

---

A.M. PAVLYUK

Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine  
 (14b, Metrolohichna Str., Kyiv 03680, Ukraine; e-mail: pavlyuk@bitp.kiev.ua)

**GENERALIZATION OF POLYNOMIAL  
 INVARIANTS AND HOLOGRAPHIC PRINCIPLE  
 FOR KNOTS AND LINKS**

PACS 02.10.Kn, 02.20.Uw

---

*We formulate the holographic principle for knots and links. For the “space” of all knots and links, torus knots  $T(2m + 1, 2)$  and torus links  $L(2m, 2)$  play the role of the “boundary” of this space. Using the holographic principle, we find the skein relation of knots and links with the help of the recurrence relation for polynomial invariants of torus knots  $T(2m + 1, 2)$  and torus links  $L(2m, 2)$ . As an example of the application of this principle, we derive the Jones skein relation and its generalization with the help of some variants of  $(q, p)$ -numbers, related with  $(q, p)$ -deformed bosonic oscillators.*

*Keywords:* holographic principle, knots, links, Jones skein relation.

**1. Introduction**

The interaction of knot theory and physics manifests itself from the knotting of physics to the physicalness of knots [1–3]. The construction of invariants of knots and links on the basis of deep physical ideas shows the physical nature of knots and links. In particular, the applications of the Jones polynomials and its generalizations [4] demonstrate that knots may play an important role in the unification of general relativity and quantum physics [5, 6]. Witten introduced a heuristic definition of the Jones polynomial invariants in terms of topological quantum field theory.

A powerful modern tool for physical investigations, the holographic principle (HP), was introduced in 1993 by 't Hooft [7], when studying the quantum mechanical features of black holes at the Planck scale. It claims that the total information contained in a volume of space is encoded on the boundary of this space. From the 't Hooft conjecture, it follows that our world at Planckian energies can be described as 2d, rather than 3d; e.g., 3d quantum gravity can be described by 2d topological field theory. In 1997, the HP was used by Maldacena [8] to develop AdS/CFT

correspondence, which manifests a holographic duality between quantum gravity in the 5d anti-de Sitter space and the conformal field theory on its 4d boundary. Verlinde [9] recently proposed that the HP provides a natural mechanism for the Newton gravity and the Einstein gravity to emerge due to entropic forces.

Thus, the HP allows one to deal with the theory in a D-dimensional space instead of studying a (D+1)-dimensional theory. Among possible generalizations of the HP, one can seek such ones that describe the properties of the whole system on the basis of the properties of its part, which can be considered as a “boundary”. In this paper, we make a conjecture on the existence of the HP for the “space” of all knots and links. As the prompting point of this conjecture, we can use the fact that the transformations of 3d knot and link structures are described with the help of the transformations of their 2d planar diagrams. To express the HP for knots and links mathematically, we consider the torus knots  $T(2m + 1, 2)$  and the torus links  $L(2m, 2)$  with  $m$  to be a non-negative integer, as playing the role of the “boundary” for the “space” of all knots and links. We will show that such an important information (concern-

ing all knots and links), as the skein relation, can be obtained from the “boundary” information – the recurrence relation for the polynomial invariants of torus knots  $T(2m + 1, 2)$  and torus links  $L(2m, 2)$ , which, in turn, can be obtained with the help of the properties of  $(q, p)$ -numbers [10, 11]. The “boundary” position of these torus knots and links together with the fact that a lot of information concerning all knots and links can be “decoded” from the “boundary information”, allows us to speak about the HP for knots and links. Using this HP, we will find the Jones skein relation and its 2-variable generalization.

## 2. Holographic Principle for Knots and Links

Polynomial invariants belong to the most important characteristics of knots and links. They are defined with the help of the skein relation, which looks as

$$P_{L_+}(t) = l_1(t)P_{L_O}(t) + l_2(t)P_{L_-}(t), \quad (1)$$

where  $l_1(t)$ ,  $l_2(t)$  are the coefficients. The skein relation (1) connects three polynomials  $P_{L_+}(t)$ ,  $P_{L_O}(t)$ , and  $P_{L_-}(t)$ , corresponding to underscript knots (links). Here,  $L_+$  denotes an arbitrary knot or link (all knots and links are also called as “Links”). From the initial Link  $L_+$  by the so-called surgery operation (namely, the elimination of a crossing), we obtain the Link  $L_O$ . From the same initial Link  $L_+$  by another surgery operation switching off a crossing, we obtain another Link  $L_-$ . For (1) to be the skein relation, it must satisfy the Reidemeister moves. The skein relation (1) together with the normalization condition for the unknot,

$$P_{\text{unknot}} = 1, \quad (2)$$

put a definite polynomial in correspondence to every knot and link.

Therefore, to operate with the skein relation (1), it is necessary to know the coefficients  $l_1$ ,  $l_2$ . The Alexander skein relation follows from (1), if

$$l_1 = t^{1/2} - t^{-1/2}, \quad l_2 = 1.$$

For the Jones skein relation,

$$l_1 = t(t^{1/2} - t^{-1/2}), \quad l_2 = t^2.$$

The 2-variable HOMFLY skein relation corresponds to

$$l_1 = tz, \quad l_2 = t^2.$$

Let us consider both torus knots and links denoted as  $L_{n,2}$ ,  $n = 0, 1, 2, \dots$ , including torus knots  $T(2m + 1, 2)$  and torus links  $L(2m, 2)$ ,  $m = 0, 1, 2, \dots$ . The surgery operation of elimination turns torus knot/link  $L_{n,2}$  into torus link/knot  $L_{n-1,2}$ , and the surgery operation of switching turns torus knot/link  $L_{n,2}$  into torus knot/link  $L_{n-2,2}$ .

Therefore, the following recurrence relation for torus knots and links  $L_{n,2}$  can be obtained from (1):

$$P_{L_{n+1,2}}(t) = l_1(t)P_{L_{n,2}}(t) + l_2(t)P_{L_{n-1,2}}(t).$$

Rewriting it in a simpler form

$$P_{n+1,2}(t) = l_1 P_{n,2}(t) + l_2 P_{n-1,2}(t) \quad (3)$$

and comparing with (1), we see that there exists the correspondence

$$\begin{aligned} P_{L_+}(t) &\leftrightarrow P_{n+1,2}(t), & P_{L_O}(t) &\leftrightarrow P_{n,2}(t), \\ P_{L_-}(t) &\leftrightarrow P_{n-1,2}(t), \end{aligned} \quad (4)$$

which allows one, having (3), to write the skein relation (1). For this, it is enough to take  $l_1$  and  $l_2$  from (3) and to put them into (1). For the verification, we use the Reidemeister moves. This is an example of the HP for knots and links. For the “space” of all knots and links, the torus knots and links  $L_{n,2}$  are considered as the “boundary” of this space.

From the recurrence relation (3) for the class of torus knots and links  $L_{n,2}$ , one finds the recurrence relation for its subclass. For example, only for torus knots  $T(2m + 1, 2)$  (or only for torus links  $L(2m, 2)$ ) (which allows one to find the skein relation (3) as well) [10], we have

$$P_{n+2,2}(t) = k_1 P_{n,2}(t) + k_2 P_{n-2,2}(t), \quad (5)$$

where

$$k_1 = l_1^2 + 2l_2, \quad k_2 = -l_2^2. \quad (6)$$

With the help of the normalization condition (2), we also find

$$P_{1,2} = 1, \quad P_{3,2} = k_1 + k_2. \quad (7)$$

## 3. Jones Skein Relation from the Holographic Principle

The  $(q, p)$ -number (or the so-called structural function for a  $(q, p)$ -deformed bosonic oscillator) is defined

as [12]

$$[n]_{q,p} = \frac{q^n - p^n}{q - p}. \quad (8)$$

The recurrence relation for  $(q, p)$ -numbers (8) looks as

$$[n + 1]_{q,p} = (q + p)[n]_{q,p} - qp[n - 1]_{q,p}. \quad (9)$$

Let us make the following substitution in (8):

$$q \rightarrow q^3, \quad p \rightarrow q. \quad (10)$$

It reduces the  $(q, p)$ -numbers to the  $(q^3, q)$ -numbers

$$[n]_{q^3,q} = \frac{q^{3n} - q^n}{q^3 - q}, \quad (11)$$

which satisfy the recurrence relation

$$[n + 1]_{q^3,q} = (q + q^3)[n]_{q^3,q} - q^4[n - 1]_{q^3,q}. \quad (12)$$

With the help of the three-step algorithm [10, 11] and (12), we can find the Jones skein relation.

First, we introduce the polynomials  $J_{n,2}(q)$  satisfying the recurrence relation repeating (12) (i.e., the recurrence relation only for torus knots  $T(2m + 1, 2)$  or only for torus links  $L(2m, 2)$ ):

$$J_{n+2,2}(q) = (q + q^3)J_{n,2}(q) - q^4J_{n-2,2}(q). \quad (13)$$

From (7), we obtain

$$J_{1,2}(q) = 1, \quad J_{3,2}(q) = q + q^3 - q^4. \quad (14)$$

Second, we formulate the recurrence relation for all polynomials  $J_{n,2}(q)$ , which leads to the corresponding skein relation. From (5) and (13), we have  $k_1 = q + q^3$ ,  $k_2 = -q^4$ . Using (6), one finds

$$l_1 = q(q^{1/2} - q^{-1/2}), \quad l_2 = q^2. \quad (15)$$

Therefore, we have the recurrence relation for torus knots  $T(2m + 1, 2)$  and for torus links  $L(2m, 2)$ , taken together:

$$J_{n+1,2}(q) = q(q^{1/2} - q^{-1/2})J_{n,2}(q) + q^2J_{n-1,2}(q). \quad (16)$$

From (16) in correspondence with the HP for knots and links in the form (4), we obtain the Jones skein relation [13]

$$J_+(q) = q(q^{1/2} - q^{-1/2})J_O(q) + q^2J_-(q). \quad (17)$$

Third, we find the expression for the Jones polynomials of torus knots  $T(2m + 1, 2)$  in terms of the  $(q^3, q)$ -numbers:

$$J_{2m+1,2}(q) = b_1(q)[m + 1]_{q^3,q} - b_2(q)[m]_{q^3,q}. \quad (18)$$

Substituting (14) into (18), we find  $b_1 = 1$  and  $b_2 = q^4$ . Thus,

$$J_{2m+1,2}(q) = [m + 1]_{q^3,q} - q^4[m]_{q^3,q}. \quad (19)$$

It is easy to verify that formula (19) can be derived from the known general formula for the Jones polynomials of torus knots,

$$J_{n,k}(q) = q^{\frac{(n-1)(k-1)}{2}} \frac{1 - q^{n+1} - q^{k+1} + q^{n+k}}{1 - q^2}, \quad (20)$$

if  $k = 2$ .

#### 4. Generalized Jones Polynomials

Let us make the substitution

$$q \rightarrow q^3 \quad (21)$$

in (8). From  $(q^3, p)$ -numbers

$$[n]_{q^3,p} = \frac{q^{3n} - p^n}{q^3 - p}, \quad (22)$$

using the three-step procedure as in the previous section, we obtain the 2-parameter generalized Jones skein relation:

$$J_+(q, p) = (q^{3/2} - p^{1/2})J_O(q, p) + q^{3/2}p^{1/2}J_-(q, p). \quad (23)$$

Formula (23) can be rewritten as

$$\begin{aligned} q^{-3/4}p^{-1/4}J_+(q, p) - q^{3/4}p^{1/4}J_-(q, p) = \\ = (q^{3/4}p^{-1/4} - q^{-3/4}p^{1/4})J_O(q, p). \end{aligned} \quad (24)$$

By putting  $p = q$ , this generalized Jones skein relation turns into the usual Jones skein relation. At last, we have the analytical formula for the simplest torus knots:

$$J_{2m+1,2}(q, p) = [m + 1]_{q^3,p} - q^3p[m]_{q^3,p}. \quad (25)$$

It turns into formula (19) for the Jones polynomial invariants if  $p = q$ .

Comparing (24) with the skein relation for the HOMFLY polynomial invariants [14]

$$a^{-1}H_+(a, z) - aH_-(a, z) = zH_O(a, z), \quad (26)$$

we find that the substitution

$$a = q^{3/4}p^{1/4}, \quad z = q^{3/4}p^{-1/4} - q^{-3/4}p^{1/4} \quad (27)$$

turns the 2-parameter generalized Jones skein relation into the HOMFLY one.

## 5. Concluding Remarks

In this paper, we state that there exists the holographic principle for knots and links. We have presented a simple example that demonstrates obtaining the skein relation of knots and links (1) from the recurrence relation (3) (or (5)) for the simplest torus knots  $T(2m+1, 2)$  and torus links  $L(2m, 2)$  due to the direct correspondence between their coefficients  $l_1$  and  $l_2$ . The formulas expressing this correspondence depend on the used surgery operations in the definition of skein relation.

We note that the derivation of the formulas for the polynomial invariants of arbitrary knots and links on the basis of the known formulas for torus knots  $T(2m+1, 2)$  and torus links  $L(2m, 2)$  would support the validity of the HP for knots and links. As the first step in this direction, it is necessary to solve the simpler problem concerning arbitrary torus knots and links. In fact, we need the key for decoding the “boundary information” and for turning it into the “volume information”.

As one more example of the HP for knots and links, we mention the expansion of an arbitrary polynomial invariant into the sum of polynomial invariants of torus knots  $T(2m+1, 2)$  and torus links  $L(2m, 2)$ . An attempt to do this for torus knots  $T(n, 3)$  and  $T(n, 4)$  was made in [15, 16].

1. *The Interface of Knots and Physics*, edited by L.H. Kauffman (Amer. Math. Soc., Providence, RI, 1995).

2. L.H. Kauffman, *Knots and Physics* (World Scientific, Singapore, 2001).
3. M.F. Atiyah, *The Geometry and Physics of Knots* (Cambridge Univ. Press, Cambridge, 1990).
4. E. Witten, *Comm. Math. Phys.* **121**, 351 (1989).
5. *Knots and Quantum Gravity*, edited by J. Baez (Oxford Univ. Press, Oxford, 1994).
6. J. Baez and J.P. Muniain, *Gauge Fields, Knots and Gravity* (World Scientific, Singapore, 1994).
7. G.'t Hooft, in *Proceedings of the Salamfest* (ICTP, Trieste, 1993), p. 283; gr-qc/9310026.
8. J. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231-252 (1998); hep-th/9802150.
9. E. Verlinde, *JHEP* **1104**, 029 (2011); arXiv:1001.0785 [hep-th].
10. A.M. Gavrilik and A.M. Pavlyuk, *Ukr. J. Phys.* **55**, 129 (2010); arXiv:0912.4674v2 [math-ph].
11. A.M. Pavlyuk, *Algebras, Groups and Geometries* **29**, 151 (2012).
12. A. Chakrabarti and R. Jagannathan, *J. Phys. A: Math. Gen.* **24**, L711 (1991).
13. V.F.R. Jones, *Bull. AMS* **12**, 103 (1985).
14. P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K. Millet, and A. Ocneanu, *Bull. AMS* **12**, 239 (1985).
15. A.M. Gavrilik and A.M. Pavlyuk, *Ukr. J. Phys.* **56**, 680 (2011); arXiv:1107.5516v1 [math-ph].
16. A.M. Pavlyuk, *Ukr. J. Phys.* **57**, 439 (2012).

Received 18.04.2013

*А.М. Павлюк*

УЗАГАЛЬНЕННЯ ПОЛІНОМІАЛЬНИХ  
ІНВАНІАНТІВ І ГОЛОГРАФІЧНИЙ ПРИНЦИП  
ДЛЯ ВУЗЛІВ І ЗАЧЕПЛЕНЬ

Резюме

Сформульовано голографічний принцип для вузлів і зачеплень. Для “простору” всіх вузлів і зачеплень торичні вузли  $T(2m+1, 2)$  і торичні зачеплення  $L(2m, 2)$  відіграють роль “межі” цього простору. Використовуючи голографічний принцип для вузлів і зачеплень, знаходимо skein-співвідношення для вузлів і зачеплень за допомогою рекурентних співвідношень для поліноміальних інваріантів торичних вузлів  $T(2m+1, 2)$  і торичних зачеплень  $L(2m, 2)$ . Як приклад застосування цього принципу, отримано skein-співвідношення Джонса і його узагальнення за допомогою певних варіантів  $(q, p)$ -чисел, що пов'язані з  $(q, p)$ -деформованими бозонними осциляторами.