed. For d = 3, the system with Hamiltonian (1) is proposed, where V is the 8×8 matrix

$$V = \frac{2\lambda}{x}\begin{pmatrix} 1 & & & \\ & M_1 & & \\ & & M_2 & \\ & & & 1 \end{pmatrix}$$

with 3×3 matrices M_1 and M_2, whose entries are

$$M_{1ab} = n_a n_b, \quad M_{1ab} = \delta_{ab} - (-1)^{a+b} n_a n_b.$$

This potential is completely reducible, so the corresponding Schrödinger equation is decoupled into four independent subsystems. In addition, this potential is not invariant with respect to the rotation group. So, its symmetry declared in [10] is seemed to be violated.

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СУПЕРІНТЕГРОВНІ
СИСТЕМИ З ДОВІЛЬНИМ СПІНОМ

Р е з ю м е

Розглянуто суперінтегровні квантовомеханічні системи зі спіном. Ці системи допускають узагальнення вектора Лапласа–Рунге–Ленца і мають динамічну симетрію відносно групи $SO(4)$ для зв'язаних станів. Наш розгляд стосується в основному двовимірних і тривимірних систем, але також представлено і спеціальний клас систем в довільних розмірностях.