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BOSONIC SYMMETRIES, SOLUTIONS, AND CONSERVATION LAWS FOR THE DIRAC EQUATION WITH NONZERO MASS

In addition to the well-known Fermi properties of the Dirac equation, the hidden bosonic properties of this equation are found. The bosonic symmetries, solutions, and the conservation laws are under consideration. Such new features of the Dirac equation with nonzero mass are found with the help of the 64-dimensional extended real Clifford–Dirac algebra and 29-dimensional proper extended real Clifford–Dirac algebra. In this case, the start from the Foldy–Wouthuysen representation is of importance. It is shown that the Dirac equation can describe not only the fermionic but also the bosonic states.

Keywords: bosonic symmetries, Dirac equation, Clifford–Dirac algebra.

1. Introduction

It is well-known that the Dirac equation is invariant with respect to the transformations, which are determined by the spin $s = \frac{1}{2}$ representation of the Poincaré group. On the basis of this fact, the conclusion that the Dirac equation describes the spin $s = \frac{1}{2}$ fields and particles (fermions) is formulated. The representations of the proper orthochronous Poincaré group (inhomogeneous Lorentz group) have the principal importance.

Below, we are able to demonstrate another hidden half of the Dirac equation possibilities. We will consider the bosonic symmetries, solutions, and conservation laws for the Dirac equation with nonzero mass.

Note that the Dirac equation itself (as well as every other field equation) does not carry the complete information about which field (particle) is described by this equation. The complete information is given only by the pair of conceptions: the equation and the transformation law of the field function. Therefore, the transformations, which are determined not by the 1/2 eigenvalues of the particle spin operator, have a special importance and the physical meaning among the additional transformations, with respect to which the Dirac equation is invariant.

Thus, in addition to the well-known fermionic spin $s = 1/2$ characteristics of the Dirac equation with nonzero mass, we consider the bosonic spin $s = (1, 0)$ symmetries [1–3], solutions, and conservation laws for this equation. Such hidden property of the Dirac equation has been called by us as the Fermi–Bose duality of this equation. We referred also to the steps of other authors [4–13] in this direction, which were published before our investigations [14–17]. The authors of works [18–25] continued our researches and referred to our papers. In another approach [26–29], the quadratic relations between the fermionic and bosonic amplitudes were found and used. In our papers [1–3, 14–17] and here, we discuss linear relations between fermionic and bosonic amplitudes.

For the most simple case of the massless Dirac equation (as well as for the slightly generalized original Maxwell equations), the bosonic symmetries, solutions, and conservation laws were found by us more than 10 years ago (see, e.g., [14–17] and references therein). Only the consideration of the 64-dimensional extended real Clifford–Dirac (ERCD) algebra A_{64} (our generalization [1–3] of the standard

16-dimensional Clifford–Dirac (CD) algebra) enabled us to extend the results to the general case where the mass in the Dirac equation is nonzero. Note that here 7 gamma-matrices obey the anticommutation relations of the CD algebra.

Hence, now we start from the 29-dimensional proper extended real CD algebra (proper ERCD algebra), which is a subalgebra of A_{64} and is generated by 7 (not 5) gamma-matrices. Such proper ERCD algebra realizes a representation of the algebra of SO(8) group.

To use the additional possibilities (additional orts) of the extended CD algebra is the first principal idea, which is the basis of our consideration. Another important idea is our start from the Foldy–Wouthuysen (FW) [30–32] representation of the Dirac equation. Note that the FW representation has important advantages in comparison with the standard so-called local representation of this equation. The quantum-mechanical operators of coordinate, velocity and spin are well defined in the FW representation and have there the simplest forms. Moreover, the standard spin operator in the FW representation commutes with the Hamiltonian of the FW equation (in the Dirac representation, the spin operator, which commutes with the Dirac Hamiltonian, is non-local, and the corresponding well-defined operators of coordinate and velocity are non-local too). One more advantage of the FW representation is as follows. The proof of the pure matrix symmetry properties is much easier and more convenient here in comparison with the standard Pauli–Dirac (PD) representation. The symmetries in the PD representation, i.e. for the Dirac equation, follow from the symmetries of the FW equation on the basis of the FW transformation.

The additional elements of the proper extended real CD algebra lead to the additional possibilities. The application of this new mathematical object enabled us to prove the hidden bosonic properties (symmetries, solutions, and conservation laws) of the Dirac equation with nonzero mass in both standard and FW [30] representations. It is the basis for our dual Fermi–Bose consideration of the spinor field.

The concept of the Fermi–Bose (FB) duality of a spinor field has been mentioned first by L. Foldy [31]. The extended consideration has been given in [33, 34]. P. Garbaczewski proved [33, 34] that the Fock space $\mathcal{H}^{\rm F}(\rm H^{3,M})$ over the quantum mechanical space $\rm L_2(R^3)\otimes C^{\otimes M}$ of the particle, which is described by the field $\phi:\rm M(1,N)\rightarrow C^{\otimes N}$, allows one to fulfill the dual FB quantization of the field ϕ in $\mathcal{H}^{\rm F}$. Both the canonical commutation relations (CCR) and anticommutation relations (CAR) were used to realize the above-mentioned quantization. Moreover, for the both types of quantization, the uniqueness of the vacuum in $\mathcal{H}^{\rm F}$ was proved. The dual FB quantization was illustrated for different examples and in the spaces $\rm M(1,N)$ of arbitrary dimensions. The massless spinor field was considered in details in [34].

In our publications, the consideration of the FB duality concept of the spinor field was extended by applying the group-theoretic approach to the problem. We have found the bosonic symmetries, solutions, and conservation laws as a consequence.

The corresponding results are given below in Sections 2–12.

2. Notations and Definitions

The system of units $\hbar=c=1$ and the metric $g^{\mu\nu}= g_{\mu\nu}=g^{\mu}_{\nu}$, $(g^{\mu}_{\nu})=\mathrm{diag}\,(1,-1,-1,-1)$, $a^{\mu}=g^{\mu\nu}a_{\nu}$, are taken. The Greek indices vary in the region $0,1,2,3\equiv\overline{0,3}$, Latin – $\overline{1,3}$, the summation over the twice repeated index is implied. The Dirac γ^{μ} matrices in the standard (PD) representation are used. Our consideration is fulfilled in the rigged Hilbert space $\rm S^{3,4}\subset H^{3,4}\subset S^{3,4*}$ where $\rm H^{3,4}$ is given by

$$\rm H^{3,4}=L_2(R^3)\otimes C^{\otimes 4}=$$

$$=\left\{\phi=(\phi^{\mu}):\rm R^3\rightarrow C^{\otimes 4};\quad \int d^3x|\phi(t,\mathbf{x})|^2<\infty\right\},\tag{1}$$

and the symbol "*" in $\rm S^{3,4*}$ means that the space of the Schwartz generalized functions $\rm S^{3,4*}$ is conjugated to the Schwartz test function space $\rm S^{3,4}$ by the corresponding topology (for more details, see [2]).

We consider the ordinary CD algebra of 4×4 Dirac matrices in the standard PD representation in terms of the standard 2×2 Pauli matrices:

$$\gamma^0=\begin{vmatrix}1&0\\0&-1\end{vmatrix},\ \gamma^k=\begin{vmatrix}0&\sigma^k\\-\sigma^k&0\end{vmatrix};$$

$$\sigma^1=\begin{vmatrix}0&1\\1&0\end{vmatrix},\ \sigma^2=\begin{vmatrix}0&-i\\i&0\end{vmatrix},\ \sigma^3=\begin{vmatrix}1&0\\0&-1\end{vmatrix};\ k=1,2,3.\tag{2}$$

The γ^μ matrices (2) together with $\gamma^4 \equiv \gamma^0\gamma^1\gamma^2\gamma^3$ satisfy the anticommutation relations of the CD algebra

$$\gamma^{\bar\mu}\gamma^{\bar\nu} + \gamma^{\bar\nu}\gamma^{\bar\mu} = 2g^{\bar\mu\bar\nu}, \quad \bar\mu, \bar\nu = \overline{0,4}, \tag{3}$$

and $\left(g^{\bar\mu}_{\bar\nu}\right) = \text{diag}\,(1, -1, -1, -1, -1)$. Here and in our publications, we use the $\gamma^4 \equiv \gamma^0\gamma^1\gamma^2\gamma^3$ matrix instead of the γ^5 matrix of other authors. Our γ^4 is equal to $i\gamma^5_{\text{standard}}$. Below, we use the notation γ^5 for a completely different matrix $\gamma^5 \equiv \gamma^1\gamma^3\hat{C}$.

The group $\mathcal{P}$ is the universal covering $\mathcal{P} \supset \mathcal{L} =$ = SL(2,C) of the proper orthochronous Poincaré group $\mathrm{P}^\uparrow_+ = \mathrm{T}(4)\times)\mathrm{L}^\uparrow_+ \supset \mathrm{L}^\uparrow_+ = \mathrm{SO(1,3)}$. The group $\mathcal{L}$ = SL(2,C) is the universal covering of the proper orthochronous Lorentz group $\mathrm{L}^\uparrow_+ \supset \mathrm{L}^\uparrow_+ = \mathrm{SO(1,3)}$.

For the purposes related to physics, it is useful to consider the corresponding groups and algebras with real parameters (e.g., the parameters $a = (a^\mu)$, $\omega = (\omega^{\mu\nu})$ of the translations and rotations for the group $\mathrm{P}^\uparrow_+$). Therefore, the corresponding generators are anti-Hermitian. The mathematical correctness of such choice of generators was verified in [35, 36].

3. Extended Real Clifford–Dirac Algebra

The ERCD algebra has been found in [1–3] as a complete set of operators of a standard CD algebra together with the operators of a Pauli–Gürsey–Ibragimov algebra [37, 38]:

$$\{\gamma^2\hat{C},\, i\gamma^2\hat{C},\, \gamma^2\gamma^4\hat{C},\, i\gamma^2\gamma^4\hat{C},\, \gamma^4,\, i\gamma^4,\, i,\, \mathrm{I}\}. \tag{4}$$

Hence, such generalization of the real CD algebra is constructed with the help of an imaginary unit $i = \sqrt{-1}$ together with the operator $\hat{C}$ of complex conjugation, i.e., the operators $i = \sqrt{-1}$ and $\hat{C}$ are the nontrivial orts of the algebra; $\hat{C}$ is the involution operator in the space $\mathrm{H}^{3,4}$.

It is known from [1–3] that 16 orts of the standard CD algebra can be written in the form

$$\{\text{ind} \quad \text{CD}\} \equiv \{\mathrm{I},\, \alpha^{\tilde\mu\tilde\nu} = 2s^{\tilde\mu\tilde\nu}\},\ \tilde\mu, \tilde\nu = \overline{0,5}, \tag{5}$$

where

$$s^{\bar\mu\bar\nu} \equiv \frac{1}{4}\left[\gamma^{\bar\mu}, \gamma^{\bar\nu}\right],\ s^{\bar\mu5} = -s^{5\bar\mu} \equiv \frac{1}{2}\gamma^{\bar\mu}; \qquad \gamma^4 \equiv \gamma^0\gamma^1\gamma^2\gamma^3, \tag{6}$$

and $\bar\mu, \bar\nu = \overline{0,4}$. Matrices (6) satisfy the commutation relations of nontrivial generators of the SO(1,5) algebra in the form

$$\left[s^{\tilde\mu\tilde\nu}, s^{\tilde\rho\tilde\sigma}\right] = -g^{\tilde\mu\tilde\rho}s^{\tilde\nu\tilde\sigma} - g^{\tilde\rho\tilde\nu}s^{\tilde\sigma\tilde\mu} - g^{\tilde\nu\tilde\sigma}s^{\tilde\mu\tilde\rho} - g^{\tilde\sigma\tilde\mu}s^{\tilde\rho\tilde\nu}, \tag{7}$$

where $(g^{\tilde\mu}_{\tilde\nu}) = \text{diag}(+1, -1, -1, -1, -1, -1)$.

In formulae (6) and (7), we rewrite the result in [39, 40], where the 16 orts of the standard CD algebra are presented in a form of the SO(3,3) algebra. We give this result in the form, which is useful for our purposes, i.e. as the SO(1,5) algebra (similarly to the algebra of the proper orthochronous Lorentz group $\mathrm{L}^\uparrow_+ = \mathrm{SO(1,3)}$ in the Minkowski space M(1,3)⊂M(1,5)).

Thus, in the terms of (5), a complete set of 64 orts of the ERCD algebra (the complete set of operators (4) and (5)) has the form

$$\{\text{ERCD}\} = \left\{\begin{matrix} (\text{ind CD}), & i\cdot(\text{ind CD}), \\ \hat{C}\cdot(\text{ind CD}), & i\hat{C}\cdot(\text{ind CD}) \end{matrix}\right\}. \tag{8}$$

The ERCD algebra (8) has only some partial features of the CD algebra. The direct generalization of the standard CD algebra is the proper ERCD algebra, which is a subalgebra of (8) (see the next section 4). Nevertheless, the ERCD algebra has another important subalgebra – the 32-dimensional algebra $\mathrm{A}_{32} = \mathrm{SO}(6) \oplus i\gamma^0 \cdot \mathrm{SO}(6) \oplus i\gamma^0$. The last one is the maximal set of pure matrix operators, which left the FW equation invariant (the details are given in [1–3]).

4. Proper Extended Real Clifford–Dirac Algebra

Here, the subalgebras of the ERCD algebra are considered briefly. The most important are the representations in $\mathrm{C}^{\otimes 4} \subset \mathrm{H}^{3,4}$ of the 29-dimensional proper ERCD algebra SO(8) spanned over the 7 orts

$$\gamma^1,\, \gamma^2,\, \gamma^3,\, \gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3, \qquad \gamma^5 = \gamma^1\gamma^3\hat{C},\, \gamma^6 = i\gamma^1\gamma^3\hat{C},\, \gamma^7 = i\gamma^0, \tag{9}$$

where the γ^μ matrices are given in (2). Generators (9) satisfy the anticommutation relations [1–3]

$$\gamma^{\mathrm{A}}\gamma^{\mathrm{B}} + \gamma^{\mathrm{B}}\gamma^{\mathrm{A}} = -2\delta^{\mathrm{AB}}, \quad \mathrm{A, B} = \overline{1,7}, \tag{10}$$

and the generators of the proper ERCD algebra $\alpha^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}} = 2s^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}}$ (together with the unit element, 4×4 matrix I_4, we have 29 independent orts I_4, $\alpha^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}} = 2s^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}}$)

$$s^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}} = \left\{s^{\mathrm{AB}} = \frac{1}{4}[\gamma^{\mathrm{A}}, \gamma^{\mathrm{B}}],\, s^{\mathrm{A}8} = -s^{8\mathrm{A}} = \frac{1}{2}\gamma^{\mathrm{A}}\right\},$$

$$\widetilde{\mathrm{A}}, \widetilde{\mathrm{B}} = \overline{1,8}, \tag{11}$$

satisfy the commutation relations of the SO(8) algebra

$$[s^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}}, s^{\widetilde{\mathrm{C}}\widetilde{\mathrm{D}}}] = \delta^{\widetilde{\mathrm{A}}\widetilde{\mathrm{C}}} s^{\widetilde{\mathrm{B}}\widetilde{\mathrm{D}}} + \\ + \delta^{\widetilde{\mathrm{C}}\widetilde{\mathrm{B}}} s^{\widetilde{\mathrm{D}}\widetilde{\mathrm{A}}} + \delta^{\widetilde{\mathrm{B}}\widetilde{\mathrm{D}}} s^{\widetilde{\mathrm{A}}\widetilde{\mathrm{C}}} + \delta^{\widetilde{\mathrm{D}}\widetilde{\mathrm{A}}} s^{\widetilde{\mathrm{C}}\widetilde{\mathrm{B}}}. \tag{12}$$

Namely the proper ERCD algebra SO(8) given by 29 orts (11) is our [1–3] direct generalization of the standard 16-dimensional CD algebra. It is also the basis for our dual FB consideration of a spinor field, which enabled us to prove the additional bosonic properties of this field.

5. Foldy–Wouthuysen Representation

For the physical applications, we consider the realizations of the proper ERCD algebra in the field space $\mathrm{S}^*(\mathrm{M}(1,3)) \otimes \mathrm{C}^{\otimes 4} \equiv \mathrm{S}^{4,4*}$ of Schwartz generalized functions and in the quantum mechanical Hilbert space $\mathrm{H}^{3,4}$ (1). These realizations are found with the help of the transformations $V^{+}\mathrm{SO}(8)V^{-}$, $v\mathrm{SO}(8)v$, where the operators of transformations have the form

$$V^{\pm} \equiv \frac{\pm i\boldsymbol{\gamma}\nabla + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}}, \quad v = \begin{vmatrix} \mathrm{I}_2 & 0 \\ 0 & \hat{C}\mathrm{I}_2 \end{vmatrix}, \tag{13}$$

where $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$, $\nabla \equiv (\partial_\ell)$, $\mathrm{I}_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$. In Section 6, the realizations of the proper ERCD algebra for bosonic fields will be presented.

We now consider the ERCD algebra (64 orts) and the proper ERCD algebra (29 orts) in the FW representation of the spinor field [30] (the advantages in comparison with the standard Dirac equation in definitions of coordinate, velocity, and spin operators are well known from [30]; see also the brief discussion in Introduction). In this representation, the equation for the spinor field (the FW equation) has a form

$$(\partial_0 + i\gamma^0\widehat{\omega})\phi(x) = 0, \quad x \in \mathrm{M}(1,3), \quad \phi \in \mathrm{H}^{3,4}. \tag{14}$$

It is linked with the Dirac equation

$$(\partial_0 + iH)\psi(x) = 0, \quad H \equiv \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m, \tag{15}$$

by the FW transformation $V^{\pm}$:

$$\phi(x) = V^{-}\psi(x), \quad \psi(x) = V^{+}\phi(x), \tag{16}$$

and

$$V^{+}\gamma^0\widehat{\omega}V^{-} = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m. \tag{17}$$

Below, the ERCD algebra and the proper ERCD algebra (11) are essentially used in our proofs of bosonic properties of the Dirac and FW equations. The proper ERCD algebra has 29 independent orts given in (11). In comparison with 16 independent orts of the standard CD algebra, we can operate now with additional elements. These additional generators of the SO(8) algebra enabled us to prove the additional bosonic symmetries of the FW and Dirac equations [1–3] and to construct the additional bosonic solutions of these equations (see Sections 7, 8). In calculations, we used the anticommutation relations (10).

6. Representations of the Proper Extended Real Clifford–Dirac Algebra

In **the fundamental FW representation**, 29 orts of the proper ERCD algebra SO(8) are given by formulae (11), where 7 generating operators have the form (9).

In a standard **Pauli–Dirac representation**, the so-called local representation, the corresponding 29 orts are the consequences of the FW transformation $V^{\pm}$ (13), (17) and are given by the elements ($\alpha^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}} = 2\widetilde{s}^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}}$, I), where

$$\widetilde{s}^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}} = \left\{ \widetilde{s}^{\mathrm{AB}} = \frac{1}{4}[\widetilde{\gamma}^{\mathrm{A}}, \widetilde{\gamma}^{\mathrm{B}}], \ \widetilde{s}^{\mathrm{A}8} = -\widetilde{s}^{8\mathrm{A}} = \frac{1}{2}\widetilde{\gamma}^{\mathrm{A}} \right\}, \tag{18}$$

$\widetilde{\mathrm{A}}, \widetilde{\mathrm{B}} = \overline{1,8}$, $\mathrm{A}, \mathrm{B} = \overline{1,7}$. Here, 7 generating operators $\widetilde{\gamma}^{\mathrm{A}} = V^{+}\gamma^{\mathrm{A}}V^{-}$ together with operators $\widetilde{\gamma}^0 = V^{+}\gamma^0V^{-}$ and $\widetilde{C} = V^{+}\hat{C}V^{-}$ are non-local and have the form:

$$\widetilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\frac{-\boldsymbol{\gamma}\cdot\nabla + m}{\widehat{\omega}} + \mathbf{p}\frac{-\boldsymbol{\gamma}\cdot\nabla + \widehat{\omega} + m}{\widehat{\omega}(\widehat{\omega}+m)}, \tag{19}$$

$$\widetilde{\gamma}^4 = \gamma^4\frac{-\boldsymbol{\gamma}\cdot\nabla + m}{\widehat{\omega}}, \tag{20}$$

$$\widetilde{\gamma}^5 = \widetilde{\gamma}^1\widetilde{\gamma}^3\widetilde{C}, \quad \widetilde{\gamma}^6 = i\widetilde{\gamma}^1\widetilde{\gamma}^3\widetilde{C}, \quad \widetilde{\gamma}^7 = i\widetilde{\gamma}^0, \tag{21}$$

$$\widetilde{\gamma}^0 = \gamma^0\frac{-\boldsymbol{\gamma}\cdot\nabla + m}{\widehat{\omega}}, \quad \widetilde{C} = \left(\mathrm{I} + 2\frac{i\gamma^1\partial_1 + i\gamma^2\partial_2}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}}\right)\hat{C}, \tag{22}$$

where $\widehat{\omega} \equiv \sqrt{-\triangle + m^2}$. In **bosonic representation**, where the proof of the bosonic properties of the FW and Dirac equations is most convenient, the corresponding 29 orts of the proper ERCD algebra (SO(8) algebra) are given by the elements ($\alpha^{\widetilde{A}\widetilde{B}} = 2\breve{s}^{\widetilde{A}\widetilde{B}}$, I):

$$\breve{s}^{\widetilde{A}\widetilde{B}} = \left\{\breve{s}^{AB} = \frac{1}{4}[\breve{\gamma}^A, \breve{\gamma}^B],\ \breve{s}^{A8} = -\breve{s}^{8A} = \frac{1}{2}\breve{\gamma}^A\right\}, \quad (23)$$

where $\widetilde{A}, \widetilde{B} = \overline{1,8}$, $A, B = \overline{1,7}$. In formulae (23), 7 generating operators $\breve{\gamma}^A$ (together with operators $\breve{\gamma}^0$, i, and $\breve{C}$) have the form

$$\breve{\gamma}^0 = \begin{vmatrix} \sigma^3 & 0 \\ 0 & \sigma^1 \end{vmatrix}, \quad \breve{\gamma}^1 = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & i & i \\ -1 & i & 0 & 0 \\ 1 & i & 0 & 0 \end{vmatrix}, \quad (24)$$

$$\breve{\gamma}^2 = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & -i & i \\ 0 & 0 & -1 & -1 \\ -i & 1 & 0 & 0 \\ i & 1 & 0 & 0 \end{vmatrix}, \quad \breve{\gamma}^3 = -\begin{vmatrix} \sigma^2 & 0 \\ 0 & i\sigma^2 \end{vmatrix}\hat{C}, \quad (25)$$

$$\breve{\gamma}^4 = \begin{vmatrix} i\sigma^2 & 0 \\ 0 & -\sigma^2 \end{vmatrix}\hat{C}, \quad \breve{\gamma}^5 = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & i & -i \\ 1 & i & 0 & 0 \\ 1 & -i & 0 & 0 \end{vmatrix}, \quad (26)$$

$$\breve{\gamma}^6 = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & -i & -i \\ 0 & 0 & 1 & -1 \\ -i & -1 & 0 & 0 \\ -i & 1 & 0 & 0 \end{vmatrix}, \quad \breve{\gamma}^7 = \gamma^7 = i\gamma^0, \quad (27)$$

$$\breve{i} = \begin{vmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^1 \end{vmatrix}, \quad \breve{C} = \begin{vmatrix} \sigma^3 & 0 \\ 0 & I_2 \end{vmatrix}\hat{C}. \quad (28)$$

The transition from the fundamental representation of the proper ERCD algebra to the bosonic representation is fulfilled by the transformation $\breve{\gamma}^A = W\gamma^A W^{-1}$ with the help of the operator W:

$$W = \frac{1}{\sqrt{2}}\begin{vmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{2}\hat{C} & 0 \\ 0 & -\hat{C} & 0 & 1 \\ 0 & -\hat{C} & 0 & -1 \end{vmatrix},$$

$$W^{-1} = \begin{vmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -\hat{C} & -\hat{C} \\ 0 & i\sqrt{2}\hat{C} & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix}, \quad (29)$$

$WW^{-1} = W^{-1}W = I_4$.

In **the relativistic canonical quantum mechanics** [31, 32] (axiomatic foundations are given briefly in [41]), 7 generating $\bar{\gamma}^A$ matrices are given by $\bar{\gamma}^A = v\gamma^A v$ (operator v is known from (13)) and have the explicit form:

$$\bar{\gamma}^1 = \gamma^1\hat{C},\ \bar{\gamma}^2 = \gamma^0\gamma^2\hat{C},\ \bar{\gamma}^3 = \gamma^3\hat{C},\ \bar{\gamma}^4 = \gamma^0\gamma^4\hat{C}, \quad (30)$$

$$\bar{\gamma}^5 = \gamma^1\gamma^3\hat{C},\ \bar{\gamma}^6 = -i\gamma^2\gamma^4\hat{C},\ \bar{\gamma}^7 = i. \quad (31)$$

Thus, the representation of the proper ERCD algebra SO(8) in relativistic canonical quantum mechanics is given similarly to the form (18), (23) with 7 generating operators (30), (31).

7. Bosonic Spin $s = (1, 0)$ Symmetry of the Foldy–Wouthuysen and Dirac Equations

We now consider an example of the construction of the important bosonic symmetry of the FW and Dirac equations. The fundamental assertion is that the subalgebra SO(6) of the proper ERCD algebra (11), which is determined by the operators

$$\{I,\ \alpha^{\bar{A}\bar{B}} = 2s^{\bar{A}\bar{B}}\}, \quad \bar{A}, \bar{B} = \overline{1,6}, \quad (32)$$

$$\{s^{\bar{A}\bar{B}}\} = \left\{s^{\bar{A}\bar{B}} \equiv \frac{1}{4}[\gamma^{\bar{A}}, \gamma^{\bar{B}}]\right\} \quad (33)$$

is the algebra of invariance of the Dirac equation in the FW representation (14) (in (33), six matrices $\{\gamma^{\bar{A}}\} = \{\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5$, and $\gamma^6\}$ are known from (2), (9)). The algebra SO(6) contains two different realizations of the SU(2) algebra for the spin $s = 1/2$ doublet. By taking the sum of two independent sets of SU(2) generators from (33), one can obtain the SU(2) generators of the spin $s = (1, 0)$ multiplet, which generate the transformation of invariance of the FW equation (14). These operators can be presented in the form

$$\breve{\mathbf{s}} \equiv (\breve{s}^j) = (\breve{s}_{mn}) =$$
$$= \frac{1}{2}(\breve{\gamma}^2\breve{\gamma}^3 - \breve{\gamma}^0\breve{\gamma}^2\breve{C},\ \breve{\gamma}^3\breve{\gamma}^1 + \breve{i}\breve{\gamma}^0\breve{\gamma}^2\breve{C},\ \breve{\gamma}^1\breve{\gamma}^2 - \breve{i}), \quad (34)$$

where the corresponding orts of the ERCD algebra in the bosonic representation are given in (24)–(28).

The spin operators (34) of the SU(2) algebra, which commute with the operator $\partial_0 + i\gamma^0\widehat{\omega}$ of the FW equation (14), can be presented in the explicit form

$$\breve{s}^1 = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & i\hat{C} & 0 \\ 0 & 0 & -\hat{C} & 0 \\ -i\hat{C} & \hat{C} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\breve{s}^2 = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & \hat{C} & 0 \\ 0 & 0 & -i\hat{C} & 0 \\ -\hat{C} & i\hat{C} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \breve{s}^3 = \begin{vmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}. \quad (35)$$

The calculation of the Casimir operator for the SU(2) generators (35) gives the result $\breve{\mathbf{s}}^2 = -1(1+1)\begin{vmatrix} \mathrm{I}_3 & 0 \\ 0 & 0 \end{vmatrix}$.

On the basis of the spin operators (34) and (35), the bosonic spin (1,0) representation of the Poincaré group $\mathcal{P}$ can be constructed. It is easy to show (after our consideration in [1–3] and above) that the generators

$$p_0 = -i\gamma_0\widehat{\omega},\ p_n = \partial_n,\ j_{ln} = x_l\partial_n - x_n\partial_l + \breve{s}_{ln},$$
$$j_{0k} = x_0\partial_k + i\gamma_0\left\{x_k\widehat{\omega} + \frac{\partial_k}{2\widehat{\omega}} + \frac{(\breve{\mathbf{s}}\times\boldsymbol{\partial})_k}{\widehat{\omega}+m}\right\}, \quad (36)$$

of the group $\mathcal{P}$ commute with the operator of the FW equation (14) and satisfy the commutation relations of the Lie algebra of the group $\mathcal{P}$ for the anti-Hermitian generators in a manifestly covariant form:

$$[p_\mu, p_\nu] = 0, \quad [p_\mu, j_{\rho\sigma}] = g_{\mu\rho}p_\sigma - g_{\mu\sigma}p_\rho,$$
$$[j_{\mu\nu}, j_{\rho\sigma}] = -\left(g_{\mu\rho}j_{\nu\sigma} + g_{\rho\nu}j_{\sigma\mu} + g_{\nu\sigma}j_{\mu\rho} + g_{\sigma\mu}j_{\rho\nu}\right). \quad (37)$$

In the space $\mathrm{H}^{3,4}$, operators (36) generate a unitary $\mathcal{P}$ representation, which is different from the fermionic $\mathcal{P}^{\mathrm{F}}$-generators D-64–D-67 of [31], i.e., the bosonic $\mathcal{P}^{\mathrm{B}}$ representation of the group $\mathcal{P}$, with respect to which the FW equation (14) is invariant. For generators (36), the Casimir operators have the form:

$$p^\mu p_\mu = m^2,$$
$$W^{\mathrm{B}} = w^\mu w_\mu = m^2\breve{\mathbf{s}}^2 = -1(1+1)m^2\begin{vmatrix} \mathrm{I}_3 & 0 \\ 0 & 0 \end{vmatrix}. \quad (38)$$

Hence, according to the Bargmann–Wigner classification, we consider the spin $s = (1,0)$ representation of the group $\mathcal{P}$.

The corresponding bosonic spin $s = (1,0)$ symmetries of the Dirac equation (15) can be found from generators (36) with the help of the FW operator (13), i.e., as $V^+\,(p_\mu, j_{\mu\nu})\,V^-$.

Some other details of the consideration of the bosonic symmetries of the FW and Dirac equations were given in [1–3].

8. Bosonic Spin $s = (1,0)$ Multiplet Solution of the Foldy–Wouthuysen and Dirac Equations

As the next step in the FB duality investigation, we consider the bosonic solution of the Dirac (FW) equation. A bosonic solution of the FW equation (14) is found completely similarly to the procedure of construction of the standard fermionic solution. Thus, the bosonic solution is determined by some stationary diagonal complete set of operators of bosonic physical quantities for the spin $s = (1,0)$-multiplet in the FW representation, e.g., by the set "momentum-spin projection $\breve{s}^3$":

$$(\mathbf{p} = -\nabla,\ \breve{s}^3), \quad (39)$$

where the spin operators $\breve{\mathbf{s}}$ and $\breve{s}^3$ for the spin $s = (1,0)$-multiplet are given in (34), (35). The fundamental solutions of Eq. (14), which are the common eigensolutions of the bosonic complete set (39), have the form

$$\varphi^-_{\mathbf{k}\mathrm{r}}(t,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}e^{-ikx}\mathrm{d}_{\mathrm{r}},$$
$$\varphi^+_{\mathbf{k}\acute{\mathrm{r}}}(t,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}e^{ikx}\mathrm{d}_{\acute{\mathrm{r}}}, \quad (40)$$

where $kx = \omega t - \mathbf{kx}$ and $\mathrm{d}_\alpha = (\delta^\beta_\alpha)$ are the Cartesian orts in the space $\mathrm{C}^{\otimes 4} \subset \mathrm{H}^{3,4}$, numbers $\mathrm{r} = (1,2)$, $\acute{\mathrm{r}} = (3,4)$ mark the eigenvalues $(+1,-1,0,\underline{0})$ of the operator $\breve{s}^3$ from (34), (35).

The bosonic solutions of Eq. (14) are the generalized states belonging to the space $\mathrm{S}^{3,4*}$; they form a complete orthonormalized system of bosonic states. Therefore, any bosonic physical state of the FW field ϕ from the manifold $\mathrm{S}^{3,4}$ dense in $\mathrm{H}^{3,4}$ (the general bosonic solution of Eq. (14)) is uniquely presented in the form

$$\phi_{(1,0)}(x) =$$
$$= \frac{1}{(2\pi)^{3/2}}\int d^3k[\xi^{\mathrm{r}}(\mathbf{k})\mathrm{d}_{\mathrm{r}}e^{-ikx} + \xi^{*\acute{\mathrm{r}}}(\mathbf{k})\mathrm{d}_{\acute{\mathrm{r}}}e^{ikx}], \quad (41)$$

where $\xi(\mathbf{k})$ are the coefficients of the expansion of the bosonic solution of the FW equation (14) with respect to the Cartesian basis $\mathrm{d}_\alpha = (\delta^\beta_\alpha)$ (40). The relationships of the amplitudes $\xi(\mathbf{k})$ with the quantum-mechanical bosonic amplitudes $b(\mathbf{k})$ of probability distribution according to the eigenvalues of the stationary diagonal complete set of operators of the quantum-mechanical bosonic $s = (1,0)$-multiplet are given by

$$\xi^1 = b^1, \quad \xi^2 = -\frac{1}{\sqrt{2}}(b^3 + b^4), \quad \xi^3 = -ib^2,$$

$$\xi^4 = \frac{1}{\sqrt{2}}(b^3 - b^4), \qquad (42)$$

where 4 amplitudes $b^{1,2,3,4}(\mathbf{k}) \equiv b^{+,-,0,\underline{0}}(\mathbf{k})$ are the quantum-mechanical momentum-spin amplitudes with the eigenvalues (+1,-1,0,$\underline{0}$) of the quantum-mechanical (1,0) multiplet $\breve{s}^3$ operator projections, respectively (last eigenvalue $\underline{0}$ is related to the proper zero spin). If $\phi_{(1,0)}(x) \in \mathrm{S}^{3,4}$, then the bosonic amplitudes $\xi(\mathbf{k})$ belong to the Schwartz complex-valued test function space too.

Moreover, the set $\{\phi_{(1,0)}(x)\}$ of solutions (41) is invariant just with respect to the unitary bosonic representation of the group $\mathcal{P}$, which is determined by generators (36) and Casimir operators (38). Therefore, the Bargmann–Wigner analysis of the Poincaré symmetry of the set $\{\phi_{(1,0)}(x)\}$ of solutions (41) completes the demonstration that it is the set of Bose-states $\phi_{(1,0)}$ of the field ϕ, i.e. the $s = (1, 0)$-multiplet states. Hence, the existence of bosonic solutions of the FW equation is proved.

In the terms of the quantum-mechanical momentum-spin amplitudes $b^\alpha(\mathbf{k})$ from (42), the bosonic spin (1,0)-multiplet solution $\psi = V^+\phi$ of the Dirac equation (15) is given by

$$\psi_{(1,0)}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k\{e^{-ikx}[b^1 v_1^-(\mathbf{k}) - \\ -\frac{1}{\sqrt{2}}(b^3 + b^4)v_2^-(\mathbf{k})] + \\ + e^{ikx}[ib^{*2}v_1^+(\mathbf{k}) + \frac{1}{\sqrt{2}}(b^{*3} - b^{*4})v_2^+(\mathbf{k})]\}, \qquad (43)$$

where the 4-component spinors are the same as in the Dirac theory of a fermionic doublet

$$v_\mathrm{r}^-(\mathbf{k}) = N \begin{vmatrix} (\widehat{\omega} + m)\mathrm{d_r} \\ (\boldsymbol{\sigma} \cdot \mathbf{k})\mathrm{d_r} \end{vmatrix}, \\ v_\mathrm{r}^+(\mathbf{k}) = N \begin{vmatrix} (\boldsymbol{\sigma} \cdot \mathbf{k})\mathrm{d_r} \\ (\widehat{\omega} + m)\mathrm{d_r} \end{vmatrix}, \qquad (44)$$

and $N \equiv \frac{1}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}}$, $\mathrm{d}_1 = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$, $\mathrm{d}_2 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$.

The well-known (standard) Fermi solution of the Dirac equation for the spin $s = 1/2$ doublet has the form

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k[e^{-ikx}a_\mathrm{r}^-(\mathbf{k})v_\mathrm{r}^-(\mathbf{k}) + \\ +e^{ikx}a_\mathrm{r}^+(\mathbf{k})v_\mathrm{r}^+(\mathbf{k})], \qquad (45)$$

where the physical meaning of the amplitudes $a_\mathrm{r}^-(\mathbf{k})$, $a_\mathrm{r}^+(\mathbf{k})$ is explained in [42].

All the above-given assertions about the FB duality of the spinor field are valid both in the FW and PD representations, i.e., for the FW and Dirac equations [(14) and (15), respectively]. The transition between the FW and PD representations is fulfilled by the FW transformation $V^\pm$ (13).

9. Lagrangian for the Foldy–Wouthuysen Equation

Before the Noether analysis of conservation laws, we must consider the Lagrange approach (L-approach) for the spinor field $\phi(x)$ in the FW representation. The L-approach in this representation was formulated first in [43], [44]. The representation of the operator $\widehat{\omega} \equiv \sqrt{-\triangle + m^2}$ in the form of a series in the Laplace operator $\triangle$ powers was used. W. Krech applied a nonstandard formulation of the least action principle in the terms of infinite-order derivatives of the field functions. The mathematical correctness was not considered.

Therefore, we present below briefly a well-defined L-approach for the spinor field in the FW representation, which is based on the standard formulation of the least action principle. The quantum-mechanical rigged Hilbert space (both in the coordinate and momentum realizations of this space) is used, but the start is well-defined from the momentum realization. In such realization, the rigged Hilbert space is given by

$$\widetilde{\mathrm{S}}^{3,4} \subset \widetilde{\mathrm{H}}^{3,4} \subset \widetilde{\mathrm{S}}^{3,4*}; \quad \widetilde{\mathrm{H}}^{3,4} = \mathrm{L}_2(\mathrm{R}_k^3) \otimes \mathrm{C}^{\otimes 4} = \\ = \{\widetilde{\phi} = (\widetilde{\phi}^\mu) : \mathrm{R}_k^3 \to \mathrm{C}^{\otimes 4}; \int d^3k|\widetilde{\phi}(t, \mathbf{k})|^2 < \infty\}\}. \qquad (46)$$

Here R_k^3 is the momentum operator $\mathbf{p}$ spectrum, which is canonically conjugated to the coordinate $\mathbf{x}$, ($[x^j, p^l] = i\delta^{jl}$). The corresponding $\mathbf{x}$-realization is connected with (46) by a 3-dimensional Fourier transformation. The alternative use of both realizations is based on the principle of heredity with classical and non-relativistic quantum mechanics of a single mass point and with the mechanics of continuous media. The Lagrange function and the action (in alternative $\mathbf{x}$ or $\mathbf{k}$-realizations) are constructed in the complete analogy with their consideration in the classical mechanics of a system with finite number of freedom degrees $q = (q_1, q_2, ...)$. The difference is only in the fact that here the continuous variable $\mathbf{k} \in \mathrm{R}^3_k$ is the carrier of freedom degrees.

In the **k**-realization, where this analogy is maximally clear, the Lagrange function has the form

$$L = L(\widetilde{\phi}, \widetilde{\phi}^{\dagger}, \widetilde{\phi}_{,0}, \widetilde{\phi}^{\dagger}{}_{,0}) =$$
$$= \frac{i}{2}[\widetilde{\phi}^{\dagger}(\widetilde{\phi}_{,0} + i\gamma^0\widetilde{\omega}\widetilde{\phi}) - (\widetilde{\phi}^{\dagger}{}_{,0} - i\widetilde{\omega}\widetilde{\phi}^{\dagger}\gamma^0)\widetilde{\phi}]. \quad (47)$$

In the **x**-realization, this function can be found from (47) by the Fourier transformation. The Euler–Lagrange equations coincide with the FW equation in both realizations. For example, in the **k**-realization, the Euler–Lagrange equations

$$\frac{\delta W}{\delta\widetilde{\phi}^{\dagger}} \equiv \frac{\partial L}{\partial\widetilde{\phi}^{\dagger}} - \frac{\partial}{\partial t}\frac{\partial L}{\partial\widetilde{\phi}^{\dagger}{}_{,0}} = 0,$$
$$\frac{\delta W}{\delta\widetilde{\phi}} \equiv \frac{\partial L}{\partial\widetilde{\phi}} - \frac{\partial}{\partial t}\frac{\partial L}{\partial\widetilde{\phi}_{,0}} = 0, \quad (48)$$

coincide with the FW equation for the vectors $\widetilde{\phi} \in \widetilde{\mathrm{H}}^{3,4}$:

$$(i\partial_0 - \gamma^0\widetilde{\omega})\widetilde{\phi}(t, \mathbf{k}) = 0;$$
$$\widetilde{\omega} \equiv \sqrt{\mathbf{k}^2 + m^2}, \quad \mathbf{k} \in \mathrm{R}^3_k, \quad (49)$$

and with conjugated equation for $\widetilde{\phi}^{\dagger}$.

The well-defined L-approach for the FW field becomes the essentially actual problem after the construction of the quantum electrodynamics in the FW representation in [45].

10. Fermi–Bose Conservation Laws for a Spinor Field

Note briefly the FB conservation laws (CLs) for the spinor field. It is preferable to calculate them in the FW (not local PD) representation too. In FW representation, the Fermi spin s^{12} from (33) (together with the "boost spin") is the independent symmetry operator for the FW equation. The orbital angular momentum and the pure Lorentz angular momentum (the carriers of external statistical degrees of freedom) are the independent symmetry operators in this representation too (one can find the corresponding independent spin and angular momentum symmetries in the PD representation for the Dirac equation too, but the corresponding operators are essentially nonlocal). Hence, one obtains 10 Poincaré and 12 additional (3 spin, 3 pure Lorentz spin, 3 angular momentum, 3 pure angular momentum) CLs.

Therefore, in the FW representation, one can find very easily 22 fermionic and 22 bosonic CLs. The separation into bosonic and fermionic sets is caused by the existence of the FB symmetries and solutions. Indeed, if the substitution of the bosonic $\mathcal{P}$ generators q (36) and the bosonic solutions (41) into the Noether formula

$$Q = \int d^3x\phi^{\dagger}(x)q\phi(x) \quad (50)$$

is made, then automatically the bosonic CLs for $s = (1, 0)$-multiplet are obtained. The standard substitution of the corresponding well-known fermionic generators and solutions gives fermionic CLs.

We illustrate briefly the difference in fermionic and bosonic CLs on the example of the corresponding spin conservation. For the fermionic spin

$$\mathbf{s} = (s_{23}, s_{31}, s_{12}) \equiv (s^{\ell}) = \frac{1}{2}\begin{vmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{vmatrix}, \quad (51)$$

$$s_z \equiv s^3 = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \quad (52)$$

and bosonic spin (34), (35), the CLs are given by

$$S^{\mathrm{F}}_{mn} = \int d^3x\phi^{\dagger}(x)s_{mn}\phi(x) =$$
$$= \int d^3kA^{\dagger}(\mathbf{k})s_{mn}A(\mathbf{k})), \quad (53)$$

$$S^{\mathrm{B}}_{mn} = \int d^3x\phi^{\dagger}(x)\breve{s}_{mn}\phi(x) =$$
$$= \int d^3kB^{\dagger}(\mathbf{k})\breve{s}_{mn}B(\mathbf{k})), \quad (54)$$

where

$$A(\mathbf{k}) = \mathrm{column}(a^-_+,\, a^-_-,\, a^{*+}_-,\, a^{*+}_+), \quad (55)$$

$$B(\mathbf{k}) = \mathrm{column}(b^1,\, b^2,\, b^{*3},\, b^{*4}). \quad (56)$$

We present these CLs in terms of quantum-mechanical Fermi and Bose amplitudes. Such explicit quantum-statistical form is inherent in all integral conserved quantities.

11. Links between the Fermionic and Bosonic Amplitudes and Some Interpretation

Here, we continue the consideration of bosonic solutions of the Dirac equation. A comparison with the standard fermionic solutions is given.

The adequate statistical quantum-mechanical sense of the coefficients $a_{\rm r}^-(\mathbf{k})$, $a_{\rm r}^+(\mathbf{k})$ in expansion (45) in the basis solutions (44) of the Dirac equation is found identically only with the help of the transition $\phi(x) = V^-\psi(x)$ (13), (16) to the FW representation [30]. Indeed, the statistical sense of the FW field $\phi(x)$ is evidently related to the statistical sense of the particle-antiparticle doublet in relativistic canonical quantum mechanics [31, 32], [41]. It is shown in [31] that

$$\phi = \left|\begin{matrix}\phi^- \\ 0\end{matrix}\right| + \left|\begin{matrix}0 \\ \phi^{*+}\end{matrix}\right|, \tag{57}$$

where $\phi^{\mp}(x)$ are the relativistic quantum-mechanical wave functions of a particle-antiparticle doublet.

The solution of the FW equation (14) expanded in the eigenvectors of the quantum-mechanical fermionic stationary diagonal complete set of operators (momentum $\mathbf{p}$, projection s^3 of the spin $\mathbf{s}^{\rm quant.-mech.}$, and sign of the charge $g = -\gamma^0$) has the form

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int d^3k\{e^{-ikx}[a_+^-(\mathbf{k})\mathrm{d}_1 + a_-^-(\mathbf{k})\mathrm{d}_2] +$$
$$+ e^{ikx}[a_-^{*+}(\mathbf{k})\mathrm{d}_3 + a_+^{*+}(\mathbf{k})\mathrm{d}_4]\}, \tag{58}$$

where the coefficients of the expansion $a_+^-(\mathbf{k})$, $a_-^-(\mathbf{k})$, $a_-^+(\mathbf{k})$, $a_+^+(\mathbf{k})$ have the meaning of the statistical quantum-mechanical amplitudes of probability distribution over the eigenvalues of the above-mentioned fermionic stationary complete set of operators. The 4-columns $\mathrm{d}_\alpha = (\delta_\alpha^\beta)$ are the Cartesian orts in the space $\mathrm{C}^{\otimes 4} \subset \mathrm{H}^{3,4}$. In order to obtain the most adequate and obvious statistical quantum-mechanical interpretation of the amplitudes and the solutions, the spin projection operator in the complete set (momentum $\mathbf{p}$, projection s^3 of the spin $\mathbf{s}^{\rm quant.-mech.}$, and sign of the charge $g = -\gamma^0$) is taken in the quantum-mechanical form [41]

$$\mathbf{s}^{\rm quant.-mech.} = \frac{1}{2}\left|\begin{matrix}\boldsymbol{\sigma} & 0 \\ 0 & -C\boldsymbol{\sigma}C\end{matrix}\right|, \tag{59}$$

$$s_z^{\rm quant.-mech.} \equiv s^3 = \frac{1}{2}\left|\begin{matrix}1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1\end{matrix}\right|, \tag{60}$$

rather than in the canonical field theory form (51), (52). The statistical sense of the amplitudes is conserved in the solution ($\psi(x) = V^+\phi(x)$ (9), (6))

$$\psi(x) = \frac{1}{(2\pi)^{3/2}}\int d^3k\{e^{-ikx}[a_+^-(\mathbf{k})v_1^-(\mathbf{k}) + a_-^-(\mathbf{k})v_2^-(\mathbf{k})] +$$
$$+ e^{ikx}[a_-^{*+}(\mathbf{k})v_1^+(\mathbf{k}) + a_+^{*+}(\mathbf{k})v_2^+(\mathbf{k})]\}, \tag{61}$$

of the Dirac equation (15) in its standard local representation. The amplitudes $a_+^-(\mathbf{k})$, $a_-^-(\mathbf{k})$, $a_-^+(\mathbf{k})$, $a_+^+(\mathbf{k})$ in the fermionic solutions (58) and (61) of the FW and Dirac equations are the same. Thus, $a_+^-(\mathbf{k})$, $a_-^-(\mathbf{k})$ are the quantum-mechanical momentum-spin amplitudes of a particle with charge $-e$ and the spin projection eigenvalues $+1/2$ and $-1/2$; $a_-^+(\mathbf{k})$, $a_+^+(\mathbf{k})$ are the quantum-mechanical momentum-spin amplitudes of an antiparticle with charge $+e$ and the spin projection eigenvalues $-1/2$ and $+1/2$, respectively.

The statistical quantum mechanical sense of the bosonic amplitudes $b^\alpha(\mathbf{k})$ of the bosonic solution (43) of the Dirac equation (15) is found similarly and is explained in Section 5 in the course of construction of this solution.

The relationship between the fermionic $a_+^-(\mathbf{k})$, $a_-^-(\mathbf{k})$, $a_-^+(\mathbf{k})$, $a_+^+(\mathbf{k})$ and bosonic $b^{1,2,3,4}(\mathbf{k}) \equiv b^{+,-,0,\underline{0}}(\mathbf{k})$ amplitudes in the same (arbitrarily fixed) physical state of a FB dual field ψ is given by the unitary operator U in the form:

$$\left|\begin{matrix}a_+^- \\ a_-^- \\ a_-^+ \\ a_+^+\end{matrix}\right| = \frac{1}{\sqrt{2}}\left|\begin{matrix}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & -i\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & -1\end{matrix}\right|\left|\begin{matrix}b^+ \\ b^- \\ b^0 \\ b^{\underline{0}}\end{matrix}\right|, \tag{62}$$

$$\left|\begin{matrix}b^+ \\ b^- \\ b^0 \\ b^{\underline{0}}\end{matrix}\right| = \frac{1}{\sqrt{2}}\left|\begin{matrix}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{2} & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1\end{matrix}\right|\left|\begin{matrix}a_+^- \\ a_-^- \\ a_-^+ \\ a_+^+\end{matrix}\right|. \tag{63}$$

Relations (62) and (63) follow directly from the comparison of solutions (43) and (61).

Note that the set of fermionic solutions $\{\psi^{\rm F}\}$ (61) of the Dirac equation is invariant with respect to the well-known induced fermionic $\mathcal{P}^{\rm F}$ representation of the Poincaré group $\mathcal{P}$ [46] (see also formula (19) in work [2]). The set of bosonic solutions $\{\psi^{\rm B}\}$ (43) of the Dirac equation is invariant with respect to the induced bosonic $\mathcal{P}^{\rm B}$ representation of the Poincaré group $\mathcal{P}$ (formula (21) in work [3]). However, the relations (62) and (63) between the fermionic $a_+^-(\mathbf{k})$, $a_-^-(\mathbf{k})$, $a_-^+(\mathbf{k})$, $a_+^+(\mathbf{k})$ and bosonic $b^{1,2,3,4}(\mathbf{k}) \equiv b^{+,-,0,\underline{0}}(\mathbf{k})$ amplitudes are not changed in any inertial frame of references.

12. Conclusions

The property of the Fermi–Bose duality of the Dirac equation is proved on three levels: bosonic symmetries, bosonic solutions, and the corresponding conservation laws. The role of the proper ERCD algebra SO(8) in the proof of this assertions is demonstrated. Thus, the property of the Fermi–Bose duality of the Dirac equation (both in the Foldy–Wouthuysen and the Pauli–Dirac representations), whose proof was started in [1–3], where the bosonic symmetries of this equation were found, is demonstrated on the next step, where the spin (1,0) bosonic solutions of this equation and the corresponding bosonic conservation laws exist.

The 64-dimensional ERCD and 29-dimensional proper ERCD algebras considered in [1–3] are the useful generalizations of the standard 16-dimensional CD algebra. Their application enabled us to prove the existence of additional bosonic symmetries, solutions, and conservation laws for the spinor field and for the Foldy–Wouthuysen and Dirac equations. It is evident that these new algebras can be used in all problems of theoretical and mathematical physics, where the standard CD algebra was used. New interesting results will follow from the wide-range application of the proper ERCD algebra instead of the standard CD algebra. Our experience [14–17] can guarantee new results at least in such problems as the deep analysis of the Maxwell [47] and complex Dirac–Kähler [48] equations.

The investigation of the spinor field in the Foldy–Wouthuysen representation has the independent meaning and purpose. This representation itself is of interest in connection with the recent result [45] of V. Neznamov, who developed the formalism of quantum electrodynamics in the Foldy–Wouthuysen representation (see also the results in [49]). Therefore, the new Lagrangian for the Foldy–Wouthuysen equation, which is put here in consideration in Section 9, has the independent meaning.

We do not consider here the Pauli principle. Our results are not related to this concept. In any case, we do not change the main well-known postulates and theory of the Fermi–Dirac and Bose–Einstein statistics. Our results have another new fundamental meaning. In our approach, the Fermi–Bose duality of a spinor field found on the level of amplitude relations in [33, 34] is proved in another way by the examples of the existence of the bosonic symmetries (Section 7) and solutions (Sections 8 and 11) of the Dirac equation with nonzero mass together with obtaining the bosonic conservation laws (Section 10) for the spinor field. It opens new possibilities for the application of the Dirac equation to the description of bosonic states. Thus, the property of the Fermi–Bose duality of the Dirac equation proven in our publications [1–3] and here does not break the Fermi statistics for fermions (with the Pauli principle) and the Bose statistics for bosons (with the Bose condensation). We also never mixed the Fermi and Bose statistics between each other. Our assertion is the following one. One can apply both the Fermi and Bose statistics for the same Dirac equation and the same spinor field with equal success, i.e., the Dirac equation can describe both fermionic and bosonic states.

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БОЗОННІ СИМЕТРІЇ, РОЗВ'ЯЗКИ ТА ЗАКОНИ ЗБЕРЕЖЕННЯ ДЛЯ РІВНЯННЯ ДІРАКА З НЕНУЛЬОВОЮ МАСОЮ

Р е з ю м е

У доповнення до добре відомих Фермі властивостей рівняння Дірака знайдено приховані бозонні властивості цього рівняння. Розглянуто бозонні симетрії, розв'язки та закони збереження. Такі нові характеристики рівняння Дірака з ненульовою масою знайдено за допомогою 64-вимірної розширеної дійсної алгебри Кліффорда–Дірака та 29-вимірної власної розширеної дійсної алгебри Кліффорда–Дірака. При цьому важливим є старт з представлення Фолді–Вотхойзена для спінорного поля. Показано, що рівняння Дірака може описувати не лише ферміонні, а й бозонні стани.