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SPIN WAVES IN ARRAYS OF MAGNETIC NANODOTS WITH MAGNETODIPOLAR COUPLING

A general theory of collective spin-wave excitations in finite and infinite periodic arrays of magnetic nanodots with magnetodipolar coupling has been developed. Non-uniform profiles of static and dynamic magnetizations in a dot are taken into account. The theory allows the spectra of collective excitations, their damping rates, excitation efficiencies by an external microwave field, and so on to be calculated and the stability of a stationary magnetic array configuration to be analyzed. An efficient technique has been proposed to calculate the spin-wave spectra in periodic arrays using the method of projection onto the eigenmodes of a solitary nanodot. The results obtained are compared with experimental data.

Keywords: spin wave, magnetic nanodot, magnonic crystal, Gilbert damping parameter, Brillouin zone, Landau–Lifshitz equation.

1. Introduction

Magnetic dots with submicronic transverse dimensions had been considered for a long time from the viewpoint of their applications as elements of magnetic random-access memory (MRAM) [1, 2]. During recent years, there appeared a capability to fabricate large arrays of magnetic dots, with the distance between them being sufficient for an appreciable magnetodipolar interaction to take place. Such arrays are now actively studied as promising artificial materials in ultrahigh frequency (UHF) electronics and magnon crystals (MCs) [3–7].

In general, MCs are structures with periodically changing magnetic parameters, such as the applied magnetic field [8, 9], magnetization [4, 5], and so forth. The variation of a MC geometry can give rise to substantial modifications in the structure of corresponding collective excitations, spin waves (SWs). In particular, the SW spectra can be obtained with properties that are not observed in conventional continuous magnetic materials. Recently, a possibility to control the properties of MCs dynamically by switching their ground state (the static magnetic configuration) has been predicted theoretically [10] and demonstrated experimentally [11, 12]. This possibility is based on the multistability of arrays of discrete magnetic elements; in other words, the same array can be in one of the different ground states under identical external conditions.

Theoretical researches of collective excitations in arrays are complicated because of a necessity to consider the long-range magnetodipolar interaction between nanodots. For today, the theory of collective excitations in arrays with arbitrary geometry has been developed in the macrospin approximation [6]. The latter is valid for magnetic dots in a quasiuniform saturated state and only for the fundamental mode, which is uniform over the nanodot volume. Although this case is one of the most important ones, because uniform modes are the most intense (they can be excited most easily by an external UHF signal and give the largest response), researches of a more general case are also necessary. In particular, non-uniform modes can possess the lowest frequency and, respectively, be responsible for the stability of the ground state [13] (a uniform state is not always the ground state of a magnetic dot [14]).

Modes with profiles that are non-uniform over the nanodot volume were considered in the literature for a few specific geometries and, for the most part, numerically: with the use of a micromagnetic simulation [12, 15], applying the method of dynamic matrix diagonalization [16, 17], by solving a system of integral equations [18, 19], and so on. The analytical consideration was carried out in the weak coupling approximation for arrays of uniformly magnetized ferromagnetic spheres [20, 21] and disks in the vortex state [22]. Note that the ana-

lytical researches of SW spectra are important not only from the viewpoint of designing MCs with required properties. This way may turn out useful in obtaining information concerning the stability region of the array ground state, the type of instability, and the most probable new ground state beyond the limits of the stability region. Such information is required for the creation of dynamic MCs [10].

In this work, the theory of collective excitations in two-dimensional arrays of magnetic dots, which was developed earlier in work [6], is generalized to the case of a non-uniform profile of static or/and dynamic magnetization inside a nanodot. One can see that going beyond the scope of the macrospin approximation does not change the properties inherent to collective SW modes (Section 2), but considerably complicates the calculation of SW spectra. In particular, in the general case, the problem can be reduced to a system of integro-differential equations, which can be solved exactly only numerically. In Section 3, one of the possible ways to solve this system approximately with the use of the projection method is discussed. The spectra calculated within this method are compared with experimental data taken from the literature.

2. General Properties of Collective Excitations in an Array of Magnetic Dots

2.1. Basic equations

We consider a two-dimensional array of magnetic dots arranged on a plane (for definiteness, let it be the $x-y$ one). The position of a nanodot (more exactly, of its center) is given by the radius-vector $\mathbf{R}_j$, where the subscript j enumerates the dots in the array. Let the array be embedded into a uniform external field $\mathbf{B}_e$. A further simplification consists in that we neglect the crystallographic anisotropy. This approximation is valid for the magnetic elements consisting of soft magnetic materials, such as permalloy. In any case, the consideration of this anisotropy does not insert any crucial changes to the theory expounded below.

The state of a magnetic nanodot is completely described by the magnetization distribution $\mathbf{M}_j(\mathbf{r}_j)$ determined within the j-th nanodot boundaries, with $|\mathbf{M}_j(\mathbf{r}_j)| = M_s$, where M_s is the saturation magnetization. The conservative dynamics of magnetization is described by the system of equations [23],

$$\frac{\partial \mathbf{M}_j(\mathbf{r}_j,t)}{\partial t} = \gamma\,(\mathbf{B}_{\mathrm{eff},j} \times \mathbf{M}_j(\mathbf{r}_j,t)), \tag{1}$$

where $\gamma \approx 2\pi \times 28$ GHz/T is the absolute value of gyromagnetic ratio for the electron spin. The effective field that acts on the j-th nanodot equals

$$\mathbf{B}_{\mathrm{eff},j} = \mathbf{B}_e - \mu_0 \sum_l \hat{\mathbf{G}}_{jl} * \mathbf{M}_l, \tag{2}$$

where $\mu_0 = 4\pi \times 10^{-7}$ H/m is the magnetic permeability of vacuum. The tensor operator $\hat{\mathbf{G}}_{jl} = \hat{\mathbf{G}}^{(\mathrm{ex})}\delta_{jl} + \hat{\mathbf{G}}^{(\mathrm{d})}_{jl}$ consists of the exchange energy operator [23]

$$\hat{\mathbf{G}}^{(\mathrm{ex})} = -\alpha_{\mathrm{ex}}^2 \nabla^2 \hat{\mathbf{I}}, \tag{3}$$

where α_{ex} is the exchange length, $\mathbf{I}$ is the identity matrix, and $\hat{\mathbf{G}}^{(\mathrm{d})}_{jl}$ is the integral operator of the magnetodipolar interaction, whose components in the Cartesian coordinate system are

$$\left[\hat{\mathbf{G}}^{(\mathrm{d})}_{jl}\right]_{\alpha\beta} = -\frac{1}{4\pi}\int d^3\mathbf{r}'_l \frac{\partial^2}{\partial x_\alpha \partial x'_\beta}\frac{1}{|\mathbf{r}_j - \mathbf{r}'_l|}, \tag{4}$$

where $\alpha, \beta = x, y, z$ [24]. While deriving Eq. (4), the magnetostatic approximation was applied. This approximation is valid if $k \gg \omega/c$, where k and ω are the wave number and the frequency of SWs, respectively, and c is the light speed. This condition is satisfied in the case of UHF oscillations and waves in ferromagnets. Note also that the total magnetic energy of the array is expressed as follows:

$$W = -\sum_j \int \mathbf{B}_e \mathbf{M}_j d^3\mathbf{r}_j + \frac{\mu_0}{2}\sum_{j,l}\int \mathbf{M}_j \hat{\mathbf{G}}_{jl} * \mathbf{M}_l d^3\mathbf{r}_j. \tag{5}$$

2.2. Collective excitation in finite nanodot arrays

Let the magnetization of the j-th magnetic dot in a stationary state be denoted as $\mathbf{M}_j = M_s\boldsymbol{\mu}_j$, where $\boldsymbol{\mu}_j(\mathbf{r}_j)$ is the unit vector. The stationary state of the array is determined from the condition that all the time derivatives in system (1) equal zero, which is reduced to a condition that the stationary magnetization and the effective field are parallel to each other in every magnetic dot,

$$B_j\boldsymbol{\mu}_j = \mathbf{B}_e - \mu_0 M_s \sum_l \hat{\mathbf{G}}_{jl} * \boldsymbol{\mu}_l, \tag{6}$$

where B_j are the Lagrange multipliers (they emerge in a natural way if the stationary states are determined with the use of the equivalent method, namely, by finding the extreme points of the total energy functional (5) in the space of functions $\boldsymbol{\mu}_j$ provided that $|\boldsymbol{\mu}_j|^2 = 1$). The quantity B_j stands for the absolute value of effective field in the j-th nanodot in the stationary state $\boldsymbol{\mu}_j$. In the general case, B_j is a function of coordinates, $B_j = B_j(\mathbf{r}_j)$. In principle, by solving the system of equations (6) provided that $|\boldsymbol{\mu}_j| = 1$, all probable stationary, but not necessarily stable, states of the array and the corresponding distributions of the effective field B_j can be determined. However, in many cases, the problem of searching the set of stationary states can be simplified. In particular, in the case of a quasiuniform state of magnetic dots, it is pertinent to take advantage of the macrospin approximation [6]; for a vortex state, the model of rigid vortex [14] can be applied; and so forth. Then, the system of equations (6) is reduced to a simple system of algebraic equations. Only in intermediate cases, the total system of integro-differential equations should be solved numerically.

To find the dynamic equations for small (linear) excitations of a stationary state, let us express the magnetization as a sum of the ground state $\boldsymbol{\mu}_j$ and small deviations $\mathbf{m}_j$ from it, $|\mathbf{m}_j| \ll 1$:

$$\mathbf{M}_j(\mathbf{r}_j, t) = M_s \left[\boldsymbol{\mu}_j(\mathbf{r}_j) + \mathbf{m}_j(\mathbf{r}_j, t)\right] + O(|\mathbf{m}_j|^2).$$

Since the length of the magnetization vector $\mathbf{M}_j$ has to be constant everywhere, the vectors $\boldsymbol{\mu}_j$ and $\mathbf{m}_j$ must be orthogonal, i.e. $\boldsymbol{\mu}_j(\mathbf{r}_j) \times \mathbf{m}_j(\mathbf{r}_j) = 0 \,\forall\, \mathbf{r}_j$.

Substituting the expansion for the magnetization $\mathbf{M}_j$ into Eq. (1) and using Eq. (6) and the orthogonality relation, we obtain a dynamic equation for the dimensionless magnetization $\mathbf{m}_j$, which yields the following equation for the frequencies ω_ν and the profiles $\mathbf{m}_{j,\nu}$ of array eigenmodes:

$$-i\omega_\nu \mathbf{m}_{\nu,j} = \boldsymbol{\mu}_j \times \sum_l \hat{\boldsymbol{\Omega}}_{jl} * \mathbf{m}_{\nu,l}. \tag{7}$$

Here, only the terms linear in $\mathbf{m}_j$ are retained, and the subscript ν enumerates different array modes. The tensor operator $\hat{\boldsymbol{\Omega}}_{jl}$ is defined as follows:

$$\hat{\boldsymbol{\Omega}}_{jl} = \gamma B_j \delta_{jl} \hat{\mathbf{I}} + \omega_M \hat{\mathbf{G}}_{jl}, \tag{8}$$

where $\omega_M = \gamma \mu_0 M_s$.

The operator $\hat{\boldsymbol{\Omega}}_{jl}$ is real-valued and self-adjoint. In other words, for any vector functions $\mathbf{f}(\mathbf{r}_j)$ and $\mathbf{g}(\mathbf{r}_l)$ determined within the j-th and l-th nanodots, respectively, the following equality is satisfied:

$$\int \mathbf{f}^*(\mathbf{r}_j) \hat{\boldsymbol{\Omega}}_{jl} * \mathbf{g}(\mathbf{r}_l) d^3\mathbf{r}_j = \int \mathbf{g}(\mathbf{r}_l) \hat{\boldsymbol{\Omega}}_{lj} * \mathbf{f}^*(\mathbf{r}_j) d^3\mathbf{r}_l.$$

Using this equality, we obtain the orthogonality relation for collective SWs modes,

$$\sum_j \langle \mathbf{m}^*_{\nu',j} \boldsymbol{\mu}_j \times \mathbf{m}_{\nu,j} \rangle_j = -iA_\nu \delta_{\nu,\nu'}, \tag{9}$$

where the symbol $\langle \ldots \rangle_j$ means the averaging over the volume of the j-th magnetic dot,

$$\langle f \rangle_j = \frac{1}{V_j} \int\limits_{V_j} f d^3\mathbf{r}_j.$$

Relation (9) is obeyed for the SW modes of the array in any stable state. It is violated only in unstable states with a saddle-point type of instability (when the eigenfrequencies become imaginary numbers), which are evidently not realized in practice.

In view of the real character of the operator $\hat{\boldsymbol{\Omega}}_{jl}$, it follows from Eq. (7) that, if $\mathbf{m}_{\nu,j}$ is an eigenfunction with the frequency ω_ν, the quantity $\mathbf{m}^*_{\nu,j}$ is also an eigenfunction with the frequency $-\omega_\nu$. Such a "duplication" of solutions is related to the fact that the dynamic magnetization $\mathbf{m}_j$ is a real quantity. Therefore, its spectrum contains both positive and negative frequencies. Hence, only a half of the formal solutions of Eq. (7) describe "physical" modes $\mathbf{m}_{\nu,j}$, whereas the others correspond to formal conjugate modes $\mathbf{m}^*_{\nu,j}$. We emphasize that the classification of modes as true and conjugate ones is ambiguous. In what follows, the modes with positive frequencies ω_ν will be referred to as true. Then, the dynamic component of the magnetization is expressed as a sum of eigenmodes:

$$\mathbf{m}_j(t) = \sum_\nu \left(\mathbf{m}_{\nu,j} c_\nu(t) + \text{c.c.}\right),$$

where the summation is carried out only over the "physical" modes, $c_\nu(t)$ is the complex amplitude of the ν-th mode, and *c.c.* means the complex conjugation. Substituting this expansion into Eq. (5) with regard for the orthogonality relation, we obtain that

the changes in the magnetic energy of the system owing to the excitation of its eigenmodes are expressed in terms of mode amplitudes as follows:

$$\Delta W = \frac{M_s V}{\gamma} \sum_{\nu} \omega_\nu A_\nu |c_\nu|^2. \tag{10}$$

If the stationary magnetic configuration $\boldsymbol{\mu}_j$ corresponds to a local minimum of the total magnetic energy W, the value of ΔW must be positive. This requirement is ensured if the "physical" modes have positive norms, $A_\nu > 0$. However, if we formally consider a stationary state that corresponds to the energy maximum, then $\Delta W < 0$, which is possible if at least some of the true modes have negative norms. Hence, Eq. (10) gives a convenient way to research the stability of stationary states of nanodot arrays, because only those states are stable, for which all the frequencies of eigenmodes, ω_ν, are real-valued (otherwise, the saddle-point instability is realized). In addition, the inequality $\omega_\nu A_\nu > 0$ must be satisfied for all modes (otherwise, the state corresponds to a special point of the "unstable focus" type). Here, we took into account that the norm of the conjugate mode is opposite by sign to the norm of the physical mode (see Eq. (9)).

The described formalism can be used in the numerical calculations of collective excitation spectra for finite arrays of magnetic dots that stay in any stationary state. However, in the general case, the task of determining the SW spectra is reduced to a complicated problem of finding the eigenvalues and eigenfunctions for a system of integro-differential equations. Below, one of the approximate methods to solve this problem is considered.

2.3. SWs in periodic arrays

Periodic ground states are an important case of the stationary configurations of arrays of magnetic dots arranged into a periodic lattice. In particular, those states include the ferromagnetic state, in which the magnetization distributions are identical in all nanodots, as well as various antiferromagnetic states, which in the absence of external magnetic field are usually the true ground states of arrays in the absence of an external magnetic field (i.e., they correspond to the global minimum of the total energy) [25, 26]. If the period of a state is substantially shorter than the array dimensions, it is reasonable to consider such arrays as infinite. For this purpose, the formalism developed in the previous subsection has to be a little modified.

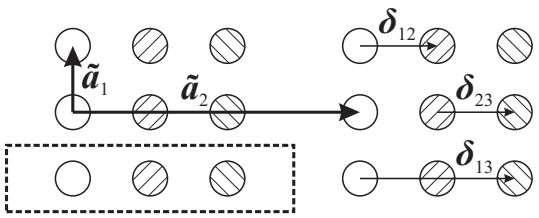

Fig. 1. Schematic diagram of a complicated periodic lattice containing $P = 3$ dots in an elementary cell. Circles with different hatchings correspond to magnetic dots belonging to different superlattices. An elementary cell in the superlattice is marked by a dashed line

In the general case of a periodic ground state, the elementary cell in the magnetic superlattice includes $P \geq 1$ magnetic dots. Note that $P = 1$ only for an array of dots that are arranged into a simple (oblique) lattice and stay in the ferromagnetic state. In all other cases, the magnetic superlattice is complicated ($P > 1$), and a complicated periodicity can be induced by both a complicated structure of the nanodot lattice and the complicated (non-ferromagnetic) ground state. The period in the array is given by the basis vectors $\tilde{\mathbf{a}}_1$ and $\tilde{\mathbf{a}}_2$ that form the superlattice $\mathcal{SL}$,

$$\mathcal{SL} = \{n_1\tilde{\mathbf{a}}_1 + n_2\tilde{\mathbf{a}}_2 \,|\, n_1 \in \mathbb{Z},\, n_2 \in \mathbb{Z}\}. \tag{11}$$

Every magnetic dot at a position j_p belongs to a certain, $p \in [1, P]$, superlattice (in the sense of Eq. (11)). Different superlattices are shifted with respect to one another by vectors $\boldsymbol{\delta}_{pq}$; the sum of all P superlattices gives the positions of all magnetic dots in the array (see the illustration in Fig. 1). It is clear that the stationary distributions of the magnetization $\boldsymbol{\mu}_j(\mathbf{r}_j)$ and the internal magnetic field B_j depend only on the superlattice index p, i.e., $\boldsymbol{\mu}_{j_p} = \boldsymbol{\mu}_p$ and $B_{j_p} = B_p$. Hence, the general condition of equilibrium (6) is reduced to P equations that look like

$$B_p\boldsymbol{\mu}_p = \mathbf{B}_e - \mu_0 M_s \sum_q \hat{\mathbf{G}}_{\mathbf{0}}(\boldsymbol{\delta}_{pq}) * \boldsymbol{\mu}_q, \tag{12}$$

where

$$\hat{\mathbf{G}}_{\mathbf{k}}(\boldsymbol{\delta}_{pq}) = \sum_{\mathbf{R}_{j_p} \in \mathcal{SL}} \hat{\mathbf{G}}_{j_p j_q} e^{-i\mathbf{k}\cdot(\mathbf{R}_{j_p} - \mathbf{R}_{j_q})}. \tag{13}$$

The operator $\hat{\mathbf{G}}_{\mathbf{k}}(\boldsymbol{\delta})$ is self-adjoint; in addition, it is evident that $\hat{\mathbf{G}}_{\mathbf{k}}(\boldsymbol{\delta}) = \hat{\mathbf{G}}^*_{-\mathbf{k}}(\boldsymbol{\delta})$. Moreover, for any

vector in the reciprocal superlattice $\mathbf{k}_S \in \mathcal{SL}^*$, we have $\hat{\mathbf{G}}_{\mathbf{k}+\mathbf{k}_S}(\boldsymbol{\delta}) = \hat{\mathbf{G}}_{\mathbf{k}}(\boldsymbol{\delta})e^{-i\mathbf{k}_S\cdot\boldsymbol{\delta}}$.

Linear SW excitations are sought in the form of plane waves,

$$\mathbf{m}_{j_p} = \mathbf{m}_{\mathbf{k},p}e^{i\mathbf{k}\mathbf{R}_{j_p}},$$

where the wave vector $\mathbf{k}$ belongs to the first Brillouin zone of the reciprocal superlattice $\mathcal{SL}^*$. Using this ansatz in Eq. (7), we obtain a finite-dimensional eigenvalue problem for the frequencies $\omega_{\mathbf{k}}$ and the profiles $\mathbf{m}_{\mathbf{k},p}$ of SW modes,

$$-i\omega_{\mathbf{k}}\mathbf{m}_{\mathbf{k},p} = \boldsymbol{\mu}_p \times \sum_q \hat{\boldsymbol{\Omega}}_{\mathbf{k},pq} * \mathbf{m}_{\mathbf{k},q}, \tag{14}$$

in which the tensor operator $\hat{\boldsymbol{\Omega}}_{\mathbf{k}}$ is defined as follows:

$$\hat{\boldsymbol{\Omega}}_{\mathbf{k},pq} = \gamma B_p\delta_{pq}\hat{\mathbf{I}} + \omega_M\hat{\mathbf{G}}_{\mathbf{k}}(\boldsymbol{\delta}_{pq}). \tag{15}$$

For a given wave vector $\mathbf{k}$, the eigenvalue problem (14) comprises a $2P$-dimensional system of linear integro-differential equations. By solving it, we obtain the complex amplitudes of SW modes $\mathbf{m}_{\mathbf{k},p}$ and the corresponding frequencies $\omega_{\mathbf{k}}$, i.e. $2P$ solutions altogether. The true SW mode looks like

$$\mathbf{m}_{j_p} \sim \cos[\mathbf{k}\mathbf{R}_{j_p} - \omega_{\mathbf{k}}t + \phi]$$

and can be written down in terms of complex amplitudes as follows:

$$\mathbf{m}_{j_p} = \mathbf{m}_{\mathbf{k},p}e^{i(\mathbf{k}\mathbf{R}_{j_p} - \omega_{\mathbf{k}}t)} + \text{c.c.}$$

The function $\mathbf{m}^*_{\mathbf{k},p}$ is a solution of the problem that is conjugate to problem (14). However, as follows from the properties of the operator $\hat{\mathbf{G}}_{\mathbf{k}}(\boldsymbol{\delta})$, the function $\mathbf{m}^*_{\mathbf{k},p}$ is also an eigenfunction of the problem, but with the opposite direction of the wave vector $\mathbf{k}$, because $\hat{\boldsymbol{\Omega}}_{\mathbf{k}} = \hat{\boldsymbol{\Omega}}^*_{-\mathbf{k}}$. As a result, the eigenvalue problem (14) simultaneously describes waves propagating in parallel with (these are solutions with $\omega_{\mathbf{k}} > 0$) and in the opposite direction to ($\omega_{\mathbf{k}} < 0$) the wave vector $\mathbf{k}$. Therefore, the classification of modes into true and formal (conjugate) ones, which was valid in the case of finite arrays, now has no sense. Of course, if we consider waves propagating only in a given direction, only a half of the solutions should be taken into consideration; i.e., in the general case, the SW spectrum for the array has P branches.

Spin waves in periodic arrays are characterized by properties that are similar to those for collective modes in finite arrays. In particular, the SWs belonging to different spectral branches are orthogonal in the sense of the equation

$$\sum_p \langle \mathbf{m}^*_{\mathbf{k},\lambda,p}\,\boldsymbol{\mu}_p \times \mathbf{m}_{\mathbf{k},\lambda',p}\rangle = -iA_{\mathbf{k},\lambda}\delta_{\lambda,\lambda'},$$

where the subscript λ enumerates the spectral branches. The condition of stability of a stationary state also remains valid: the state is stable only if all the frequencies $\omega_{\mathbf{k},\lambda}$ are real-valued, and the inequality $\omega_{\mathbf{k},\lambda}A_\lambda > 0$ holds true for all wave vectors in the first Brillouin zone.

2.4. Effect of small perturbations

Above, we have considered an ideal conservative system. It is convenient to examine the dissipation effect in the framework of perturbation theory. Certainly, the perturbation theory can be applied not only to SW damping, but also to many other small effects: e.g., the excitation of SW modes by an external UHF field, the influence of thermal noise, and others. Here, we will develop an analog of the well-known quantum-mechanical non-stationary perturbation theory [27] for arrays of magnetic dots. For the sake of definiteness, let us consider finite arrays. In the case of periodic arrays, there are no basic modifications to the scheme: only the sums over the modes should be substituted by the sums over the spectral branches and the wave-vectors.

In the general case, the perturbed Landau–Lifshitz equation for the nanodot magnetization $\mathbf{M}_j(t)$ can be written down in the form

$$\frac{\partial\mathbf{M}_j}{\partial t} = \gamma\,(\mathbf{B}_{\text{eff},j} \times \mathbf{M}_j) + \gamma\,(\mathbf{b}_j \times \mathbf{M}_j),$$

where $\mathbf{b}_j$ is the effective perturbation field, which can depend on the time or/and the magnetization $\mathbf{M}_l$. Only linear processes are taken into consideration. In this case, the magnetization of every nanodot can be expanded in a series in collective eigenmodes of the array (certainly, only the true modes with positive frequencies must be taken into account). Then we have

$$\mathbf{M}_j(t) = M_s\left[\boldsymbol{\mu}_j + \sum_\nu(\mathbf{m}_{\nu,j}c_\nu(t) + \text{c.c})\right].$$

Substituting this series into the perturbed dynamic equation with regard for the orthogonality relation (9) for collective modes $\mathbf{m}_{\nu,j}$, we obtain the following dynamic equation for the amplitudes of SW modes, $c_\nu(t)$:

$$\frac{dc_\nu}{dt} = -i\omega_\nu c_\nu + i\gamma b_\nu - i\gamma \sum_{\nu'} \left(S_{\nu,\nu'} c_{\nu'} + \tilde{S}_{\nu,\nu'} c^*_{\nu'} \right), \tag{16}$$

with the coefficients

$$b_\nu = \frac{1}{A_\nu} \sum_j \langle \mathbf{m}^*_{\nu,j} \mathbf{b}_j \rangle_j, \tag{17a}$$

$$S_{\nu,\nu'} = \frac{1}{A_\nu} \sum_j \langle (\mathbf{m}^*_{\nu,j} \mathbf{m}_{\nu',j})(\boldsymbol{\mu}_j \mathbf{b}_j) \rangle_j, \tag{17b}$$

$$\tilde{S}_{\nu,\nu'} = \frac{1}{A_\nu} \sum_j \langle (\mathbf{m}^*_{\nu,j} \mathbf{m}^*_{\nu',j})(\boldsymbol{\mu}_j \mathbf{b}_j) \rangle_j. \tag{17c}$$

In the case where the perturbation $\mathbf{b}_j$ is induced by an external UHF field, the term $i\gamma b_\nu$ in Eq. (16) describes a linear excitation of SW modes, and two other terms correspond to the parametric processes occurring at a parallel pumping. Note that if the perturbation field depends on the dynamic component of the magnetization, only the terms of the zeroth and first orders in the amplitudes of SW modes c_ν should be retained, because the expansion of the magnetization in a series in eigenmodes is not complete for nonlinear processes and can give rise to incorrect results.

Now consider two practically important examples of the application of perturbation theory: the account of the dissipation and the excitation of collective modes by an external field. In the former case, taking the Gilbert form for the dissipative term [23], we can write the perturbation field in the form

$$\mathbf{b}_j = -\frac{\alpha_G}{\gamma M_s} \frac{\partial \mathbf{M}_j}{\partial t},$$

where α_G is the Gilbert damping parameter. By calculating coefficients (17) for this perturbation, we obtain an equation for the amplitudes of SW modes,

$$\frac{dc_\nu}{dt} = -i\omega_\nu c_\nu - \sum_{\nu'} \Gamma_{\nu,\nu'} c_{\nu'}, \tag{18}$$

in which the damping constants are

$$\Gamma_{\nu,\nu'} = \alpha_G \omega_{\nu'} \left(\frac{1}{A_\nu} \sum_j \langle \mathbf{m}^*_{\nu,j} \times \mathbf{m}_{\nu',j} \rangle_j \right). \tag{19}$$

One can see that, in the general case, the damping can bring about a coupling between various SW modes. However, if the collective modes are not degenerate in the frequency, this connection can be neglected, so that the standard equation for a dissipative oscillator with damping constant $\Gamma_\nu = \Gamma_{\nu,\nu}$ can be used. In the case of degenerate modes, the description of the damping will be correct, if the non-diagonal terms $\Gamma_{\nu,\nu'}$ ($\nu' \neq \nu$) are considered for all modes with identical frequencies, $\omega_\nu = \omega_{\nu'}$. As is seen from Eq. (19), the damping constant depends on the ellipticity of a SW mode. The damping constant equals $\Gamma_\nu = \alpha_G \omega_\nu$ only for a mode with circular polarization at each magnetic dot. In other cases, $\Gamma_\nu > \alpha_G \omega_\nu$. Strictly speaking, for $\nu = \nu'$, the proportionality coefficient in the parentheses in Eq. (19) changes within the limits $[1, \infty)$ and grows with the precession ellipticity ε [23]. However, the explicit dependence between the damping constant and the SW mode ellipticity cannot be obtained in the general case.

The direct excitation of SWs by an external UHF with frequency ω is described by the perturbation field $\mathbf{b}_j = (\mathbf{b}_{e,j} e^{-i\omega t} + \text{c.c.})$. With regard for the dissipation (in the nondegenerate case), the dynamic equation for the amplitudes c_ν looks like

$$\frac{dc_\nu}{dt} = -i\omega_\nu c_\nu - \Gamma_\nu c_\nu + i\gamma b_{e,\nu} e^{-i\omega t},$$

where the external force amplitude equals

$$b_{e,\nu} = \frac{1}{A_\nu} \sum_j \langle \mathbf{m}^*_{\nu,j} \mathbf{b}_{e,j} \rangle_j.$$

Using the equations given above, it is simple to calculate the spectrum of UHF radiation absorption by an array of interacting nanodots. In particular, in the practically important case where the excitation is induced by a spatially uniform UHF field ($\mathbf{b}_{e,j} = \mathbf{b}_e$), the power absorbed by the array is expressed as follows:

$$P = \frac{\omega V N_d}{\mu_0} \mathbf{b}^*_e \hat{\boldsymbol{\chi}}''(\omega) \mathbf{b}_e, \tag{20}$$

where N_d is the number of nanodots in the array. The tensor of effective permeability of the magnetic

dot array, $\hat{\boldsymbol{\chi}}(\omega) = \hat{\boldsymbol{\chi}}'(\omega) + i\hat{\boldsymbol{\chi}}''(\omega)$, can be expressed in terms of the frequencies and the profiles of eigenmodes as follows:

$$\hat{\boldsymbol{\chi}}(\omega) = \gamma\mu_0 M_s \sum_{\nu} \frac{\hat{\boldsymbol{\chi}}_\nu}{(\omega_\nu - \omega) - i\Gamma_\nu}, \tag{21a}$$

$$\hat{\boldsymbol{\chi}}_\nu = \frac{1}{N_d A_\nu} \sum_{j,l} \langle \mathbf{m}_{\nu,j} \rangle_j \otimes \langle \mathbf{m}^*_{\nu,l} \rangle_l, \tag{21b}$$

where the symbol $\otimes$ denotes the direct (Cartesian) product of vectors.

Hence, all parameters of a nanodot array, which are important in practice, can be determined if the frequencies ω_ν of collective SW modes and their profiles $\mathbf{m}_{\nu,j}$ are known. Note that, in the case of infinite periodic arrays, Eqs. (19) and (21b) remain almost the same: the difference is in that the sum over the nanodots in the array is substituted by the sum over the superlattices, and only the SWs with the zero wave vector are evidently taken into account in the expression for the effective permittivity tensor.

3. Calculation of SW Spectra

3.1. Projection method

Now let us consider how the spectra of collective excitations in the array can be approximately calculated with regard for a non-uniformity of the static or dynamic magnetization over the magnetic dot volume. We assume that the ground state of the array, $\boldsymbol{\mu}_j$, is known (the issue concerning its determination, i.e., the procedure of solving Eq. (6), goes beyond the scope of this work). A further simplification consists in that all magnetic dots are considered identical (this assumption is not crucial for the application of the method described below to be valid).

We use the projection method. Specifically, we project the true solution of Eq. (7) onto a certain complete basis of vector functions $\mathbf{m}_\lambda(\mathbf{r})$,

$$\mathbf{m}_j(\mathbf{r}) = \sum_\lambda a_{\lambda,j} \mathbf{m}_\lambda(\mathbf{r}_j).$$

As a basis, it is convenient to use the eigenmodes of a solitary nanodot or any other orthogonal basis (in the sense of Eq. (9); of course, without the summation over the nanodots). Note that, in the first case, it is necessary to consider both "physical" and formal conjugate modes, because only this basis is complete.

By projecting Eq. (7) onto the basis, we obtain the system of equations

$$\sum_{\lambda'} a_{\lambda',j} A_\lambda \tilde{\omega}_{\lambda\lambda'} + \omega_M \sum_{\lambda',l} a_{\lambda',l} N_{\lambda\lambda'}(\mathbf{R}_{jl}) = \omega a_\lambda A_\lambda \tag{22}$$

with the coefficients

$$\tilde{\omega}_{\lambda\lambda'} = \frac{1}{A_\lambda} \langle \mathbf{m}^*_\lambda(\mathbf{r}_j)(\gamma B_j + \omega_M \hat{\mathbf{G}}^{(\mathrm{ex})}) * \mathbf{m}_{\lambda'}(\mathbf{r}_j) \rangle_j,$$

$$N_{\lambda\lambda'}(\mathbf{R}_{jl}) = \langle \mathbf{m}^*_\lambda(\mathbf{r}_j) \hat{\mathbf{G}}^{(\mathrm{d})}_{jl} * \mathbf{m}_{\lambda'}(\mathbf{r}_l) \rangle_j.$$

where A_λ is the norm of the basis mode $\mathbf{m}_\lambda$, and $\mathbf{R}_{jl} = = \mathbf{R}_j - \mathbf{R}_l$. The most difficult procedure is the calculation of the coefficients $N_{\lambda\lambda'}(\mathbf{R})$. Actually, every of them is expressed as an integral over six (or, in the case of quasi-two-dimensional modes, four) variables (see Eq. (4)). Let us demonstrate how they can be effectively calculated.

In work [28], it was shown that the quantities $N_{\lambda\lambda'}(\mathbf{R})$ can be expressed with the use of the inverse Fourier transformation as follows:

$$N_{\lambda\lambda'}(\mathbf{R}) = \frac{1}{V} \int \mathbf{D}_\lambda(\boldsymbol{\kappa}) \frac{\boldsymbol{\kappa} \otimes \boldsymbol{\kappa}}{\kappa^2} \mathbf{D}^*_{\lambda'}(\boldsymbol{\kappa}) e^{i\boldsymbol{\kappa}\times\mathbf{R}} \frac{d^3\boldsymbol{\kappa}}{(2\pi)^3},$$

where $\mathbf{D}_\lambda(\boldsymbol{\kappa})$ is the Fourier transform of the $\mathbf{m}_\lambda(\mathbf{r})$-mode profile. Let us consider the case of flat magnetic dots with the constant height h (just such nanodots are fabricated today). We also assume the modes to be uniform across the nanodot thickness (only such modes can be excited experimentally). Under those conditions, the previous expression becomes simpler:

$$N_{\lambda\lambda'}(\mathbf{R}) = \int N_{\mathbf{k},\lambda\lambda'} e^{i\mathbf{k}\cdot\mathbf{R}} \frac{d^2\mathbf{k}}{(2\pi)^2}.$$

Here,

$$N_{\mathbf{k},\lambda\lambda'} = \boldsymbol{\sigma}_\lambda(\mathbf{k}) \hat{\mathbf{N}}_\mathbf{k} \boldsymbol{\sigma}^*_{\lambda'}(\mathbf{k}), \tag{23}$$

$$\boldsymbol{\sigma}_\lambda(\mathbf{k}) = \int \mathbf{m}_\lambda(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^2\mathbf{r} \tag{24}$$

is the two-dimensional Fourier transform of the $\mathbf{m}_\lambda(\mathbf{r})$-mode profile, S is the magnetic nanodot area, and the tensor $\hat{\mathbf{N}}_\mathbf{k}$ looks like

$$\hat{\mathbf{N}}_\mathbf{k} = \frac{1}{S} \begin{pmatrix} \frac{k_x^2}{k^2} f(kh) & \frac{k_x k_y}{k^2} f(kh) & 0 \\ \frac{k_x k_y}{k^2} f(kh) & \frac{k_y^2}{k^2} f(kh) & 0 \\ 0 & 0 & 1 - f(kh) \end{pmatrix}, \tag{25}$$

where $f(kh) = 1 - (1 - \exp[-kh])/(kh)$. In fact, the coefficients $N_{\mathbf{k},\lambda\lambda'}$ are a generalization of the known tensor of mutual demagnetization [6, 28] to the case of a non-uniform dynamic magnetization profile.

In many cases, the mean-field approximation can be used. In other words, only the absolute value of the effective field (6) averaged over the nanodot volume should be calculated,

$$B_j = \langle \mathbf{B}_e \boldsymbol{\mu}_j \rangle - \mu_0 M_s \left[\langle \boldsymbol{\mu}_j \hat{\mathbf{G}}^{(\mathrm{ex})} * \boldsymbol{\mu}_j \rangle + \sum_l N_s(\mathbf{R}_{jl}) \right],$$

where

$$N_s(\mathbf{R}_{jl}) = \langle \boldsymbol{\mu}_j \times \hat{\mathbf{G}}^{(\mathrm{d})}_{jl} * \boldsymbol{\mu}_l \rangle_j.$$

This approximation is valid for modes that are not localized in regions, where the internal field strongly varies (e.g., it cannot be used for edge modes). It is clear that the coefficients $N_s(\mathbf{R})$ are calculated analogously to $N_{\lambda\lambda'}(\mathbf{R})$.

The described method of calculation is especially efficient for the consideration of periodic states of arrays. In this case, the system of equations for the coefficients $a_{p,\lambda}$ in the expansion of the dynamic superlattice magnetization $\mathbf{m}_p$ also looks like Eqs. (22), but the summation over the dots, j, is substituted by the summation over the superlattices, q. Instead of the coefficients $N_{\lambda\lambda'}$, there emerge the coefficients

$$G_{\mathbf{k},\lambda\lambda'}(\boldsymbol{\delta}_{pq}) = \sum_{\mathbf{R}\in\mathcal{SL}} N_{\lambda\lambda'}(\mathbf{R}+\boldsymbol{\delta}_{pq}) e^{-i\mathbf{k}(\mathbf{R}+\boldsymbol{\delta}_{pq})} =$$
$$= \frac{1}{S_{\mathcal{SL}}} \sum_{\mathbf{K}\in\mathcal{SL}^*} N_{\mathbf{k}+\mathbf{K},\lambda\lambda'} e^{i\mathbf{K}\boldsymbol{\delta}_{pq}}. \qquad (26)$$

Here, we used a relation well known in the solid-state physics [29] and substituted the summation over the direct superlattice $\mathcal{SL}$ by the sum over the reciprocal lattice $\mathcal{SL}^*$ ($S_{\mathcal{SL}}$ is the area of the elementary cell in the superlattice). For simple nanodot geometries, there exist simple analytical expressions for the coefficients $N_{\mathbf{k}+\mathbf{K},\lambda\lambda'}$ (see an example below). It should also be emphasized that, usually, there is no need to take plenty of basis modes $\mathbf{m}_\lambda$ into consideration. Instead, one can apply the diagonal approximation, i.e., only the corresponding basis functions $\mathbf{m}_\lambda$ and $\mathbf{m}^*_\lambda$ can be considered for every of the spectral branches. The complete system of equations has to be dealt with only at points, where different branches intersect one another [30].

3.2. An example of the application of the method

Let us illustrate the application of the calculation method described above. We intend to calculate the SW spectra for an array of flat square magnetic dots with height h and the transverse dimensions $l \times l$. The dots are arranged into a square lattice with the lattice constant a (we use the experimental setup from work [4], see Fig. 2, *a*). An external magnetic field parallel to the x-axis, $\mathbf{B}_e = B_e \boldsymbol{e}_x$, is applied to the array. The field magnetizes nanodots to the saturated state, so that the array is in the ferromagnetic (FM) state.

Let us use the diagonal approximation; i.e. only one type of eigenmodes for a solitary nanodot is taken into account for every spectral branch,

$$\mathbf{m}_\mathbf{k} = a_\lambda \mathbf{m}_\lambda + a^*_{\lambda'} \mathbf{m}^*_{\lambda'}.$$

Here, $\mathbf{m}^*_{\lambda'} = (\mathbf{m}_\lambda)^*$, and the notation λ' in the subscript was introduced for convenience to distinguish between the physical and conjugate modes. We choose a circularly polarized basis $\mathbf{m}_\lambda = (0, 1, i) f_\lambda$, where the profile function f_λ corresponds to the profiles of solitary nanodot modes. For the Damon–Eshbach (DE) modes (these are modes with the nodes of dynamic magnetization arranged in the direction perpendicular to the static magnetization vector) in the approximation of fixed boundary conditions, the functions f_λ are

$$f_{1\mathrm{DE}} = \cos\frac{\pi x}{l} \sin\frac{2\pi y}{l}, \quad f_{2\mathrm{DE}} = \cos\frac{\pi x}{l} \cos\frac{3\pi y}{l},$$

and so on (the coordinate system origin coincides with the nanodot center). For the backward (BA) magneto-static modes (the nodes are arranged in the direction parallel to $\boldsymbol{\mu}$), the substitutions $x \leftrightarrow y$ should be made in the expressions given above [31,32]. The corresponding generalized shape can be obtained

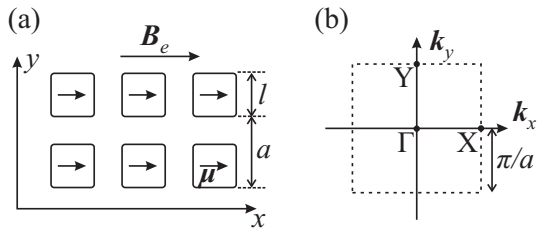

Fig. 2. Sketch of the array under consideration (*a*). Boundaries of the first Brillouin zone for a square lattice (dashed line) (*b*)

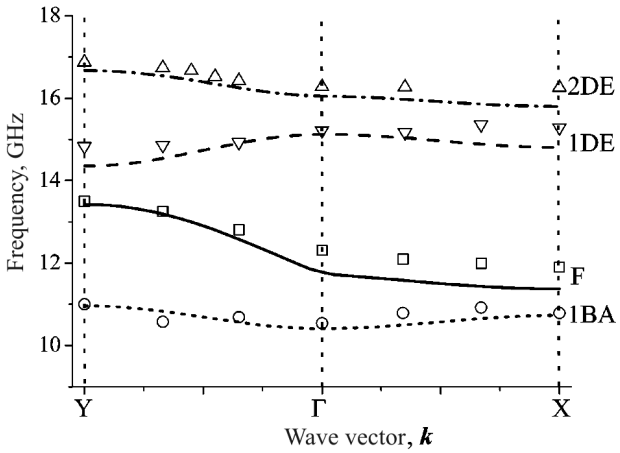

Fig. 3. Spectra of collective SW modes in an array of square nanodots arranged in a square lattice in the ferromagnetic state. Curves demonstrate the results of calculations, symbols correspond to experimental data of work [4]. For the notation of the symmetry points in the first Brillouin zone (Γ, X, Y) see Fig. 2,*b*. The experimental parameters: the transverse size of a magnetic dot $l = 450$ nm, the height $h = 30$ nm, the lattice constant $a = 520$ nm, and the external field $B_e = 0.15$ T. Material parameters of permalloy were used in calculations: the saturation magnetization $\mu_0 M_s = 1.03$ T, the gyromagnetic ratio $\gamma = 29.3$ GHz/T, and the squared exchange length $\alpha_{ex}^2 = 28$ nm^2

analytically:

$$\boldsymbol{\sigma}_{1\mathrm{DE}}(\mathbf{k}) = (0,1,i)\,\frac{4\pi^2 l^2 \cos(k_x l/2)\cos(k_y l/2)}{(\pi^2 - k_x^2 l^2)\left(\pi^2 - k_y^2 l^2\right)},$$

and so forth. Similar expressions were obtained for the BA modes.

The choice of a circularly polarized basis zeroes the non-diagonal coefficients $\tilde{\omega}_{\lambda\lambda'}$. Taking this fact into account, we obtain the following equation for the amplitudes a_λ:

$$a_\lambda\left(\tilde{\omega}_{\lambda\lambda}A_\lambda + \omega_M G_{\mathbf{k},\lambda\lambda}(\mathbf{0})\right) +$$

$$+\, a_{\lambda'}\omega_M G_{\mathbf{k},\lambda\lambda'}(\mathbf{0}) = \omega_{\mathbf{k}} a_\lambda A_\lambda.$$

Bearing in mind that $\tilde{\omega}_{\lambda'\lambda'} = -\tilde{\omega}_{\lambda\lambda}$ and $A_{\lambda'} = -A_\lambda$, we obtain the ultimate expression for the dispersion law of a collective SW mode in the case of a nanodot array in the FM state,

$$\omega_{\mathbf{k},\lambda} = \sqrt{\left(\tilde{\omega}_{\lambda\lambda} + \omega_M \frac{G_{\mathbf{k},\lambda\lambda}(\mathbf{0})}{A_\lambda}\right)^2 - \omega_M^2 \frac{|G_{\mathbf{k},\lambda\lambda'}(\mathbf{0})|^2}{A_\lambda^2}}. \tag{27}$$

If the mode is uniform over the nanodot volume,

$$G_{\mathbf{k},\lambda\lambda}(\mathbf{0})/A_\lambda = (F_{\mathbf{k}}^{(yy)} + F_{\mathbf{k}}^{(zz)})/2,$$

$$G_{\mathbf{k},\lambda\lambda'}(\mathbf{0})/A_\lambda = i(F_{\mathbf{k}}^{(yy)} - F_{\mathbf{k}}^{(zz)})/2$$

(for the definition of the tensor $\hat{\boldsymbol{F}}$, see work [6]). A direct substitution of those expressions into Eq. (27) makes it evident that the latter is equivalent to the equation obtained in the framework of the macrospin approximation (see work [6, Eq. (3.38)]).

The calculated SW spectra are depicted in Fig. 3. The spectrum of fundamental (F) mode was obtained in the macrospin approximation, because it is known from the literature that its profile in thin magnetic dots is closer to a uniform profile than to a harmonic one [5]. As one can see, the SW spectra calculated in the diagonal approximation (without any fitting parameters) coincide rather well with experimental data (it should be taken into account that the experimental error was about 0.5 GHz [4]). It is also evident that the dispersion dependences of spectral branches formed by modes with non-zero dipole moments (F, $2n$DE, $2n$BA) behave similarly to the dispersion of SWs in the in-plane magnetized film: the SW frequency decreases with the growth of $|\mathbf{k}|$, when the wave vector is parallel to the static magnetization direction (in our case, it is the x-axis), and increases if $\mathbf{k} \perp \boldsymbol{\mu}$ (along the y-axis). Collective SWs formed by modes with zero dipole moment – these are the $(2n+1)$DE and $(2n+1)$BA modes – do not change the sign of the group velocity if the direction of their propagation changes; specifically, $v_{\mathrm{gr}} > 0$ for backward magnetostatic modes, and $v_{\mathrm{gr}} < 0$ for DE ones. It is clear that the formulated rules can be violated owing to the interaction between various spectral branches if they approach close to one another.

Hence, the SW spectra of nanodot arrays in periodic states can be calculated rather accurately and easily with the use of the method of projection onto the eigenmodes of a solitary nanodot. Note that the availability of analytical expressions for the SW dispersion law allows the general properties of SWs to be to analyzed; in particular, this concerns the behavior in a vicinity of the point $\mathbf{k} = \mathbf{0}$, which would be a difficult task if only numerical methods of calculation could be applied.

4. Conclusions

In this work, the theory of collective SW excitations in arrays of magnetic nanodots is generalized to the case where the profiles of static and dynamic magnetizations in the nanodot volume are non-uniform. The general properties of collective array modes are determined, in particular, an orthogonality relation and a sufficient condition for the stationary array state to be stable are derived. The influence of small perturbations on array modes is considered; in particular, the dependence of SW damping constant on the precession ellipticity is demonstrated.

The collective excitations are considered in two cases: for finite and infinite periodic arrays. From the mathematical viewpoint, the both models are identical; therefore, the properties of modes in finite arrays and waves in periodic ones are identical too. In both cases, the problem of finding the frequencies and the structure of collective excitations is reduced to a finite-dimensional integro-differential eigenvalue problem. An effective approximate way to calculate SW spectra in the framework of the projection method by generalizing the tensor of mutual nanodot demagnetization is proposed. This method allows, in particular, the analytical expressions for SW dispersion laws in the case of periodic arrays to be obtained.

Spin waves are considered in the ferromagnetic state of an array of nanodots magnetized to saturation and arranged in a square lattice. A reasonable coincidence between the calculation results and the experimental data is revealed. It is demonstrated that the behavior of the SW dispersion law substantially depends on the presence of the average dipole mode moment in a nanodot: modes with non-zero dipole moments behave similarly to SWs in a tangentially magnetized film of a ferromagnet, whereas other modes do not change the sign of their group velocity, when the direction of their propagation changed from parallel to perpendicular to the static magnetization vector.

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СПІНОВІ ХВИЛІ У МАСИВАХ МАГНІТНИХ НАНОТОЧОК, ПОВ'ЯЗАНИХ МАГНІТОДИПОЛЬНОЮ ВЗАЄМОДІЄЮ

Р е з ю м е

Представлено загальну теорію колективних спін-хвильових збуджень у скінченних та нескінченних періодичних масивах магнітних наноточок, пов'язаних магнітодипольною взаємодією. Теорія враховує неоднорідність статичної та динамічної намагніченості в об'ємі наноточки і дозволяє розраховувати спектри колективних збуджень, їх сталі затухання, ефективність збудження зовнішнім полем, тощо та досліджувати стійкість стаціонарного стану масиву. Запропоновано ефективний спосіб розрахунку властивостей спінових хвиль у періодичних масивах методом проекцій на власні моди ізольованої наноточки; отримані результати порівняні з експериментальними даними.