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IONIZATION OF ATOMS IN A STRONG LASER RADIATION FIELD AND THE IMAGINARY TIME METHOD

The phenomenon of nonlinear relativistic ionization induced by a strong electromagnetic wave has been considered. The relativistic version of the imaginary time method is used to calculate the probability for an electron with an energy of the order of its rest energy to tunnel through a potential barrier under the action of a strong electromagnetic wave. Besides the exponential factor, the Coulomb and pre-exponential ones are also obtained with regard for the electron spin and the ionization probability. Simple analytical formulas for the momentum distributions of relativistic photo-electrons are derived. The relativistic effects are shown to result in a nonzero drift velocity of an electron, when it quits the barrier. In the nonrelativistic limit, the well-known Keldysh exponent and the Landau–Lifshitz formula for the ionization probability of a hydrogen atom in the ground state are obtained.

Keywords: relativistic tunnel and multiphonon ionization, imaginary time method, Keldysh parameter.

1. Introduction

An extensive body of the literature is devoted to the study of the ionization of atoms and ions under the action of intensive laser radiation [1–6]. The theory of these processes was started by Keldysh in his classical work [7], where the tunnel effect in an alternating electric field and the multiphoton ionization of atoms were demonstrated to be the limiting cases of the nonlinear photoionization process, the course of which considerably depends on the value of the Keldysh adiabatic parameter γ. This parameter is defined as the ratio between the frequency of laser radiation, ω, and the frequency of electron tunneling through the potential barrier, ω_i,

$$\gamma = \frac{\omega}{\omega_t} = \frac{\sqrt{2m|E_0|}\omega}{eF}, \qquad (1)$$

where E_0 is the potential of atomic level ionization, and F the electromagnetic field strength created by a laser. The tunnel ionization of atomic states occurs if $\gamma \ll 1$. In the case $\gamma \gg 1$, the process has a multiphoton character. The authors of works [1–7] used the nonrelativistic approximation, which is valid for the valence electrons in each atom between hydrogen and uranium. For today, owing to the progress in laser physics and technology, the intensity of laser pulses reaches the values up to $I \sim 10^{22}$ W/cm^2 [8], and the range of their duration was substantially expanded (femto- and even ultrashort atto-second pulses were already obtained [9, 10]). In electromagnetic fields with such intensity, the ponderomotive energy of an electron emitted due to the ionization, $E_p = e^2F^2/(4m\omega^2)$, can be of the order of its rest energy, mc^2. In addition, the laser fields that are so intense that they can exceed the atomic field $F_{\rm H} = 5.14 \times 10^9$ V/cm by several orders of magnitude are capable of creating the multivalent ions with charges $Z \sim 40$–60, for which the binding energy of the ground level is also comparable with the electron rest energy. According to the modern ideas, the sub-barrier motion of an electron in the course of ionization cannot be considered as nonrelativistic, so that the Keldysh theory needs modifications [7, 11–13]. The main relativistic effects in the final state are [14–18] the relativistic distribution of the energy of emitted electrons and the shift of their angular distribution in the propagation direction of the incident laser beam.

The process of relativistic ionization in crossed electric and magnetic fields was considered in works [19, 20]. The results obtained can be applied to the ionization only in the case of very strong laser fields, when the parameter $\varepsilon = \frac{eF}{\omega mc} \gg 1$. As the frequency ω of the light emitted by a laser increases (e.g., in the case of sensitive x-ray lasers), very high intensities are required to satisfy this condition. Hence, the result of works [19, 20] has to be generalized onto the case of nonzero frequencies. In this work, we intend to examine the effects associated with relativistic velocities in the final states and/or with low-lying initial states from a special viewpoint. We concentrate our attention on the process of nonlinear ionization of a strongly coupled electron with the binding energy E_b, the latter having an order of the electron rest energy. A requirement to be satisfied in this case is inverse in comparison with that for the case of purely classical ionization, i.e. $F \ll F_b$. In addition, we also have a quasi-classical condition $\hbar\omega \ll E_b$. No restrictions are imposed on the parameter ε. In such a manner, the both modes–relativistic tunneling and multiphoton ionization–are included into consideration. Below, we apply the relativistic version of the "imaginary time" method [20, 21]. Being a generalization of the quasi-classical WKB-approximation onto the case of fields varying in time, this method describes the tunnel transition of an electron from a bound state into the continuum with the help of the classical equations of motion, but the time is an imaginary-valued quantity.

2. Classical Relativistic Action and Imaginary Time Method

The imaginary part of the reduced classical action S_f calculated along similar trajectories determines, with an exponential accuracy, the probability for an electron to transit from a bound state with energy E_0 into the continuum,

$$W_0 \propto \exp\left\{-\frac{2}{\hbar}\,\mathrm{Im}\,(S_f(t_0)+E_0t_0)\right\}, \tag{2}$$

where $E_0 = mc^2 - E_b$, and S_f is the classical relativistic action for an electron with charge e that moves in the field of a plane electromagnetic wave with the vector potential $\boldsymbol{A}(t-x/c)$. Hereafter, the notation $\boldsymbol{A}$ means a two-dimensional vector in the $y-z$ plane. The action can be found as a solution of the Hamilton–Jacobi equation [22],

$$S_f(\xi;\xi_0) = mc^2\left\{\boldsymbol{f}\cdot\frac{\boldsymbol{r}}{c} - \alpha\frac{x}{c} - \frac{1+\alpha^2+f^2}{2\alpha}(\xi-\xi_0) + \right. \\ \left. + \frac{e}{mc^2\alpha}\boldsymbol{f}\int\limits_{\xi_0}^{\xi}\boldsymbol{A}d\xi - \frac{e^2}{2m^2c^4\alpha}\int\limits_{\xi_0}^{\xi}\boldsymbol{A}^2d\xi\right\}, \tag{3}$$

where α and $\boldsymbol{f} = (a_1, a_2)$ are constants, $\boldsymbol{r} = (y, z)$, and ξ_0 is the initial value of the variable $\xi = t - x/c$.

In the framework of the ordinary Hamilton–Jacobi method, we differentiate the action S_f with respect to the variables a_1, a_2, and α; then, equating the results to new constants β_1, β_2, and β_3, we obtain a trajectory of the electron under the wave-field action. In the case of harmonic, plane, and linearly polarized waves with the field strength $\boldsymbol{E} = F\boldsymbol{e}_y\cos\omega\xi$, the electron motion in the laboratory coordinate system is described by the expressions

$$\alpha^2(t+x/c) - \beta^2\xi + \frac{2\varepsilon}{\omega}a_1\cos\omega\xi + \frac{\varepsilon^2}{4\omega}\sin 2\omega\xi = \beta_3, \\ v_x = c\frac{f(\xi)-1}{f(\xi)+1},\ y = \beta_1 + \frac{ca_1}{\alpha}\xi - \frac{c\varepsilon}{\alpha\omega}\cos\omega\xi, \\ v_y = \frac{2c}{\alpha\,(1+f(\xi))}\left[a_1+\varepsilon\sin\omega\xi\right],\ z = \beta_2 + \frac{ca_2}{\alpha}\xi, \\ v_z = \frac{2c}{\alpha\,(1+f(\xi))}\,a_2, \\ f(\xi) = \frac{\delta^2}{\alpha^2} + \frac{2\varepsilon}{\alpha^2}a_1\sin\omega\xi + \frac{\varepsilon^2}{\alpha^2}\sin^2\omega\xi, \tag{4}$$

where the parameters β_1, β_2, and β_3, as well as a_1, a_2, and α, have to be determined from the initial conditions for the charge position and velocity. Let us introduce the notation $\beta^2 = 1 + a_1^2 + a_2^2 + \varepsilon^2/2$ and $\delta^2 = 1 + a_1^2 + a_2^2$. The complex initial time moment t_0 is determined from the classical return point on the complex half-plane,

$$E_f(t_0) = mc^2\left\{\frac{1+\alpha^2+f^2}{2\alpha} - \frac{e}{mc^2\alpha}\boldsymbol{f}\boldsymbol{A}(t_0) + \right. \\ \left. + \frac{e^2}{2m^2c^4\alpha}\boldsymbol{A}^2(t_0)\right\} = E_0. \tag{5}$$

Minimizing the imaginary part of the action, we arrive at the following boundary conditions [3, 21]:

$$\boldsymbol{r}(t_0) = 0, \quad \mathrm{Im}\,\boldsymbol{r}(t=0) = 0. \tag{6}$$

The former corresponds to the start of the electron motion under the barrier at a time moment t_0, and the latter means that the most probable (extreme) trajectory becomes actual at $t = 0$. At $t > 0$, it describes the electron motion to the infinity in the classically allowed region.

While searching for simple analytical results, let us consider the case of laser-emitted linearly polarized light. By minimizing the action, we obtain from Eqs. (5) and (6) that $\boldsymbol{f} = 0$. We intend to derive a system of nonlinear equations that determines the complex initial time t_0 and the constant α,

$$t_0 = i\tau_0 = -\frac{i}{\omega}\text{arcsinh}\left(\eta\sqrt{1+\alpha^2-2\alpha\varepsilon_0}\right),$$

$$\alpha^2 = 1 + \frac{1}{2\eta^2}\left[1 - \frac{\eta\sqrt{1+\alpha^2-2\alpha\varepsilon_0}}{\text{arcsinh}\left(\eta\sqrt{1+\alpha^2-2\alpha\varepsilon_0}\right)} \times \right.$$

$$\left. \times\sqrt{1+\eta^2\left(1+\alpha^2-2\alpha\varepsilon_0\right)}\right] \quad (7)$$

with the dimensionless initial energy $\varepsilon_0 = E_0/mc^2$ and the relativistic adiabatic parameter $\eta = \varepsilon^{-1} = \omega mc/(eF)$. Substituting the quantities t_0 and α into the action for the final state, we obtain the probability of relativistic quasi-classical ionization in the field of a linearly polarized laser beam. With an exponential accuracy,

$$W_0 \propto \exp\left\{-\frac{2E_b}{\hbar\omega}\left[\left(1 + \frac{1}{2\gamma^2\alpha} + \frac{mc^2}{E_b}\frac{(1-\alpha)^2}{2\alpha} \times \right.\right.\right.$$

$$\left.\left.\left. \times\,\text{arcsinh}\gamma(\alpha)\right) - \frac{1}{2\gamma^2\alpha}\gamma(\alpha)\sqrt{1+\gamma^2(\alpha)}\right]\right\}, \quad (8)$$

where α is the solution of Eq. (7). Hereafter, $\gamma = \sqrt{2mE_b}\omega/(eF)$ is the nonrelativistic Keldysh parameter corresponding to the binding energy E_b, and $\gamma(\alpha) = \eta\sqrt{1+\alpha^2-2\alpha\varepsilon_0}$ is the α-dependent adiabatic parameter. Equation (8) is the most general expression for the ionization rate in the quasi-classical regime and at the field strength lower than the over-barrier threshold. It describes both the tunnel and multiphoton ionization. It is a relativistic generalization of the well-known result obtained by Keldysh [7].

Now, let us analyze some limiting cases. Near the tunnel ionization limit, $\eta \ll 1$, we reproduce the static result [19, 20] with the frequency correction,

$$W_0 \propto \exp\left\{-\frac{F_s}{F}\Phi\right\},$$

$$\Phi = \frac{2\sqrt{3}\left(1-\alpha_0^2\right)^{3/2}}{\alpha_0} - \frac{3\sqrt{3}\left(1-\alpha_0^2\right)^{5/2}}{5\alpha_0}\eta^2 + O(\eta^4), \quad (9)$$

where $F_s = m^2c^3/e\hbar = 1.32 \times 10^{16}$ V/cm is the Schwinger field in quantum electrodynamics [23] and $\alpha_0 = (\varepsilon_0 + \sqrt{\varepsilon_0^2+8})/4$. In the nonrelativistic limit, $\varepsilon_b = E_b/mc^2 \ll 1$, we obtain that the parameter $\alpha_0 = 1 - \varepsilon_b/3 + \varepsilon_b^2/27$, and the expression for the probability of the nonrelativistic tunnel ionization, which takes the first relativistic and the frequency corrections into account, looks like

$$W_0 \propto$$

$$\propto \exp\left\{-\frac{4}{3}\frac{\sqrt{2m}E_b^{3/2}}{e\hbar F}\left[1 - \frac{\gamma^2}{10} - \frac{E_b}{12mc^2}\left(1 - \frac{13}{30}\gamma^2\right)\right]\right\}. \quad (10)$$

Here, the first two terms in the brackets describe the rate of ordinary nonrelativistic ionization and the first frequency correction to it [15]; the last term is the first relativistic correction. From Eq. (9), it follows that the account of relativistic effects overestimates the ionization rate in comparison with the nonrelativistic case. However, even for the binding energy of the order of the electron rest energy, the relativistic correction in the exponential factor is rather small. Near the "vacuum" limit $\varepsilon_0 = -1$ (i.e. for a level that is shifted down to the limit of a lower continuum, which corresponds to the critical nucleus charge $Z_{cr} = 173$), Eq. (9) transforms into $W_0 \propto \exp\left\{-9F_s/2F(1 - 9\eta^2/40)\right\}$. The maximum deviation in the exponent argument from the Keldysh formula reaches 18%. Here, the "vacuum" limit should not be confused with the creation of pairs in vacuum. It is known that the nonlinear vacuum phenomena are absent for a plane wave [23]. In contrast, we deal with the ionization of an atom that is at rest in the laboratory coordinate system. Note also that the one-particle approximation is used. Therefore, the process of pair generation remains beyond the scope of our consideration.

Now, let us proceed to the multiphoton limit, $\eta \gg 1$. In this case, the parameter $\alpha = 1 - \varepsilon_b/2\ln 2\gamma$, and the probability of relativistic ionization reads

$$W_0 \propto \exp\left\{-\frac{2E_b}{\hbar\omega}\left[\ln 2\gamma - \frac{1}{2} - \frac{E_b}{8mc^2\ln 2\gamma}\right]\right\}. \quad (11)$$

The first two terms in the brackets display, as above, the nonrelativistic result [7]. Relativistic effects giving rise to the increase of the ionization probability are contained in the last term.

It was shown above that, in the relativistic theory, the ionization rate grows irrespective of whether the parameter η is large or small. This result can be compared with those obtained by Crawford and Reiss [14, 17]. In their numerical calculations, the cited authors also found that the ionization rate increases in the field of a circularly polarized wave if $\eta \gg 1$; however, in the case $\eta \ll 1$, their results demonstrate a substantial decrease of the ionization probability [14]. In the case of linearly polarized light, it was shown [17] that the ionization rate is depressed by relativistic effects. However, Crawford and Reiss studied the over-barrier ionization of a hydrogen atom in the strong field approximation. In contrast to them, we analyze the sub-barrier ionization from a strongly coupled electron level, which results in an increase of the ionization rate. This growth is associated with a shift of the initial time t_0 toward earlier moments. As a result, the sub-barrier complex trajectory becomes shorter, and the ionization rate increases in comparison with that in the nonrelativistic theory. In Fig. 1, the dependence of the relativistic ionization rate on the binding energy ε_b (Eq. (8)) and the results of calculation using the Keldysh nonrelativistic formula are shown. Figure 1 should be regarded only as an illustration of the increment effect for two parameters, $\eta < 1$ and $\eta > 1$, because the values of frequency and intensity used at calculations remain not accessible for experimenters till now.

The transition from the multiphoton regime to the tunnel one as the field strength increases can be studied near the nonrelativistic limit $\varepsilon_b \ll 1$. Here, in the first approximation with respect to ε_b and at $\alpha = 1 - (\varepsilon_b/2\gamma^2)[(\gamma/\mathrm{arcsinh}\gamma)\sqrt{1+\gamma^2} - 1]$, the ionization probability is estimated as follows:

$$W_0 \propto \exp\left\{-\frac{2E_b}{\hbar\omega} f(\gamma)\right\},$$

$$f(\gamma) = \mathrm{arcsinh}\gamma + \frac{1}{2\gamma^2}\left[\mathrm{arcsinh}\gamma - \gamma\sqrt{1+\gamma^2}\right] - $$

$$- \varepsilon_b \frac{\gamma^4 + \gamma^2 - 2\gamma\sqrt{1+\gamma^2}\,\mathrm{arcsinh}\gamma + \mathrm{arcsinh}^2\gamma}{8\gamma^4 \mathrm{arcsinh}\gamma}. \qquad (12)$$

The terms in the function $f(\gamma)$, which survive at $\varepsilon_b \to 0$, describe the rate of nonrelativistic quasi-classical Keldysh ionization [7], whereas the term proportional to ε_b is the first relativistic correction to the Keldysh formula. Equation (12) is valid in the whole interval of γ-variation, i.e. both in the multiphoton,

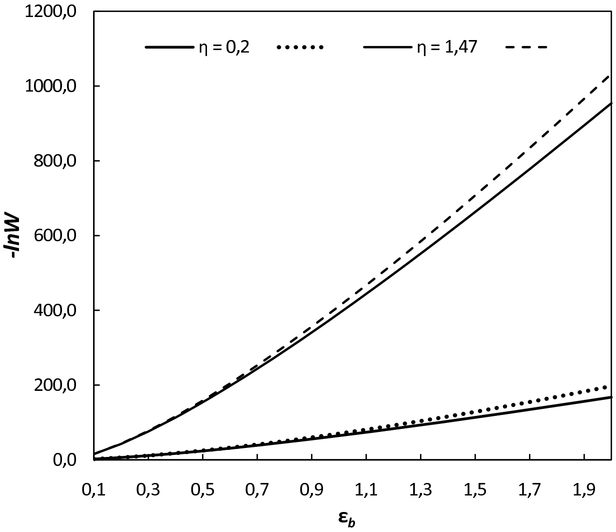

Fig. 1. Dependences of the absolute value of ionization rate logarithm on the binding energy $\varepsilon_b = E_b/mc^2$. Solid curves show the results of relativistic calculations by formula (8), dashed curves are the results of nonrelativistic calculations by the Keldysh formula (formula (12) without the relativistic correction). The parameter values (in atomic units) are $\omega = 100$ and $I = 8.5 \times 10^7$

$\gamma > 1$, and tunnel, $\gamma < 1$, limits. For a small adiabatic parameter, $\gamma \to 0$, it coincides with Eq. (10) and transforms in the case of large γ into Eq. (11). Note that Eq. (12) reproduces very accurately the complete relativistic formula (8) at $E_b < mc^2$.

Now, let us consider the changes in the energy spectrum induced by relativistic effects. In the nonrelativistic theory and in the case of linear light polarization, the most probable value of electron momentum at $t = 0$, i.e. at the electron departure moment, equals zero. Electrons are emitted mainly in the direction of the laser beam polarization. In the relativistic theory, which is considered in this work, we may put $a_1 = a_2 = 0$ in Eq. (4). Then, for the most probable velocity of electron departure in the laboratory coordinate system, we obtain

$$v_x = c\frac{1-\alpha^2}{1+\alpha^2}, \quad v_y = v_z = 0, \qquad (13)$$

where α is the solution of the second equation in system (7). In the static limit, $\omega \to 0$, the result of work [20] is reproduced.

From those equations, it follows that the strongly coupled electron is emitted along the laser beam propagation direction, i.e. perpendicularly to the polar-

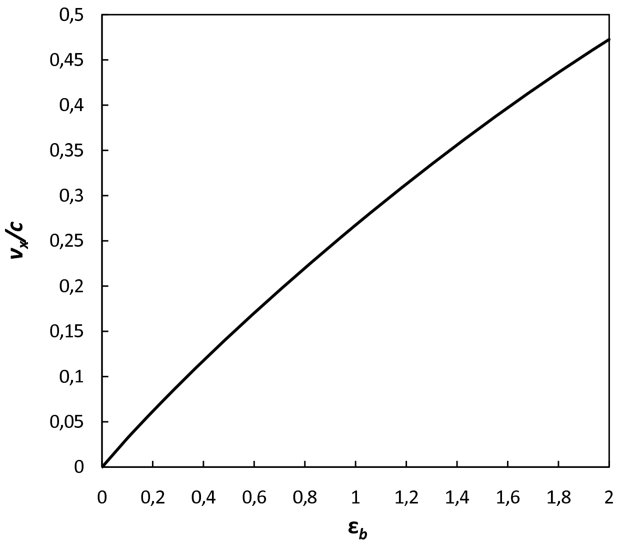

Fig. 2. Dependence of the x-component of the emission velocity, v_x/c, on the binding energy of the initial level $\varepsilon_b = E_b/mc^2$. In the nonrelativistic theory, the emission velocity equals zero. The parameter values (in atomic units) are $\omega = 100$ and $I = 8.5 \times 10^7$

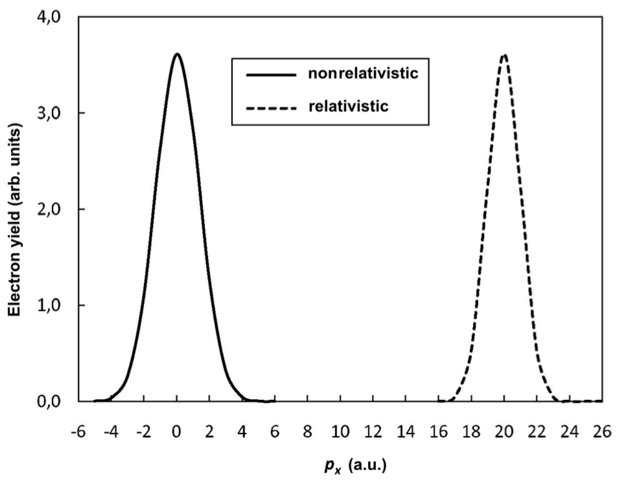

Fig. 3. Spectra of electron momentum projections on the beam propagation direction calculated by formula (14). The field strength $I = 2.5 \times 10^{10}$ V/cm

ization direction. For a nonrelativistic initial state with $\varepsilon_b \ll 1$, the average velocity of departure along the beam propagation direction, $v_x = ce_b/3$, is low. Nevertheless, this quantity seems to be a more sensitive criterion of relativistic effects in the initial state. The dependence of the x-component of the emission velocity, v_x/c, on the binding energy of the initial level is shown in Fig. 2.

The electron energy spectrum undergoes the influence of relativistic effects in the final state as well. Let us put $a_1 = p_{y,0}/mc$, $a_2 = p_{z,0}/mc$, and $\alpha = (-p_{x,0} + \sqrt{1 + p_{x,0}^2 + p_{y,0}^2 + p_{z,0}^2})/mc$, and confine the consideration to the tunnel limit, $\gamma \ll 1$. Supposing that the relativistic effects are weak in the initial state, $\varepsilon_b \ll 1$ and $(p_{y,0}, p_{z,0}) \ll mc$, it is possible to obtain the following formula:

$$W_p = W_0 \exp\left\{-\frac{\gamma}{m\hbar\omega}\left[(p_{x,0} - < p_{x,0} >)^2 + p_{z,0}^2\right]\right\} \times$$

$$\times \exp\left\{-\frac{p_{y,0}^2}{m\hbar\omega}\left(\frac{\gamma^2}{3} + \frac{p_{y,0}^2}{4m^2c^2}\right)\gamma\right\}, \qquad (14)$$

where W_0 is the total ionization rate (Eq. (10)) in the weakly relativistic tunnel limit. The first exponential function in formula (14) describes the momentum distribution in the plane that is perpendicular to the polarization axis. In the weakly relativistic limit, there is only one relativistic effect; this is the appearance of the average momentum $\langle p_{x,0} \rangle = E_b/3c$ at the departure moment. A nonzero average velocity of electron departure along the propagation vector breaks the symmetry in the (x, z)-plane, the violation taking place in the nonrelativistic theory as well. The first term in the second exponential function in Eq. (14) is responsible for the nonrelativistic energy spectrum of low-energy electrons moving along the polarization axis, whereas the second (relativistic) term becomes important at large energies, $p_{y,0}^2 > 4\gamma^2 m^2 c^2/3$. Only if the adiabatic parameter is small, $\gamma \leq 1$, the requirement of high energies does not contradict the condition $p_{y,0} < mc$. Note that the second term in the second exponential function agrees with the corresponding Krainov's term [15, 16]. The spectrum of electron momentum projections on the beam propagation direction is depicted in Fig. 3.

3. Coulombic Correction

Above, we neglected the Coulomb interaction between the departing electron and the atomic core, so that the formulas obtained concern the case of negative ion ionization (ions of the type H^-, Na^-, and so forth). In the case of ionization of neutral atoms and positive ions, the Coulomb interaction of the arisen electron with the atomic (ionic) remnant has to be taken into account. With that end in view, we intend to apply the quasi-classical perturbation theory in order to calculate the correction to the classical action, $\delta S = Z\int dt/r(t)$. However, since this integral diverges at $r \to 0$, let us use the procedure of matching with the asymptotic wave function of a free atom, $\chi_k(r) \simeq \exp\{-[(kr) - \eta\ln(kr) + O(1)]\}$ (see details in work [13]). This approximation gives us the Coulomb factor [3]

$$Q(z_0) = 2\lambda z_0 \exp\{J(z_0)\},$$
$$J(z_0) = \int\limits_0^1 \left[\frac{\gamma z_0}{|\boldsymbol{r}([1-s]z_0)|} - \frac{1}{s}\right] ds \quad (15)$$

in the expression for the tunnel ionization probability W [20],

$$W = \omega_b\, C_{\kappa l}^2 S_\pm\, P_0(\tau_0)\, Q^{2n^*}(\tau_0) W_0, \quad (16)$$

where $z_0 = \omega\tau_0$, $\boldsymbol{r}$ is trajectory (4) corresponding to the imaginary time (7) at the sub-barrier motion of an electron,$\omega_b = E_b/\hbar = 0.776(1-\varepsilon_0)\times 10^{21}\ \text{s}^{-1}$ is the frequency corresponding to the binding energy of the level, the parameter W_0 is defined by formula (8), and $n^* = Z\tilde{\alpha}\varepsilon_0/\sqrt{1-\varepsilon_0^2}$ is a relativistic analog of the Sommerfeld parameter (Z is the atomic core charge, and $\tilde{\alpha} = e^2/\hbar c = 1/137$). The parameter n^* is close to 1, as a rule (for a hydrogen atom, $n^* = 1$). In addition, $C_{\kappa l}$ is the asymptotic coefficient in the atomic wave function at infinity (in particular, $C_{\kappa l} = 1$ for the 1s-state of a hydrogen atom [12]); P_0 and $S_\pm$ are the exponential and spin factors, respectively, which can also be determined in the framework of the imaginary time method. Collecting all multipliers and carrying out rather cumbersome calculations, we obtain the following formula for the ionization probability of a relativistic s-level in the adiabatic approximation:

$$W = \omega_b\, C_{\kappa l}^2 E^{3/2-2n^*} S_\pm\, P\, W_0, \quad (17)$$

where

$$E = (1+\zeta^2)F/(3\sqrt{3}\zeta^3 F_s);$$
$$S_\pm = \exp\left\{\pm\frac{\sqrt{3}\zeta}{\sqrt{1+\zeta^2}}\left(1-\frac{\mu}{\mu_\text{B}}\right)\right\}; \quad (18)$$

the subscripts $\pm$ correspond to the spin projections ($s_z = \pm\hbar/2$) on the magnetic field direction of the wave (we suppose that the magnetic field of the wave is directed along the z-axis), so that the states with different s_z-values have different ionization rates;

$$\zeta = \left[1+\frac{\varepsilon_0}{2}\left(\varepsilon_0 - \sqrt{8+\varepsilon_0^2}\right)\right]^{1/2},$$

and, for the ground level $1s_{1/2}$ of a hydrogen-like atom with the charge $Z = 60$, we have $\mu = 0.933\mu_\text{B}$ [24], where $\mu_\text{B} = e\hbar/2mc$ is the Bohr magneton. Formula (18) makes allowance for the spin rotation in an external electromagnetic field (its magnitude is determined by the Bargmann–Michel–Telegdi equation [25]), as well as the split of the initial level by the wave magnetic field. The factor P is expressed by the formula

$$P = \frac{\sqrt{3}(1+\varepsilon_0)}{\sqrt{\pi(1-\zeta^4/9)}}\left[2\left(1-\frac{\zeta^2}{3}\right)^2\right]^{2n^*} \times$$
$$\times \exp\left(6Z\tilde{\alpha}\arcsin\frac{\zeta}{\sqrt{3}}\right). \quad (19)$$

In the nonrelativistic limit, relation (17) gives rise to the well-known Landau–Lifshitz formula [26]

$$W_\text{H} = 8|E_\text{H}|\frac{F_\text{H}}{F}\exp\left\{-\frac{2F_\text{H}}{3F}\right\} \quad (20)$$

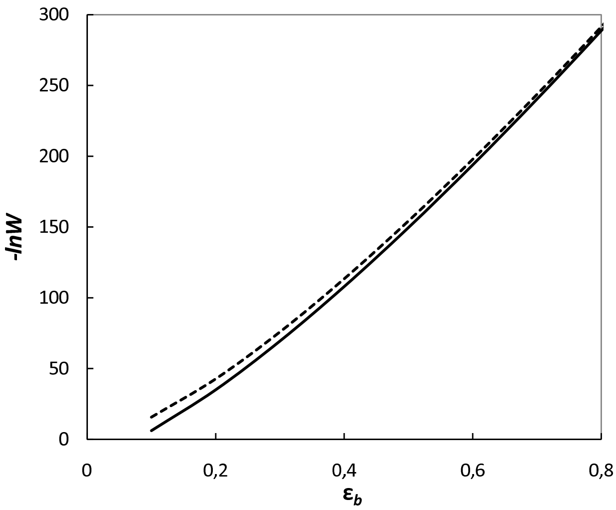

Fig. 4. Logarithm of the ionization probability (formula (17), solid curve) and Keldysh formula (expression (12) without the relativistic correction, dashed curve). $\eta = 1.47$. The account for the Coulomb interaction increases the ionization probability

for the ionization probability of the ground state in a hydrogen atom ($E_{\rm H}$ = 13.6 eV). Figure 4 demonstrates the result of taking the Coulomb interaction into account.

4. Conclusions

On the basis of the Hamilton–Jacobi equation, the classical action and the trajectories of relativistic motion are obtained for an electron moving in the field of a linearly polarized electromagnetic wave. With the help of the imaginary time method, the sub-barrier motion of an electron is studied, and the simple formulas for the probability of atomic level ionization in the field of the strong laser radiation with the energy comparable with that of an electron at rest are derived. The expressions for the ionization probability presented in this work cover a wide interval of adiabatic Keldysh parameter variation ranging from the multiphoton ionization to the tunnel mode. In the nonrelativistic limit, they coincide with the known relations obtained by other authors. The momentum distribution of outcoming electrons is studied, and it is shown, in particular, that the electron can possess a nonzero drift velocity after it quits the sub-barrier region in the relativistic case. In the adiabatic limit, the Coulomb interaction between a relativistic electron and the atomic core and its influence on the ionization probability are taken into account. In the nonrelativistic limit, the Landau–Lifshitz formula for the ionization probability of a hydrogen atom in the ground state is obtained, and the expression for a spin relativistic correction is given. The results obtained can also be applied to nuclear physics and quantum chromodynamics.

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ІОНІЗАЦІЯ АТОМІВ
У ПОЛІ СИЛЬНОГО ЛАЗЕРНОГО
ВИПРОМІНЮВАННЯ ТА МЕТОД УЯВНОГО ЧАСУ

Р е з ю м е

Розглянуто феномен нелінійної релятивістської іонізації, спричиненої потужним лінійно поляризованим полем лазера. За допомогою релятивістської версії методу уявного часу розраховано ймовірність тунелювання електрона, енергія зв'язку якого може бути порядку енергії спокою, крізь потенціальний бар'єр під дією поля сильної електромагнітної хвилі. Окрім експоненційного множника розраховано також кулонівський та передекспоненційний фактори з урахуванням спіну електрона та ймовірності іонізації. Отримано прості аналітичні формули для імпульсних розподілів релятивістських фотоелектронів. Показано, що релятивістські ефекти приводять до появи ненульової (дрейфової) швидкості електрона після виходу з-під бар'єра. На нерелятивістській границі отримано відому експоненту Келдиша, а також формулу Ландау–Ліфшица для іонізації основного стану атома водню.