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ON BEHAVIOR OF QUANTUM PARTICLES IN AN ELECTRIC FIELD IN SPACES OF CONSTANT CURVATURE, HYPERBOLIC AND SPHERICAL MODELS

In the Lobachevsky hyperbolic and Riemann spherical spaces, generalized potentials describing a uniform electric field are introduced as solutions of the covariant Maxwell equations. Exact solutions of the Schrödinger equation in the presence of the electric field are constructed in both models. The similarity of the energy spectra of the particle against the background of a spherical space with the electric field and in the Coulomb field is noted.

Keywords: Electric field, spaces of constant curvature, Schrödinger equation, Dirac equation, exact solutions.

1. Introduction

The motion of a quantum mechanical particle in a homogeneous magnetic and electric fields is a classical problem in quantum mechanics [1–4]. In [5–12], the problem of the motion of a particle in a magnetic field in two-dimensional Lobachevsky and Riemann spaces was investigated. The generalization to three-dimensional geometric models was performed recently: exact solutions of the Schrödinger equation for a particle in an external magnetic field in the three-dimensional Lobachevsky H_3 and Riemann S_3 spaces were found in [13]. The generalized magnetic fields are considered as an analogue of the uniform magnetic field in the flat space: these fields [13] are simple solutions of Maxwell's equations in models of H_3 and S_3; in addition, they are invariant with respect to the transverse shifts in the curved models (such shifts cause only special gauge transformations of the corresponding 4-potentials). In the limit of vanishing curvature, the generalized magnetic fields reduce to the known expression in the flat space. Generalized Landau energy levels modified by the presence of a space curvature were found. A characteristic feature of the spectrum in the case of the Lobachevsky space is the finite number of bound states for the Schrödinger particle. The energy spectrum of a particle in the Riemann space is discrete. In [14] and [15], other possibilities to generalize the concept of the uniform magnetic field for spaces of constant curvature were additionally analyzed in other coordinate systems.

The corresponding system in classical mechanics was investigated in [16]. In the generalized cylindrical coordinates, three integrals of motion were found, and the exact solutions of classical equations of motion for a particle in Lobachevsky and Riemann spaces were constructed. A symmetry between the different trajectories was established and described in detail. In the Lobachevsky space, trajectories in the external magnetic field belong to generalized cylinder surfaces, and the angular speed of rotation varies in time. In the Riemann space, particles move along closed trajectories; the motion is periodic, and the geometric parameters of the trajectories depend on the physical characteristics of a particle and the curvature radius of space models. The constructed solutions may be of interest to describe the behavior of charged particles in a macroscopic magnetic fields in cosmological models, to simulate the behavior of a plasma in the magnetic field of a special configuration, as well as to model the behavior of particles on surfaces in nanophysics problems.

In the present paper, we will solve the problem of the motion of Schrödinger's particle in the three-dimensional Lobachevsky and Riemann spaces in an external electric field, which is a generalization of the uniform electric field in a flat space. A similar analysis is perfomed for a Dirac particle.

2. Schrödinger Equation in the Electric Field in the Lobachevsky Space H_3

In the hyperbolic Lobachevsky space, the following system of cylindrical coordinates exists (we use dimensionless quantities):

$$ds^2 = dt^2 - \cosh^2 z(dr^2 + \sinh^2 r\, d\phi^2) - dz^2,$$

$$z \in (-\infty,\ +\infty), \quad r \in [0,\ +\infty), \quad \phi \in [0,\ 2\,\pi],$$

$$u^1 = \cosh z \sinh r \cos\phi, \quad u^2 = \cosh z \sinh r \sin\phi, \quad (1)$$

$$u^3 = \sinh z, \quad u^0 = \cosh z \cosh r,$$

$$(u^0)^2 - (u^1)^2 - (u^2)^2 - (u^2)^2 = -1, \quad u^0 \geq +1.$$

We introduce an external electric field along the axis z by the 4-potential

$$A_0 = \nu \tanh z, \quad A_r = 0, \quad A_z = 0,\ A_\phi = 0, \qquad (2)$$

which is a solution of Maxwell's equations in the Lobachevsky space

$$A_0 = \nu\, \tanh\, z, \quad F_{z0} = \frac{\nu}{\mathrm{ch}^2 z},$$

$$\frac{1}{\sqrt{-g}}\, \frac{\partial}{\partial x^\beta} \sqrt{-g} F^{\alpha\beta} = 0;$$

in a flat space, we have $A_0 = \nu \tanh z \approx \nu\ z$,

$$F_{z0} = \frac{4\nu}{(e^z + e^{-z})^2} \approx \frac{4\nu}{2 + (1+2z) + (1-2z)} = \nu.$$

We note that the potential of the electric field in the Lobachevsky space is very different from that in the flat space, if we compare them at large z.

The Schrödinger equation in the electric field (2) has the form

$$\left(i\, \frac{\partial}{\partial t} + e\, A_0\right) \Psi =$$

$$= -\frac{1}{2} \left(\frac{1}{\cosh^2 z}\, \frac{\partial^2}{\partial r^2} + \frac{\cosh r}{\cosh^2 z\, \sinh\, r}\, \frac{\partial}{\partial r} + \right.$$

$$\left. + \frac{1}{\cosh^2 z\, \sinh^2 r}\, \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} + 2\, \frac{\sinh\, z}{\cosh\, z}\, \frac{\partial}{\partial z} \right) \Psi. \qquad (3)$$

The variables are separated by the substitution $\Psi = = e^{-i\epsilon t}\, e^{im\phi}\, Z(z)\, R(r)$ (λ stands for a separation constant):

$$-\frac{d^2 R}{dr^2} - \frac{\cosh r}{\sinh r}\, \frac{dR}{dr} + \frac{m^2}{\sinh^2 r}\, R = \lambda\, R, \qquad (4)$$

$$\cosh^2 z\, \frac{d^2 Z}{dz^2} + 2\, \cosh z\, \sinh z\, \frac{dZ}{dz} +$$

$$+ 2\, \cosh^2 z \left(\epsilon + \nu\, \frac{\sinh z}{\cosh z} \right) Z = \lambda\, Z. \qquad (5)$$

The expression for the dimensionless parameters ϵ and ν corresponding to the energy E and the amplitude E_0 of the electric field are determined by the relations

$$\epsilon = \frac{E}{\hbar^2/M\rho^2}, \quad \nu = \frac{e\, E_0\, \rho}{\hbar^2/M\, \rho^2}. \qquad (6)$$

3. Solution of the Radial Equation

In Eq. (4), we make change of the variable, $x = = (1 + \cosh r)/2,\ x \in [1,\ +\infty)$, so that

$$x\, (1-x)\, \frac{d^2 R}{dx^2} + (1 - 2\, x)\, \frac{dR}{dx} -$$

$$- \left(\lambda + \frac{1}{4}\, \frac{m^2}{x} + \frac{1}{4}\, \frac{m^2}{1-x} \right) R = 0.$$

We also introduce the substitution $R = x^a\, (1-x)^b\, F$, which results in

$$x\, (1-x)\, \frac{d^2 F}{dx^2} + [2\, a + 1 - (2\, a + 2\, b + 2)\, x]\ \frac{dF}{dx} +$$

$$+ [-(a+b)\, (a+b+1) - \lambda +$$

$$\left. + \frac{1}{4}\, \frac{4\, a^2 - m^2}{x} + \frac{1}{4}\, \frac{4\, b^2 - m^2}{1-x} \right]\ F = 0. \qquad (7)$$

At

$$a = \pm \frac{\mid m \mid}{2}, \quad b = \pm \frac{\mid m \mid}{2}, \qquad (8)$$

the equation becomes simpler:

$$x\, (1-x)\, \frac{d^2 F}{dx^2} + [2\, a + 1 - (2\, a + 2\, b + 2)\, x]\ \frac{dF}{dx} -$$

$$- [(a+b)\, (a+b+1) + \lambda]\ F = 0. \qquad (9)$$

It is the equation for the hypergeometric function $F = = F(\alpha, \beta, \gamma; x)$ with the parameters

$$\alpha = a + b + \frac{1}{2} - \frac{i}{2}\, \sqrt{4\lambda - 1},$$

$$\beta = a + b + \frac{1}{2} + \frac{i}{2}\, \sqrt{4\lambda - 1}, \quad \lambda > \frac{1}{4}, \qquad (10)$$

$$\gamma = 2\, a + 1.$$

Let us use a solution vanishing at the point $r = 0$:

$$F = u_2 = F(\alpha,\ \beta,\ \alpha+\beta+1-\gamma;\ 1-x); \tag{11}$$

for a and b, we take positive values

$$a = +\frac{\mid m \mid}{2}, \quad b = +\frac{\mid m \mid}{2}; \tag{12}$$

the full radial function $R(r)$ is given by

$$R = x^a (1-x)^b F(\alpha,\ \beta,\ \alpha+\beta+1-\gamma;\ 1-x). \tag{13}$$

To find the behavior of solutions at infinity $r \to \to +\infty$, we use the Kummer relation

$$u_2 = \frac{\Gamma(\alpha+\beta+1-\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta+1-\gamma)\Gamma(\beta)} e^{-i\pi\alpha}\, u_3 + \\ + \frac{\Gamma(\alpha+\beta+1-\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha+1-\gamma)\Gamma(\alpha)} e^{-i\pi\beta}\, u_4, \tag{14}$$

where

$$u_2 = F(\alpha,\ \beta,\ \alpha+\beta+1-\gamma;\ 1-x),$$

$$u_3 = (-x)^{-\alpha} F\left(\alpha,\ \alpha+1-\gamma,\ \alpha+1-\beta;\ \frac{1}{x}\right), \tag{15}$$

$$u_4 = (-x)^{-\beta} F\left(\beta,\ \beta+1-\gamma,\ \beta+1-\alpha;\ \frac{1}{x}\right).$$

Therefore, the asymptotic behavior at $x \to 1$ ($r \to +\infty$) is given by

$$R \approx (-1)^{a+b}\Gamma(\alpha+\beta+1-\gamma)\times \\ \times \left(\frac{\Gamma(\beta-\alpha)}{\Gamma(\beta+1-\gamma)\Gamma(\beta)} e^{-i\pi\alpha}\, (-x)^{a+b-\alpha} + \\ + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha+1-\gamma)\Gamma(\alpha)} e^{-i\pi\beta}\, (-x)^{a+b-\beta}\right).$$

From whence, we have

$$x \approx \frac{e^r}{4}, \quad R \approx (-1)^{a+b}\Gamma(\alpha+\beta+1-\gamma)(-x)^{-1/2}\times \\ \times \left(\frac{\Gamma(\beta-\alpha)}{\Gamma(\beta+1-\gamma)\Gamma(\beta)} e^{-i\pi\alpha}\, (-x)^{+i\sqrt{\lambda-1/4}} + \\ + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha+1-\gamma)\Gamma(\alpha)} e^{-i\pi\beta}\, (-x)^{-i\sqrt{\lambda-1/4}}\right). \tag{16}$$

Thus, these solutions at infinity are standing waves of the oscillatory type (described by real functions in the whole space). The factor $e^{-r/2}$ in the radial wave function is irrelevant to the physical interpretation of the probability density

$$dW = \sqrt{-g}\ \psi^* \psi, \tag{17}$$

because this factor $e^{-r/2}$ will be compensated by the factor $\sinh r \approx e^{+r}/2$ entering the volume element $dV = \sqrt{-g}\ dr\ dz\ d\phi$.

4. Solution of the Equation for $Z(z)$

Let us consider Eq. (5) for $Z(z)$:

$$\cosh^2 z\, \frac{d^2 Z}{dz^2} + 2\cosh z \sinh z\, \frac{dZ}{dz} + \\ + 2\cosh^2 z \left(\epsilon + \nu\, \frac{\sinh z}{\cosh z}\right) Z = \lambda\, Z. \tag{18}$$

With the substitution

$$Z(z) = \frac{1}{\cosh z} f(z),$$

one removes the first derivative term:

$$\left(\frac{d^2}{dz^2} + 2\epsilon - 1 + 2\nu \tanh z - \frac{\lambda}{\cosh^2 z}\right) Z(z) = 0.$$

This is a one-dimensional Schrödinger equation with the potential

$$U(z) = -2\nu\ \tanh z + \frac{\lambda}{\cosh^2 z}.$$

We now make change of the variable $y = (1 + \tanh z)/2$. Then (18) takes the form

$$\left[y\,(1-y)\,\frac{d^2}{dy^2} + \frac{1}{2}\,\frac{\epsilon-\nu}{y} + \frac{1}{2}\,\frac{\epsilon+\nu}{1-y} - \lambda\right] Z = 0.$$

Using the substitution $Z = y^c\,(1-y)^d F$, we get

$$y\,(1-y)\,\frac{d^2}{dy^2} + [2\,c - 2\,y\,(c+d)\,]\ \frac{dF}{dy} + \\ + [(c+d)\,(1-c-d) - \lambda + \\ + \frac{1}{1-y}\left(d^2 - d + \frac{\epsilon+\nu}{2}\right) + \\ + \frac{1}{y}\left(c^2 - c + \frac{\epsilon-\nu}{2}\right)\Big] F = 0. \tag{19}$$

At

$$c = \frac{1 \pm i\sqrt{2\,\epsilon - 1 - 2\,\nu}}{2}, \quad d = \frac{1 \pm i\sqrt{2\,\epsilon - 1 + 2\,\nu}}{2}, \tag{20}$$

Eq. (19) becomes simpler

$$y\,(1-y)\,\frac{d^2}{dy^2} + [2\,c - 2\,y\,(c+d)\,]\,\frac{dF}{dy} -$$
$$-\,[(c+d)\,(c+d-1)+\lambda]\,F = 0,$$

and is the equation for the hypergeometric function $F(A,\,B,\,C;\,y)$ with parameters

$$A = c + d - \frac{1}{2} + \frac{i}{2}\,\sqrt{4\,\lambda - 1},$$

$$B = c + d - \frac{1}{2} - \frac{i}{2}\,\sqrt{4\,\lambda - 1}, \quad C = 2\,c.$$

The wave function can be represented as

$$Z = \left(\frac{1+\tanh z}{2}\right)^c \left(\frac{1-\tanh z}{2}\right)^d \times$$
$$\times\, F\left(A\,,\; B\;\, C\,;\; \frac{1+\tanh z}{2}\right). \tag{21}$$

In the area of $z \to -\infty$, these solutions behave themselves like

$$z \to -\infty, \quad Z = \left(\frac{1+\tanh z}{2}\right)^c \approx$$
$$\approx \left(\frac{1}{1+e^{-2z}}\right)^c \sim e^z e^{\pm iz\sqrt{2\,\epsilon-1-2\,\nu}}; \tag{22}$$

and the factor $\sqrt{-g}$ becomes e^{-2z}.

There are two physically different situations (for definiteness, we assume that $\nu > 0$). The first situation is such that the energy is above the potential barrier

$$\text{I}, \quad (2\epsilon - 1 - 2\nu) > 0,$$

and the far-left solutions are oscillatory (22).

The second situation is such that the energy is lower than the potential curve at $z < 0$:

$$\text{II}, \quad (2\epsilon - 1 - 2\nu) < 0;$$

in this case, there is a solution that tends to zero:

$$\text{III}, \quad z \to -\infty, \quad Z = \left(\frac{1+\tanh z}{2}\right)^c \approx$$
$$\approx \left(\frac{1}{1+e^{-2z}}\right)^c \sim e^z e^{\pm z\sqrt{-(2\,\epsilon-1-2\,\nu)}}. \tag{23}$$

To describe the behavior of solutions in the region $z \to +\infty$ $(y \to 1)$, we use the Kummer relation

$$U_1 = F(A,B,C,y),$$

$$U_2 = F(A,B,A+B+1-C,1-y),$$

$$U_6 = (1-y)^{C-A-B} \times$$
$$\times\, F(C-A,C-B,C+1-A-B,1-y),$$

$$U_1 = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}\,U_2 +$$
$$+\,\frac{\Gamma(C)\Gamma(-C+A+B)}{\Gamma(A)\Gamma(B)}U_6. \tag{24}$$

Allowing for the identities

$$y = \frac{1+\tanh z}{2}, \quad 1 - y = \frac{1-\tanh z}{2},$$

we find the behavior of the solutions U_1 as $z \to +\infty$:

$$z \to +\infty, \quad U_1(\frac{1+\tanh z}{2}) = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} +$$
$$+\,\frac{\Gamma(C)\Gamma(-C+A+B)}{\Gamma(A)\Gamma(B)}\,(1-y)^{C-A-B}. \tag{25}$$

Hence, for the full function Z (see (22)), we obtain the representation

$$Z = (1-y)^d\,\left[\frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} +\right.$$
$$\left.+\,\frac{\Gamma(C)\Gamma(-C+A+B)}{\Gamma(A)\Gamma(B)}\,(1-y)^{C-A-B}\right]. \tag{26}$$

In view of the relation (see (20))

$$C - A - B = 2c - (2c + 2d - 1) = 1 - 2d,$$

we arrive at

$$Z = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}(1-y)^d +$$
$$+\,\frac{\Gamma(C)\Gamma(-C+A+B)}{\Gamma(A)\Gamma(B)}\,(1-y)^{1-d}. \tag{27}$$

Given

$$d = \frac{1 \pm i\sqrt{2\,\epsilon - 1 + 2\,\nu}}{2},$$

$$(1-y)^d = \left(\frac{1}{1+e^{2z}}\right)^d \sim e^{-2zd} = e^{-z} e^{\mp i\sqrt{2\epsilon-1+2\nu}},$$

$$(1-y)^{1-d} = \left(\frac{1}{1+e^{2z}}\right)^{1-d} \sim$$
$$\sim e^{-2z(1-d)} = e^{-z} e^{\pm i\sqrt{2\epsilon-1+2\nu}}, \qquad (28)$$

we note that the expression under the square root is positive, so the solutions are oscillatory. Therefore, (26) takes the form

$$Z = \Gamma(C) e^{-z} \left(M e^{\mp i\sqrt{2\epsilon-1+2\nu}} + N\, e^{\pm i\sqrt{2\epsilon-1+2\nu}}\right),$$

$$M = \frac{\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}, \quad N = \frac{\Gamma(-C+A+B)}{\Gamma(A)\Gamma(B)}.$$

The signs $\pm$ at roots are independent to each other (that is, there are 4 possibilities). For the coefficients of M, N, one gets more detailed representations,

$$M =$$
$$= \frac{\Gamma(\mp i\sqrt{2\epsilon - 1 + 2\nu})}{\Gamma(c-d+\frac{1}{2}-\frac{i}{2}\sqrt{4\lambda-1})\Gamma(c-d+\frac{1}{2}+\frac{i}{2}\sqrt{4\lambda-1})},$$

$$N =$$
$$= \frac{\Gamma(\pm i\sqrt{2\epsilon - 1 + 2\nu})}{\Gamma(c+d-\frac{1}{2}+\frac{i}{2}\sqrt{4\lambda-1})\Gamma(c+d-\frac{1}{2}-\frac{i}{2}\sqrt{4\lambda-1})},$$

where

$$c = \frac{1 \pm i\sqrt{2\,\epsilon - 1 - 2\,\nu}}{2}, \quad d = \frac{1 \pm i\sqrt{2\,\epsilon - 1 + 2\,\nu}}{2}.$$

Let $2\epsilon - 1 - 2\nu < 0$, then, due to the identity

$$c^* = c, \quad \left(-d + \frac{1}{2}\right)^* = d - \frac{1}{2},$$

we get $M^* = N$. This leads, in turn, to the following statement: the reflection coefficient equals to unity for a particle going from the right site to the electric field through the potential barrier (provided $2\epsilon - 1 - 2\nu < 0$):

$$R_{\text{reflection}} = \frac{MM^*}{NN^*} = \frac{NN^*}{MM^*} = 1.$$

5. Schrödinger Equation in the Electric Field in the Riemann Space S_3

In the spherical Riemann space, the system of cylindrical coordinates can be introduced:

$$ds^2 = dt^2 - \cos^2 z \left(dr^2 + \sin^2 r\, d\phi^2\right) - dz^2,$$

$$z \in [-\pi/2,\ \pi/2], \quad r \in [0,\ \pi], \quad \phi \in [0,\ 2\,\pi],$$

$$u^1 = \cos z \sin r \cos\phi, \quad u^2 = \cos z\, \sin r \sin\phi, \qquad (29)$$

$$u^3 = \sin z, \quad u^0 = \cos z \cos r,$$

$$u_0^2 + u_1^2 + u_2^2 + u_3^2 = +1.$$

The external electric field along the axis z is

$$A_0 = \nu\, \tan z, \qquad (30)$$

which is a solution of the Maxwell equations in the Riemann space. The Schrödinger equation in the presence of the electric field (30) takes the form

$$\left(i\,\frac{\partial}{\partial t} + e\,A_0\right)\,\Psi =$$
$$= -\frac{1}{2}\left(\frac{1}{\cos^2 z}\,\frac{\partial^2}{\partial r^2} + \frac{\cos r}{\cos^2 z\,\sin\,r}\,\frac{\partial}{\partial r} + \right.$$
$$\left. + \frac{1}{\cos^2 z\,\sin^2 r}\,\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} - 2\,\frac{\sin\,z}{\cos\,z}\,\frac{\partial}{\partial z}\right)\,\Psi. \qquad (31)$$

The variables are separated by the substitution $\Psi = e^{-i\epsilon t}\, e^{im\phi}\, Z(z)\, R(r)$:

$$-\frac{d^2R}{dr^2} - \frac{\cos r}{\sin r}\,\frac{dR}{dr} + \frac{m^2}{\sin^2 r}\,R = \lambda\,R, \qquad (32)$$

$$\cos^2 z\,\frac{d^2Z}{dz^2} - 2\,\cos z\,\sin z\,\frac{dZ}{dz} +$$
$$+ 2\,\cos^2 z\,\left(\epsilon + \nu\,\frac{\sin z}{\cos z}\right)\,Z = \lambda\,Z. \qquad (33)$$

6. Solution of the Radial Equation

In Eq. (32), we change the variable, $x = (1-$ $-\cos r)/2,\ x \in [0,\ 1]$:

$$x\,(1-x)\,\frac{d^2R}{dx^2} + (1-2\,x)\,\frac{dR}{dx} -$$
$$- \left(-\lambda + \frac{1}{4}\,\frac{m^2}{x} + \frac{1}{4}\,\frac{m^2}{1-x}\right)\,R = 0,$$

and introduce the substitution $R = x^a\,(1-x)^b\,F$. This gives

$$x\,(1-x)\,\frac{d^2F}{dx^2} + [2\,a+1-(2\,a+2\,b+2)\,x]\,\frac{dF}{dx} +$$
$$+\left[-(a+b)\,(a+b+1)+\lambda+\right.$$
$$\left.+\frac{1}{4}\,\frac{4\,a^2-m^2}{x}+\frac{1}{4}\,\frac{4\,b^2-m^2}{1-x}\right]\,F=0. \qquad (34)$$

At

$$a = \pm\frac{\mid m\mid}{2}, \quad b=\pm\frac{\mid m\mid}{2} \qquad (35)$$

(to get finite solutions, we will use positive values for a and b), the equation becomes simpler:

$$x\,(1-x)\,\frac{d^2F}{dx^2} + [2\,a+1-(2\,a+2\,b+2)\,x]\,\frac{dF}{dx} -$$
$$-\,[(a+b)\,(a+b+1)-\lambda]\,F=0. \qquad (36)$$

It is the equation for the hypergeometric function $F = F(\alpha,\ \beta,\ \gamma;\ x)$ with parameters

$$\alpha = a+b+\frac{1}{2}-\sqrt{\lambda+\frac{1}{4}},$$
$$\beta = a+b+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}, \qquad (37)$$
$$\gamma = 2\,a+1.$$

The condition of polynomial solutions, $\alpha = -n$ $(n = 0, 1, 2, ...)$, leads to the quantization of the parameter λ:

$$\lambda+\frac{1}{4} = \left(2\mid m\mid+\frac{1}{2}+n\right)^2. \qquad (38)$$

The corresponding functions R look as

$$R(r) = \left(\sin\frac{r}{2}\right)^{+|m|}\left(\cos\frac{r}{2}\right)^{+|m|}\times$$
$$\times\,F\left(-n,\ 2\mid m\mid+1+n,\ \mid m\mid+1;\ \sin^2\frac{r}{2}\right). \qquad (39)$$

The constructed solutions vanish at the points $r = 0$, π; they provide us with standing waves described by real functions.

7. Solution of the Equation for $Z(z)$

Consider Eq. (33) for $Z(z)$

$$\cos^2 z\,\frac{d^2Z}{dz^2} - 2\,\cos z\,\sin z\,\frac{dZ}{dz} +$$
$$+\,2\,\cos^2 z\,\left(\epsilon+\nu\,\frac{\sin z}{\cos z}\right)\,Z = \lambda\,Z. \qquad (40)$$

By means of the substitution

$$Z(z) = \frac{1}{\cos z}\,f(z),$$

we remove the first derivative term

$$\left(\frac{d^2}{dz^2}+2\epsilon+1+2\nu\tan z-\frac{\lambda}{\cos^2 z}\right)f(z)=0.$$

This is a one-dimensional Schrödinger-like equation in the field given by the potential function

$$U(z) = -2\nu\tan z.$$

Changing the variable $y = (1-i\tan z)/2$ (complex variable y ranges over a vertical line in the complex plane passing through the point $(1/2,\ 0)$) reduces Eq. (40) to

$$\left(y\,(1-y)\,\frac{d^2}{dy^2}-\frac{1}{2}\,\frac{\epsilon-i\nu}{y}-\frac{1}{2}\,\frac{\epsilon+i\nu}{1-y}+\lambda\right)Z=0.$$

Using the substitution $Z = y^c(1-y)^dF$, we arrive at

$$y\,(1-y)\,\frac{d^2F}{dy^2}+[2\,c-2\,y\,(c+d)\,]\,\frac{dF}{dy}+$$
$$+\left[(c+d)\,(1-c-d)+\lambda+\right.$$
$$+\frac{1}{1-y}\left(d^2-d-\frac{\epsilon+i\nu}{2}\right)+$$
$$\left.+\frac{1}{y}\left(c^2-c-\frac{\epsilon-i\nu}{2}\right)\right]F=0. \qquad (41)$$

At

$$c=\frac{1\pm\sqrt{2\,\epsilon+1-2\,i\nu}}{2}, \quad d=\frac{1\pm\sqrt{2\,\epsilon+1+2\,i\nu}}{2}, \qquad (42)$$

Eq. (41) reads

$$y\,(1-y)\,\frac{d^2F}{dy^2}+[2\,c-2\,y\,(c+d)\,]\,\frac{dF}{dy}-$$
$$-\left[\left(c+d-\frac{1}{2}\right)^2-\left(\lambda+\frac{1}{4}\right)\right]F=0.$$

It is an equation of the hypergeometric type with parameters

$$\alpha = c + d - \frac{1}{2} + \sqrt{\lambda + \frac{1}{4}}, \quad \beta = c + d - \frac{1}{2} - \sqrt{\lambda + \frac{1}{4}},$$
$$\gamma = 2\,c.$$

Let

$$c = \frac{1 - \sqrt{2\,\epsilon + 1 - 2\,i\nu}}{2}, \quad d = \frac{1 - \sqrt{2\,\epsilon + 1 + 2\,i\nu}}{2}.$$

In this case, the combination $c+d$ in the expression for parameters of the hypergeometric function permits the quantization condition

$$\frac{1}{2} - \frac{\sqrt{2\,\epsilon + 1 - 2\,i\nu} + \sqrt{2\,\epsilon + 1 + 2\,i\nu}}{2} + + \frac{1}{2}\sqrt{1 + 4\lambda} = -n. \tag{43}$$

Let us introduce the notation $N = 1 + 2n + \sqrt{1 + 4\lambda}$. From (43), we find the formula for the energy spectrum and the corresponding functions Z:

$$2\epsilon + 1 = \frac{N^2}{4} - \frac{4\nu^2}{N^2}, \quad Z = y^c(1-y)^d F(\alpha,\,\beta,\,\gamma;\;y),$$
$$c = \frac{1}{2}\left(1 - \frac{N}{2} + \frac{2\,i\nu}{N}\right), \quad d = \frac{1}{2}\left(1 - \frac{N}{2} - \frac{2\,i\nu}{N}\right). \tag{44}$$

To find the behavior of the solutions at the singular points

$$z \to +\pi/2, \quad y \to \frac{1 - i\infty}{2}, \quad 1 - y \to \frac{1 + i\infty}{2},$$
$$z \to -\pi/2, \quad y \to \frac{1 + i\infty}{2}, \quad 1 - y \to \frac{1 - i\infty}{2}, \tag{45}$$

we use the Kummer identity

$$F(\alpha,\,\beta,\,\gamma;\,y) = (1-y)^{-\alpha} F\left(\alpha, \gamma - \beta, \gamma; \frac{y}{y-1}\right). \tag{46}$$

We note that

$$\frac{y}{y-1} = -e^{-2iz}. \tag{47}$$

Therefore, this argument at $z = \pm\pi/2$ has no special features: it is finite. By (46), solution (44) can be written as

$$Z = y^c(1-y)^{d-\alpha}\; F\left(\alpha, \gamma - \beta, \gamma; \frac{y}{y-1}\right) =$$
$$= \left(\frac{e^{-iz}}{2\cos z}\right)^c \left(\frac{e^{+iz}}{2\cos z}\right)^{d-\alpha} F(\alpha, \gamma - \beta, \gamma; -e^{-2iz}).$$

At the singular points $z = \pm\pi/2$, the factor

$$(\cos z)^{\alpha - c - d} = (\cos z)^{(-1/2 + \sqrt{\lambda + 1/4}\;)}$$

tends to zero (according to (38), $\sqrt{\lambda + 1/4} > 1/2$).

Note that the energy spectrum of particles in an electric field in the spherical space is very similar to the energy spectrum in the Kepler problem on the sphere [17–25]: compare the formulas

$$2\epsilon + 1 = -\frac{4\nu^2}{N^2} + \frac{N^2}{4}$$

and

$$E = -\frac{e^2}{2M\hbar^2}\frac{1}{n^2} + \frac{\hbar^2}{M\rho^2}\frac{n^2 - 1}{1};$$

this suggests the existence of other connections between these systems.

The solutions against the background of a compact spherical space can be used to model the localized systems (composite particles, quantum dots, *etc.*) in the presence of electric fields [24, 25].

This approach can be extended to the case of a spin-1/2 Dirac particle.

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ПРО РУХ КВАНТОВИХ ЧАСТИНОК В ЕЛЕКТРИЧНОМУ ПОЛІ В ПРОСТОРАХ ПОСТІЙНОЇ КРИВИЗНИ, ГІПЕРБОЛІЧНІЙ ТА СФЕРИЧНІЙ МОДЕЛЯХ

Р е з ю м е

У сферичному просторі Рімана і гіперболічному просторі Лобачевського введено поняття однорідного електричного поля як розв'язки загальноковаріантних рівнянь Максвелла в цих просторах. Знайдено точні розв'язки рівняння Шредінгера за наявності електричного поля в обох моделях. Відзначено подібність спектра енергій частинки на фоні сферичного простору в електричному полі і спектра енергій частинки в кулонівському полі.