

## PARTICLE'S DIFFUSION IN A TWO-DIMENSIONAL RANDOM VELOCITY FIELD

*The two approaches to describe the diffusion process of test particles in a two-dimensional random velocity field are compared to each other: the method of decorrelation trajectories and the moment approximation. The frozen turbulence case is considered, because it is the most complicated test for statistical theories. The results of considered analytical approaches are verified by direct numerical simulation.*

*Keywords:* test-particle diffusion, random field, numerical simulation.

### 1. Introduction

The problem of turbulent transport arises widely in various phenomena in the fluid and plasma physics and can be formulated as a problem of statistical description of the evolution of an ensemble of particles in a random velocity field. There are two ways to describe such evolution of a system: the self-consistent description, when particle's motion influences the random velocity field, and a problem with external fields, when it is not. The latter one is a simplified description in contrast to the self-consistent problem. On the other hand, due to the full control over parameters of the problem and a more tractable analysis of the results, it gives a chance to estimate the quality of analytical approximations.

The evolution of the probability function distribution in the problem with external fields is described by a nonlinear integro-differential equation [1] that defines a nonlocal dependence of the time derivative of the probability distribution function on its values at the previous time moments. There are no methods to obtain the exact solution for that equation. That is why, the simplified parabolic diffusion equation local in the time with time-dependent diffusion coefficient is used. The diffusion coefficient can be expressed in terms of the correlation function of velocity components along particle trajectories by the Taylor formula [2, 5]. We can reformulate the problem as a problem of relation between Lagrangian and Eulerian statistical quantities. The correlation functions of velocity

components along particles trajectories are unknown for us and called Lagrangian correlation functions. At the same time, the Eulerian correlation functions are defined at the fixed points of the laboratory frame and considered as known ones. Thus, the problem is to obtain the unknown Lagrangian correlation function for the velocity component using the known Eulerian one.

The importance of the dependence of the trajectories of particles on a random field, the so-called Lagrangian nonlinearity, is usually characterized by the Kubo number  $K$ . It relates the mean free path of a particle for the correlation time to the correlation length and describes a possibility for a particle to explore the spatial structure of a random field till its complete change.

Small values of Kubo number  $K < 1$  correspond to a weak Lagrangian nonlinearity in the problem. In this case, the motion of particles can be considered as random, because the random field changes rapidly in time. For that values of Kubo number, the asymptotic scaling for the diffusion coefficient as a  $D \sim K^2$  obtained by the known Corrsin approximation [3, 5] is agreed to be correct. But, in the case of a strong Lagrangian nonlinearity  $K > 1$ , all the methods that are based explicitly or implicitly on the Corrsin approximation give an asymptotic scaling law for the diffusion coefficient, as  $D \sim K$  [4]. This result contradicts the result of numerical simulations [6]. The reason for this contradiction is the account for the temporal, not spatial, decay of correlations only in the Corrsin approximation. While the Corrsin approximation is not valid for the description of diffusion processes in a random fields with long correlations, we will pay at-

tention to it. This will grant a possibility to demonstrate the differences in statistical approaches, which are studied in this work.

The frozen turbulence is of special interest in the validation of statistical approaches [1]. In this case, the random field is static, Kubo number  $K \rightarrow \infty$ , and motion of particles is completely described by the spatial structure of a random field. According to the recent numerical simulations [6] for a strong Lagrangian nonlinearity  $1 < K < \infty$ , the percolation scaling law [7] for the diffusion coefficient  $D \sim K^{0.7}$  is observed. The exponent less than one corresponds to the particle trapping effects. The percolation approach cannot describe the time evolution of the diffusion process. It is important to point out that, in the frozen turbulence limit, the asymptotic scaling law gives that the diffusion coefficient is zero, rather than infinity.

In order to describe the time evolution of the diffusion coefficient, the method of decorrelation trajectories was proposed [8, 9] with the aim to take into account the effects of particle trapping caused by the spatial structure of a random field. The authors [8] pointed out that the method could not be obtained from the first principles. Instead of, its validity was demonstrated by the comparison of its results with a numerical simulation [6]. This approximation was able to recover a percolation asymptotic scaling law for the diffusion coefficient and to qualitatively recover the time dependence of the Lagrangian correlation function of the velocity components in the cases of the large  $K > 1$  and small  $K < 1$  Kubo numbers. But the comparison of results of the decorrelation trajectory method with a direct numerical simulation for the frozen turbulence for a very strong Lagrangian nonlinearity, as  $K \rightarrow \infty$ , was not presented. Only the qualitative considerations of the correctness of the method was given.

The moment approximation [10] for the frozen turbulence was proposed recently. The results of this method were in a qualitative agreement with a direct numerical simulation. Let us underline that this method uses no free parameters: there is only the assumption about the closure of an equation.

We will give the detailed comparison of these approaches with each other and verify their predictions by the results of a direct numerical simulation for a frozen turbulence, as  $K \rightarrow \infty$ . We are also interested in the best way to an approximate Lagrangian

correlation function of velocity components, by using corresponding Eulerian ones.

The work is organized as follows. The basic concepts are introduced in Section 2, the approximations are considered in Sections 3–5, and the equations for a numerical simulation are described in Section 6. The obtained results and their discussion are presented in Sections 7–10, and the conclusions are given in Section 11.

## 2. Taylor Relation

Studying the motion of particles  $\chi(\tau)$  in a random velocity field  $\mathbf{v}(\chi(\tau), \tau)$

$$\frac{d\chi(\tau)}{d\tau} = \mathbf{v}(\chi(\tau), \tau), \quad (1)$$

Taylor [2, 5] proposed a statistical approach to obtain a mean square displacement with the use the an integral relation

$$\Delta_i(\tau) = \langle \chi_i^2(\tau) \rangle = \int_0^\tau d\tilde{\tau}_1 \int_0^{\tilde{\tau}_1} d\tilde{\tau}_2 C_{v_i v_i}^L(\tilde{\tau}_1, \tilde{\tau}_2), \quad (2)$$

where  $\langle \dots \rangle$  describes the averaging over the ensemble of realizations, and  $C_{v_i v_i}^L(\tilde{\tau}_1, \tilde{\tau}_2)$  is a Lagrangian correlation function of velocity components. In a homogeneous stationary case, a Lagrangian correlation function of velocity components depends only on the difference between two correspondent moments time  $\tilde{\tau} = |\tilde{\tau}_1 - \tilde{\tau}_2|$  and can be written as follows:

$$C_{v_i v_i}^L(\tilde{\tau}) = \langle v_i(\chi(\tau_1 + \tilde{\tau}), \tau_1 + \tilde{\tau}) v_i(\chi(\tau_1), \tau_1) \rangle. \quad (3)$$

At the same time, the mean square displacement (2) is determined as

$$\Delta_i(\tau) = \int_0^\tau d\tilde{\tau} (\tau - \tilde{\tau}) C_{v_i v_i}^L(\tilde{\tau}), \quad (4)$$

and the time-dependent diffusion coefficient is

$$D_i(\tau) = \frac{1}{2} \frac{d}{d\tau} \Delta_i(\tau) = \int_0^\tau d\tilde{\tau} C_{v_i v_i}^L(\tilde{\tau}). \quad (5)$$

Using Eqs. (4) and (5), the problem of statistical description of the motion of particles in the random velocity field (1) can be reduced to a problem of obtaining the unknown Lagrangian correlation function of velocity components.

One of the possible approaches to the solution of this problem is to approximate the Lagrangian correlation function by the corresponding Eulerian ones, which can be calculated or obtained from experiment. These functions are defined at fixed points of the laboratory frame and are independent of the trajectories of particles. We have

$$C_{v_i v_i}^E(\boldsymbol{\chi}, \tilde{\tau}) = \langle v_i(\boldsymbol{\chi}_1 + \boldsymbol{\chi}, \tau_1 + \tilde{\tau}) v_i(\boldsymbol{\chi}_1, \tau_1) \rangle. \quad (6)$$

Further, we will study Eq. (1) to describe the two-dimensional motion of particles in a static incompressible Gaussian isotropic random velocity field  $\mathbf{v}(\boldsymbol{\chi}(\tau))$ . In this case, the Eulerian correlation function of random values (6) doesn't depend on the time explicitly, and the Eqs. (3)–(5) are valid.

The main purpose of this work is the comparison of different analytical approximations of the Lagrangian correlation function of velocity components by the corresponding Eulerian ones for the frozen turbulence, as  $K \rightarrow \infty$ . For such a problem, the crucial role is played by the particle trapping effects.

### 3. Corrsin Approximation

One of the first widely used approximations in plasma physics is the Corrsin approximation [3, 5]. It is based on the exact equation

$$C_{v_i v_i}^L(\tau) = \int [D\boldsymbol{\chi}(\tau)] C_{v_i v_i}^{E_c}[\boldsymbol{\chi}(\tau)] P[\boldsymbol{\chi}(\tau); \tau], \quad (7)$$

with the Eulerian correlation function of a velocity component  $C_{v_i v_i}^{E_c}$  and the probability  $P(\boldsymbol{\chi}, \tau)$  of a trajectory realization  $\boldsymbol{\chi}(\tau)$  in the position  $\boldsymbol{\chi}$  at the time moment  $\tau$ . The mentioned Eulerian correlation function is calculated in the laboratory frame, but depends on particle's trajectories as a Lagrangian one. There is no mathematically strict way to obtain the Eulerian correlation function  $C_{v_i v_i}^{E_c}$  and the trajectory realization probability  $P(\boldsymbol{\chi}, \tau)$ . Thus, the approximate values are used. The Corrsin approximation supposes the statistical independence of particle's trajectories  $\boldsymbol{\chi}(\tau)$  of the random velocity field  $\mathbf{v}(\boldsymbol{\chi})$  and the Gaussian probability distribution function for displacements  $\Delta_i(\tau)$  of particle's trajectories. These assumptions allow one to replace the conditional Eulerian correlation function  $C_{v_i v_i}^{E_c}$  by the usual Eulerian one  $C_{v_i v_i}^E$  and the unknown trajectory realization probability  $P(\boldsymbol{\chi}, \tau)$  by the Gaussian one in Eq. (7).

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Let us look at the Lagrangian correlation function of a velocity component (3), by using the integral representation

$$C_{v_i v_i}^L(\tau) = \langle v_i(\boldsymbol{\chi}(\tau_2)) v_i(\boldsymbol{\chi}(\tau_1)) \rangle = \int d\boldsymbol{\chi}_2 d\boldsymbol{\chi}_1 \times \langle v_i(\boldsymbol{\chi}_2) v_i(\boldsymbol{\chi}_1) \delta(\boldsymbol{\chi}_2 - \boldsymbol{\chi}(\tau_2)) \delta(\boldsymbol{\chi}_1 - \boldsymbol{\chi}(\tau_1)) \rangle. \quad (8)$$

The first assumption of the Corrsin approximation allows us to rewrite the averaging under the integral as the multiplication of two independent averages. Particle's trajectories  $\boldsymbol{\chi}(\tau)$  are supposed to be statistically independent of the random velocity field  $\mathbf{v}(\boldsymbol{\chi})$  in this approximation. So, we have

$$\begin{aligned} \langle v_i(\boldsymbol{\chi}_2) v_i(\boldsymbol{\chi}_1) \delta(\boldsymbol{\chi}_2 - \boldsymbol{\chi}(\tau_2)) \delta(\boldsymbol{\chi}_1 - \boldsymbol{\chi}(\tau_1)) \rangle &\approx \\ \approx \langle \delta(\boldsymbol{\chi}_2 - \boldsymbol{\chi}(\tau_2)) \delta(\boldsymbol{\chi}_1 - \boldsymbol{\chi}(\tau_1)) \rangle &\times \\ \times \langle v_i(\boldsymbol{\chi}_2) v_i(\boldsymbol{\chi}_1) \rangle. &\quad (9) \end{aligned}$$

The Eulerian correlation function of a velocity component for a static random velocity field is

$$\langle v_i(\boldsymbol{\chi}_2) v_i(\boldsymbol{\chi}_1) \rangle = C_{v_i v_i}^E(\boldsymbol{\chi}_2 - \boldsymbol{\chi}_1). \quad (10)$$

After the introduction of new variables  $\boldsymbol{\chi} = \boldsymbol{\chi}_2 - \boldsymbol{\chi}_1$ ,  $\tilde{\boldsymbol{\chi}} = \boldsymbol{\chi}_2 + \boldsymbol{\chi}_1$  and the further substitution of Eqs. (9) and (10) in (8), the Lagrangian correlation function becomes

$$\begin{aligned} C_{v_i v_i}^L(\tau) &\approx \int d\boldsymbol{\chi} C_{v_i v_i}^E(\boldsymbol{\chi}) \times \\ \times \int \frac{d\tilde{\boldsymbol{\chi}}}{2} \langle \delta(\frac{\boldsymbol{\chi} + \tilde{\boldsymbol{\chi}}}{2} - \boldsymbol{\chi}(\tau_2)) \delta(\frac{\tilde{\boldsymbol{\chi}} - \boldsymbol{\chi}}{2} - \boldsymbol{\chi}(\tau_1)) \rangle &= \\ = \int d\boldsymbol{\chi} C_{v_i v_i}^E(\boldsymbol{\chi}) \langle \delta(\boldsymbol{\chi} - \boldsymbol{\chi}(\tau)) \rangle, &\quad (11) \end{aligned}$$

where the displacement along trajectory  $\boldsymbol{\chi}(\tau) = \boldsymbol{\chi}(\tau_2) - \boldsymbol{\chi}(\tau_1)$  for the time  $\tau = \tau_2 - \tau_1$  is introduced. The second assumption is

$$\langle \delta(\boldsymbol{\chi} - \boldsymbol{\chi}(\tau)) \rangle \approx \prod_{i=x,y} \frac{\exp(-\frac{\chi_i^2}{2\Delta_i(\tau)})}{\sqrt{2\pi\Delta_i(\tau)}} = P^{CA}(\boldsymbol{\chi}, \tau). \quad (12)$$

The substitution of Eq. (12) in (11) gives us the Lagrangian correlation function of a velocity component in the Corrsin approximation in the form

$$C_{v_i v_i}^L(\tau) \approx \int d\boldsymbol{\chi} C_{v_i v_i}^E(\boldsymbol{\chi}) P^{CA}(\boldsymbol{\chi}, \tau). \quad (13)$$

In work [4], the approximate diffusion coefficient with the asymptotic scaling law  $D \sim K$  was obtained for a

strong Lagrangian nonlinearity  $K > 1$ . Later, in a numerical simulation [6], the asymptotic scaling law for the diffusion coefficient  $D \sim K^\gamma$  with  $\gamma < 1$  was obtained for the Kubo numbers  $1 < K < \infty$ . Moreover, for Eq. (13) in the case of a static random velocity field, the asymptotic scaling law for a diffusion coefficient appears to be infinite, instead of zero, as  $K \rightarrow \infty$ .

In work [8], it was pointed out that, for some cases of Gaussian displacements, there is a possibility to calculate  $C_{v_i v_i}^{Ec}$  exactly in Eq. (7). The diffusion coefficient obtained with the use of this exact Eulerian correlation function and the Gaussian trajectory realization probability (12) has the same asymptotic scaling law  $D \sim K$  for  $K > 1$ . This result contradicts the results of numerical simulations and the prediction of percolation theory, and assumption (12) is considered as a reason for the inadequacy of the Corrsin approximation for large Kubo numbers.

The numerical simulation [6] demonstrated that the probability distribution for displacements  $P(\chi, \tau)$  has a maximum at  $\chi = 0$  and a long power-law tail, which is caused by the trapping of particles in the spatial structure of a random velocity field: particles with a large absolute value of potential move periodically along equipotential surfaces along closed and rather short trajectories, while particles with a small absolute value of potential can travel for long distances.

This demonstrates also that the Gaussian distribution of the displacements of trajectories (12) is an inadequate assumption for large Kubo numbers, and there is a need to formulate an approach, which is not based on the second assumption of the Corrsin approximation (12).

#### 4. Method of Decorrelation Trajectories

The method of decorrelation trajectories was proposed in [8, 9] with the aim to take the particle trapping effects into account. This method does not involve the assumption of Gaussian displacements of trajectories (12). Instead of, it uses the set of deterministic trajectories, which are called the trajectories of spatial decorrelation or decorrelation trajectories. Every of these trajectories is determined by the conditional Eulerian correlation functions of random fields. The condition defines the equality of the initial values of random fields at the starting points of particle's trajectories to the given values. It is important

to point out that the way to organize the subensembles is not strictly fixed in this method: there is a freedom to choose a way to split an ensemble into subensembles. The authors of the method [8] used the way that reproduces quite well the results of numerical simulations [6]. The Lagrangian correlation function of a velocity component is approximated as the product of the initial velocity and the average velocity in a subensemble along the corresponding decorrelation trajectory averaged over all subensembles.

So, the method of decorrelation trajectories is based on two assumptions: there is a set of subensembles by initial values in the ensemble of realizations, and there is the unique dynamics of particles for each subensemble, which is described by a corresponding decorrelation trajectory.

The approximation of a Lagrangian correlation function proposed in work [8] is formulated as a full weighted Eulerian correlation function of the ensemble of realizations of random fields, which is determined along the decorrelation trajectories for all subensembles. This formulation of the approximation does not reflect all assumptions used in the method of decorrelation trajectories. Thus, we are going to present an interpretation of this method, which is based on the integral representation of the Lagrangian correlation function of velocity components. Consider the ensemble of realizations of random fields, which is partitioned in subensembles by their initial values, as selected in work [8]:

$$\sigma_0 = \sigma(\mathbf{0}), \quad \mathbf{v}_0 = \mathbf{v}(\mathbf{0}). \quad (14)$$

Similar to Eq. (8), we present the Lagrangian correlation function of a velocity component as the integral

$$C_{v_i v_i}^L(\tau) = \int d\sigma_0 d\mathbf{v}_0 d\mathbf{v} v_{0i} v_i \times \langle \delta(\sigma_0 - \sigma(\mathbf{0})) \delta(\mathbf{v}_0 - \mathbf{v}(\mathbf{0})) \delta(\mathbf{v} - \mathbf{v}(\chi(\tau))) \rangle. \quad (15)$$

This representation involves the conditions for the initial values of random potential and velocity field. Because the values of  $\sigma_0$ ,  $\mathbf{v}_0$ , and  $\mathbf{v}$  are numbers independent of the averaging over realizations, the average multiplication of  $\delta$ -functions is

$$\begin{aligned} & \langle \delta(\sigma_0 - \sigma(\mathbf{0})) \delta(\mathbf{v}_0 - \mathbf{v}(\mathbf{0})) \delta(\mathbf{v} - \mathbf{v}(\chi(\tau))) \rangle = \\ & = \int \frac{d\kappa_{\sigma_0} d\kappa_{\mathbf{v}_0} d\kappa_{\mathbf{v}}}{(2\pi)^5} \exp(-i\kappa_{\sigma_0} \sigma_0 - i\kappa_{\mathbf{v}_0} \mathbf{v}_0 - \kappa_{\mathbf{v}} \mathbf{v}) \times \\ & \times \langle \exp(i\kappa_{\sigma_0} \sigma(0) + i\kappa_{\mathbf{v}_0} \mathbf{v}(0) + i\kappa_{\mathbf{v}} \mathbf{v}(\chi(\tau))) \rangle. \quad (16) \end{aligned}$$

Averaging the exponent, we need to consider the relation between functions at the same time and at an arbitrary time. The average of the velocity component at the same time moment is fully described by the second cumulant, because of the supposed homogeneity, stationarity, and statistical independence of the random fields

$$\begin{aligned} \langle v_i(\boldsymbol{\chi}(\tau)) v_j(\boldsymbol{\chi}(\tau)) \rangle &= \langle v_i(\mathbf{0}) v_j(\mathbf{0}) \rangle = \\ &= \delta_{ij} C_{v_i v_j}^L(0) = \delta_{ij} C_{v_i v_j}^E(\mathbf{0}). \end{aligned} \quad (17)$$

So, the Lagrangian correlation function at the initial time moment corresponds to the Eulerian correlation function at the starting point and is known. Averaging the random functions  $\sigma(0)$ ,  $\mathbf{v}(0)$ , and  $\mathbf{v}(\boldsymbol{\chi}(\tau))$  at different time moments demands one to consider higher cumulants in general. But, we limit ourselves to the second cumulant, for simplicity:

$$\langle v_i(\boldsymbol{\chi}(\tau)) v_j(\mathbf{0}) \rangle = C_{v_i v_j}^L(\tau), \quad (18)$$

and

$$\langle v_i(\boldsymbol{\chi}(\tau)) \sigma(\mathbf{0}) \rangle = C_{v_i \sigma}^L(\tau). \quad (19)$$

These averages are described by Lagrangian correlation functions at an arbitrary nonzero time moment and are still unknown. So, taking only the second cumulant into account, we have

$$\begin{aligned} &\langle \exp(i\kappa_{\sigma_0} \sigma(0) + i\kappa_{\mathbf{v}_0} \mathbf{v}(0) + i\kappa_{\mathbf{v}} \mathbf{v}(\boldsymbol{\chi}(\tau))) \rangle \approx \\ &\approx \exp \left( -\frac{1}{2} \left( \kappa_{\sigma_0}^2 C_{\sigma\sigma}^L(0) + \sum_{i=x,y} \kappa_{v_{0i}}^2 C_{v_i v_i}^L(0) \right) \right) \times \\ &\times \exp \left( -\frac{1}{2} \sum_{i=x,y} \kappa_{v_i}^2 C_{v_i v_i}^L(0) \right) \times \\ &\times \exp \left( -\sum_{i=x,y} \kappa_{v_i} \kappa_{\sigma_0} C_{v_i \sigma}^L(\boldsymbol{\chi}(\tau)) \right) \times \\ &\times \exp \left( -\sum_{i \neq j=x,y} \kappa_{v_i} \kappa_{v_j} C_{v_i v_j}^L(\boldsymbol{\chi}(\tau)) \right). \end{aligned} \quad (20)$$

The right-hand side of Eq. (16) can be represented as the product

$$\begin{aligned} &\langle \delta(\sigma_0 - \sigma(\mathbf{0})) \delta(\mathbf{v}_0 - \mathbf{v}(\mathbf{0})) \delta(\mathbf{v} - \mathbf{v}(\boldsymbol{\chi}(\tau))) \rangle \approx \\ &\approx P_0(\sigma_0, \mathbf{v}_0) P(\mathbf{v}(\tau), \boldsymbol{\chi}(\tau); \sigma_0, \mathbf{v}_0), \end{aligned} \quad (21)$$

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with the probability of realization of a subensemble by the initial values of random fields

$$\begin{aligned} P_0(\sigma_0, \mathbf{v}_0) &= \frac{1}{\sqrt{(2\pi)^3 C_{v_x v_x}^E(\mathbf{0}) C_{v_y v_y}^E(\mathbf{0}) C_{\sigma\sigma}^E(\mathbf{0})}} \times \\ &\times \exp \left( -\frac{v_{0x}^2}{2C_{v_x v_x}^E(\mathbf{0})} - \frac{v_{0y}^2}{2C_{v_y v_y}^E(\mathbf{0})} - \frac{\sigma_0^2}{2C_{\sigma\sigma}^E(\mathbf{0})} \right). \end{aligned} \quad (22)$$

The probability of realization of the velocity  $\mathbf{v}$  in a position  $\boldsymbol{\chi}(\tau)$  for the corresponding subensemble is

$$\begin{aligned} P(\mathbf{v}(\tau), \boldsymbol{\chi}(\tau); \sigma_0, \mathbf{v}_0) &\approx \prod_{i=x,y} \frac{1}{\sqrt{2\pi \Sigma_i(\boldsymbol{\chi}(\tau); \sigma_0, \mathbf{v}_0)}} \times \\ &\times \exp \left( -\frac{(v_i - \Upsilon_i(\boldsymbol{\chi}(\tau); \sigma_0, \mathbf{v}_0))^2}{2\Sigma_i(\boldsymbol{\chi}(\tau); \sigma_0, \mathbf{v}_0)} \right). \end{aligned} \quad (23)$$

The value

$$\begin{aligned} \Upsilon_i(\boldsymbol{\chi}(\tau); \sigma_0, \mathbf{v}_0) &= \sigma_0 \frac{C_{v_i \sigma}^L(\boldsymbol{\chi}(\tau))}{C_{\sigma\sigma}^E(\mathbf{0})} + \\ &+ \sum_{j=x,y} v_{0j} \frac{C_{v_i v_j}^L(\boldsymbol{\chi}(\tau))}{C_{v_j v_j}^E(\mathbf{0})}, \end{aligned} \quad (24)$$

is determined by the unknown Lagrangian correlation functions (18) and (19). It corresponds formally to the most probable velocity in a subensemble. The dispersion is

$$\Sigma_i(\boldsymbol{\chi}(\tau); \sigma_0, \mathbf{v}_0) = C_{v_i v_i}^E(\mathbf{0}) - \sum_{\rho=\sigma, v_x, v_y} \frac{(C_{v_i \rho}^E(\boldsymbol{\chi}(\tau)))^2}{C_{\rho\rho}^E(\mathbf{0})}. \quad (25)$$

The substitution of Eqs. (20)–(25) in the weighted Lagrangian correlation function (15) leads to an expression, where the partial Lagrangian correlation function of velocity components for a subensemble is

$$\begin{aligned} C_{v_i v_i}^L(\tau) &\approx \int d\sigma_0 d\mathbf{v}_0 P_0(\sigma_0, \mathbf{v}_0) \times \\ &\times C_{v_i v_i}^{Lc}(\boldsymbol{\chi}(\tau); \sigma_0, \mathbf{v}_0). \end{aligned} \quad (26)$$

Here, the partial Lagrangian correlation function of a velocity component for a subensemble is

$$\begin{aligned} C_{v_i v_i}^{Lc}(\tau; \sigma_0, \mathbf{v}_0) &= v_{0i} \int d\mathbf{v} v_i P(\mathbf{v}(\tau), \boldsymbol{\chi}(\tau); \sigma_0, \mathbf{v}_0) = \\ &= v_{0i} \Upsilon_i(\boldsymbol{\chi}(\tau); \sigma_0, \mathbf{v}_0). \end{aligned} \quad (27)$$

Equation (21) is a consequence of the first assumption of the method of decorrelation trajectories: the existence of subensembles by the initial values of random fields. It describes the dependence of the weighted Lagrangian correlation function of a velocity component on the parameters of subensembles.

It is important to point out that Eq.(21) is in agreement with Eq. (3), when the Lagrangian correlation functions of Eq.(24) are independent of the parameters of a subensemble. This behavior is a consequence of the neglect of higher cumulants in the averaging (20).

It is worth to note that the most probable velocity (24) for a subensemble depends on the unknown trajectories of particles. These trajectories cannot be determined by the approximation of a Gaussian displacement of trajectories (12), because it leads to the asymptotically infinite diffusion coefficient for a static random field. At the same time, according to work [8], the assumption of statistical independence of the trajectories of particles on the random field (9) does not lead to a wrong asymptotic scaling law for the diffusion coefficient. Thus, we will use this assumption and perform the integration in Eq.(11) before the averaging over particles' trajectories:

$$C_{v_i v_j}^L(\tilde{\tau}) = \left\langle \int d\mathbf{x} C_{v_i v_i}^E(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}(\tau)) \right\rangle \approx \approx \langle C_{v_i v_j}^E(\mathbf{x}(\tau)) \rangle. \quad (28)$$

The unknown correlation functions (18) and (19) appear in the definition of a partial Lagrangian correlation function, and we suppose that, for all trajectories in the subensemble, condition (14) is satisfied. According to the second assumption of the method of decorrelation trajectories, the dynamics in each subensemble can be described by a characteristic trajectory. Further, we approximate the Eulerian correlation function of random fields, which is averaged over particles' trajectories, as the Eulerian correlation function along a characteristic trajectory

$$C_{v_i v_j}^L(\tilde{\tau}) \approx C_{v_i v_j}^E(\langle \mathbf{x}(\tau) \rangle). \quad (29)$$

Let us consider condition (14) defining the characteristic trajectory. It is formulated for one point of the random field and means the existence of the infinite number of random fields that satisfy this condition. The description of the statistical behavior of the infinite number of correlated particles according

to Eqs. (26) and (27) is defined by the most probable velocity in subensemble (24)), which depends on the unknown characteristic trajectory  $\langle \mathbf{x}(\tau) \rangle$ . There is no way to obtain it in a strict mathematical way. According to the second assumption of the method of decorrelation trajectories, it is natural to approximate this trajectory by a trajectory along the most probable velocity

$$\langle \mathbf{x}(\tau; \sigma(\mathbf{0}) = \sigma_0, \mathbf{v}(\mathbf{0}) = \mathbf{v}_0) \rangle \approx X_i(\tau; \sigma_0, \mathbf{v}_0). \quad (30)$$

This trajectory is called the spatial decorrelation trajectory or the decorrelation trajectory and is defined by the equation

$$\frac{d}{d\tau} X_i(\tau; \sigma_0, \mathbf{v}_0) = \Upsilon_i(\mathbf{X}(\tau; \sigma_0, \mathbf{v}_0); \sigma_0, \mathbf{v}_0) \quad (31)$$

which closes up our system for the method of decorrelation trajectories. The interpretation of the decorrelation trajectories (31) follows from the definition of diffusion coefficient (5) and the partial Lagrangian correlation function (27)

$$D_i(\tau; \sigma_0, \mathbf{v}_0) = \int_0^\tau d\tilde{\tau} C_{v_i v_i}^{L_c}(\tilde{\tau}; \sigma_0, \mathbf{v}_0) = v_{0i} X_i(\tau; \sigma_0, \mathbf{v}_0^0), \quad (32)$$

and represents the partial diffusion coefficient.

There are different ways to split up the ensemble to subensembles [8], for example, by the initial values of random potential, and different ways to close the equations, for example, by considering the decay of the Lagrangian correlation function in the time – the so-called space-time decorrelation trajectories. All of them give similar results, but there are the decorrelation trajectories with subensembles (14) by the initial values of potential and velocity with spatial decorrelation trajectories (31), which reproduces better the results of numerical simulations [6]. The authors of work [8] noted that the decorrelation trajectories are not the averaged trajectory of particles in the subensemble. According to our interpretation, these trajectories are the partial diffusion coefficients (32) for the correspondent subensembles.

The time evolution of the Lagrangian correlation function of a velocity component for an explicitly time-dependent random potential was obtained by the method of decorrelation trajectories in work [8] for Kubo numbers  $K = 4$  and  $K = 160$ . It reproduces qualitatively, but not quantitatively, the correlation function obtained by the numerical simulation [6]. The

obtained asymptotic scaling law for the diffusion coefficient appears to be correct in a wide range of Kubo numbers: the method is in agreement with the quasi-linear theory for the small values of Kubo number  $K < 1$ . For large Kubo numbers,  $K > 1$ , it reproduces the percolation scaling law with good accuracy.

There was no comparison between the method of decorrelation trajectories and the results of numerical simulations for a static random velocity field in work [8]. But the authors stated the possibility of the reproduction of a subdiffusive time evolution and the asymptotically zero diffusion coefficient by the proposed method because of fact that the decorrelation trajectories are finite closed curves for  $\sigma_0 \neq 0$ , and the diffusion coefficient is an integral over the all decorrelation trajectories.

The main assumption of the method is a possibility to describe a characteristic evolution of particles in the subensemble in terms of the decorrelation trajectory (31), as pointed out in work [8]. These deterministic trajectories are the partial diffusion coefficients (32) according to the mentioned interpretation. So, the method of decorrelation trajectories closes the system of equations, using the diffusion coefficient instead of the mean square displacement, as the Corrsin approximation does.

Decorrelation trajectories  $X_i(\tau; \sigma_0, \mathbf{v}_0)$  (31) are dependent on three parameters of the subensemble  $\sigma_0, \mathbf{v}_0$ , which are used in the further averaging in Eq. (21) as continuous values. This correlation function with continuous averaging is approximated by the discrete sum with a finite number of subensembles in numerical calculations. The asymptotic scaling law for the diffusion coefficient for a static random potential can depend on the number of decorrelation trajectories. Thus, it is important to analyze the dependence of decorrelation trajectories on the parameters of subensembles and to validate the closure, using the diffusion coefficient, by comparison with the results of direct numerical simulations.

### 5. Moment Approximation

There is another method proposed in work [10] to approximate the Lagrangian correlation function of velocity components for an isotropic frozen turbulence. The moment approximation uses the first assumption of the Corrsin approximation (9), but it closes the equations in another way. The mean square displacement is an important statistical character-

istic that describes a displacement of trajectories, when the average displacement of Lagrangian particles tends to zero in isotropic frozen turbulence.

Let us consider an integral form of the Lagrangian correlation function of a velocity component (8) with the first assumption of the Corrsin approximation (9) as

$$C_{v_i v_i}^L(\tau) = \langle C_{v_i v_i}^E(\boldsymbol{\chi}(\tau)) \rangle. \quad (33)$$

We define Eulerian correlation function along the velocity for isotropic random fields

$$C_{\mathbf{v}\mathbf{v}}^E(\boldsymbol{\chi}) = \sum_{i=x,y} C_{v_i v_i}^E(\boldsymbol{\chi}) = C_{\mathbf{v}\mathbf{v}}^E(|\boldsymbol{\chi}|). \quad (34)$$

This correlation function depends on the absolute value of difference between two sequent particle's trajectory values. The Lagrangian correlation function of velocity components is approximated by the Eulerian correlation function dependent on the mean square displacement for the isotropic frozen turbulence. Using assumption (29), we have

$$\begin{aligned} C_{v_i v_i}^L(\tau) &\approx \langle C_{v_i v_i}^E(|\boldsymbol{\chi}(\tau)|) \rangle \approx C_{v_i v_i}^E(\langle |\boldsymbol{\chi}(\tau)| \rangle) \approx \\ &\approx C_{v_i v_i}^E(\sqrt{\Delta(\tau)}). \end{aligned} \quad (35)$$

The time evolutions for the Lagrangian correlation function of velocity components, diffusion coefficient, and mean square displacement were obtained in work [10] and found to be in agreement with the results of direct numerical simulations. It is pointed out that the moment approximation has no free choices in contrast to the method of decorrelation trajectories.

### 6. Numerical Simulation

We study the motion of magnetized test particles in a random frozen electrostatic field in a plane perpendicular to a constant magnetic field. Particle's trajectories satisfy the drift motion equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} = \frac{e}{m} \frac{[\mathbf{E}(\mathbf{x}) \times \mathbf{e}_B]}{\Omega_B}, \quad (36)$$

where  $\Omega_B = \frac{eB}{mc}$ , and field is defined as a superposition of  $N$  harmonic waves with a Gaussian weight and an amplitude of the potential  $\phi_0$

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= -\frac{\partial}{\partial \mathbf{x}} \phi(\mathbf{x}) = -A\phi_0 \times \\ &\times \sum_{s=1}^N \mathbf{k}_s \exp\left(-\frac{1}{2} \left(\frac{\mathbf{k}_s}{\Delta k}\right)^2\right) \sin(-\mathbf{k}_s \mathbf{x} + \alpha_s). \end{aligned} \quad (37)$$

The set of wave vectors is given by  $N_\kappa$  different absolute values in the interval  $(0, k_{\max})$  in  $N_\theta$  different directions with dispersion  $\Delta k$

$$\begin{aligned} \mathbf{k}_s &= k_l \mathbf{e}_m, \quad \mathbf{e}_m = (\cos(\theta_m), \sin(\theta_m)); \\ k_l &= l \delta k = l \frac{k_{\max}}{N_\kappa}, \quad l = 1, N_\kappa; \\ \theta_m &= m \frac{2\pi}{N_\theta}, \quad m = 1, N_\theta; \\ N &= N_\kappa \times N_\theta. \end{aligned} \quad (38)$$

The normalization parameter is defined as

$$A = \sqrt{\frac{4k_{\max}}{\Delta k \sqrt{\pi} N_\kappa N_\theta}}, \quad (39)$$

by the condition  $\langle \mathbf{E}(\mathbf{x}) \mathbf{E}(\mathbf{x}) \rangle = 1$ . Each realization of the random field  $\mathbf{E}(\mathbf{x})$  is defined by the set of random phases  $\{\alpha_i\}$ .

We use the dimensionless spatial  $\boldsymbol{\chi} = \mathbf{x} \Delta k / (2\pi)$  and time  $\tau = t \sigma \Omega_B / (2\pi)$  variables for a convenient numerical simulation and the dimensionless amplitude of the potential  $\sigma = e \phi_0 \delta k^2 / (m \Omega_B^2)$ . The substitution of these variables in Eq. (36) gives us

$$v_i = \frac{d\chi_i}{d\tau} = \epsilon_{ik} \frac{\partial}{\partial \chi_k} \sigma(\boldsymbol{\chi}), \quad (40)$$

where  $\epsilon_{ik}$  denotes the antisymmetric second-rank tensor  $\epsilon_{xy} = -\epsilon_{yx} = 1$ . The isotropic dimensionless potential  $\sigma(\boldsymbol{\chi})$  looks as

$$\begin{aligned} \sigma(\boldsymbol{\chi}) &= \sqrt{\frac{4\kappa_{\max}}{\sqrt{\pi} N_\kappa N_\theta}} \frac{1}{2\pi} \times \\ &\times \sum_{s=1}^N \exp(-\kappa_s^2/2) \cos(\alpha_s - 2\pi \boldsymbol{\kappa}_s \boldsymbol{\chi}), \end{aligned} \quad (41)$$

where  $\boldsymbol{\kappa}_s = \mathbf{k}_s / \Delta k$  represents a set of dimensionless wave vectors.

The Eulerian correlation function of a potential in the continuous limit  $N \rightarrow \infty$  is obtained as

$$\begin{aligned} C_{\sigma\sigma}^E(\boldsymbol{\chi}) &= \langle \sigma(\boldsymbol{\chi} - \boldsymbol{\chi}_2) \sigma(\boldsymbol{\chi}_2) \rangle = \\ &= \frac{1}{4\pi^2} \exp\left(-\frac{\pi^2 \boldsymbol{\chi}^2}{2}\right) I_0\left(\frac{\pi^2 \boldsymbol{\chi}^2}{2}\right), \end{aligned} \quad (42)$$

and depends on  $\boldsymbol{\chi}^2$  only, i.e. it is isotropic. The Eulerian correlation functions of velocity components or the potential and velocity components are defined as the derivatives of the correlation function of potential (42)

$$C_{v_i \sigma}^E(\boldsymbol{\chi}) = -\epsilon_{ik} \frac{\partial}{\partial \chi_k} C_{\sigma\sigma}^E(\boldsymbol{\chi}), \quad (43)$$

$$C_{v_i v_j}^E(\boldsymbol{\chi}) = -\epsilon_{ik} \epsilon_{jm} \frac{\partial^2}{\partial \chi_k \partial \chi_m} C_{\sigma\sigma}^E(\boldsymbol{\chi}). \quad (44)$$

The integration of the system of equations (40) for different realizations gives us particle's trajectories. Their further averaging over  $N_r$  realizations of random fields allows us to obtain the mean displacement

$$\langle \boldsymbol{\chi} \rangle(\tau) = \frac{1}{N} \sum_{i=0}^{N_r} \boldsymbol{\chi}_{\{\alpha\}_i}(\tau),$$

and the mean square displacement

$$\langle \boldsymbol{\chi}^2 \rangle(\tau) = \frac{1}{N} \sum_{i=0}^{N_r} (\boldsymbol{\chi}_{\{\alpha\}_i}(\tau) - \langle \boldsymbol{\chi} \rangle(\tau))^2.$$

Using similar expressions, we can obtain higher cumulants as well. The fourth cumulant indicates an important deviation of the probability distribution function from the Gaussian one.

We choose  $N = 1440$  partial harmonic waves ( $N_\kappa = 20$ ,  $N_\theta = 72$ ) in the numerical simulation, using considerations of a smooth enough behavior of the correlation function of the potential  $C_{\sigma\sigma}^E(\boldsymbol{\chi})$ . The maximal absolute value of dimensionless wave vector in the numerical simulation is limited to the value  $\kappa_{\max} = 2$ . There are  $N = 1440$  random phases generated for each realization of the random field. The numbers of realizations  $N_r$  for a run are noted in the subscriptions to the images.

## 7. Analytical Models

It is the Lagrangian correlation function of velocity components that appears to be a key value in the chosen (two-dimensional incompressible static random velocity field) formulation of the problem. It describes the time evolution and the asymptotic scaling law for the transport coefficients. Using general considerations, it is obvious that the asymptotic value of diffusion coefficient is zero in the limit of a strong Lagrangian nonlinearity  $K \rightarrow \infty$ . This also means that, in order to have a zero integral over the time, the Lagrangian correlation function of velocity components should have a long interval of negative values for large times, because it has a positive values at the start. Such behavior of the Lagrangian correlation function reflects particle trapping effects. Particles are moving along the closed trajectories that are fully



determined by the spatial structure of a random potential. The analytical approximations are needed to reproduce this behavior of the correlation function correctly.

The evolution equations for the mean square displacement in the Corrsin approximation and in the moment approximation are obtained, by using the Taylor relation (4) after the substitution the approximate Lagrangian correlation functions (13) in the right-hand side. In the Corrsin approximation, the equation is as follows:

$$\frac{\partial^2}{\partial t^2} \Delta = \frac{\sqrt{2\pi}}{[1 + 2\pi^2 \Delta]^{3/2}}. \quad (45)$$

In the moment approximation, along with Eq. (35), we have

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \Delta &= \frac{\sigma^2}{2} \exp\left(-\frac{\pi^2 \Delta}{2}\right) \left( I_0\left(\frac{\pi^2 \Delta}{2}\right) (1 - \pi^2 \Delta) + \right. \\ &+ \left. I_1\left(\frac{\pi^2 \Delta}{2}\right) \pi^2 \Delta \right). \end{aligned} \quad (46)$$

These equations are closed up, using the same mean square displacement  $\Delta$ , but in different ways. In contrast to these approximations, the equation in the method of decorrelation trajectories is closed up by the partial diffusion coefficient  $X_{i,j}$ :

$$\begin{aligned} \frac{d}{d\tau} X_i &= \exp\left(-\frac{(\pi X)^2}{2}\right) I_1\left(\frac{(\pi X)^2}{2}\right) \left[ \pi^2 \sigma_0 \epsilon_{ij} X_j + \right. \\ &+ v_{0j} \left( 2\pi^2 X_j X_i + \frac{2X_j X_i}{X^2} - 1 \right) + \\ &+ \left. v_{0i} \left( 2\pi^2 X_j^2 + \frac{2X_j^2}{X^2} - 1 \right) \right] + \\ &+ \exp\left(-\frac{(\pi X)^2}{2}\right) I_0\left(\frac{(\pi X)^2}{2}\right) \left[ v_{0i} (1 - 2\pi^2 X_j^2) - \right. \\ &- \left. \pi^2 \sigma_0 \epsilon_{ij} X_j + v_{0j} 2\pi^2 X_i X_j \right], X^2 = \sum_{i=x,y} X_i^2. \end{aligned} \quad (47)$$

Ensemble's diffusion coefficient is obtained by the averaging of partial diffusion coefficients over all subensembles with a corresponding probability of realization of a subensemble

$$\begin{aligned} D_i &= \int d\sigma_0 d\mathbf{v}_0 \frac{v_{0i} X_i}{\sqrt{(2\pi)^3 C_{\sigma\sigma}^E(0) C_{v_x v_x}^E(0) C_{v_y v_y}^E(0)}} \times \\ &\times \exp\left(-\frac{\sigma_0^2}{2C_{\sigma\sigma}^E(0)} - \sum_{j=x,y} \frac{v_{0j}^2}{2C_{v_j v_j}^E(0)}\right). \end{aligned} \quad (48)$$

The method of decorrelation trajectories and the moment approximation don't use the second assumption of the Corrsin approximation (12) about a Gaussian displacement for particles' trajectories. These approximations describe the evolution of particles' trajectories, using the averaged characteristics: partial diffusion coefficients or mean square displacement. The method of decorrelation trajectories is based on the idea to consider the known initial values to describe the motion of particles. At the same time, the functional form of the Lagrangian correlation functions is independent of subensembles' parameters: the equations for the partial diffusion coefficient for different subensembles differs by the value of  $\sigma_0, \mathbf{v}_0$ .

During numerical calculations in the method of decorrelation trajectories, we need to switch to a discrete sum over subensembles instead of the continuous integration in Eq. (48). It is natural to suppose that the approximation will reproduce the numerical simulation results better with more subensembles taken into account. Let us look at right-hand sides of Eqs. (45)–(47). The right-hand side of Eq. (45) is positive for any values of argument, while, for Eqs. (46) and (47), it can be negative. That is why we can immediately conclude that the Corrsin approximation (45) is not able to reproduce the zero asymptotic diffusion coefficient and the subdiffusive evolution. The reason is the assumption of a Gaussian displacement for particles' trajectories, which is absent in the method of decorrelation trajectories and in the moment approximation. Let us demonstrate that, in the moment approximation, the Lagrangian correlation function of velocity components (35) tends to zero, as the mean square displacement tends to infinity  $\tau \rightarrow \infty, \Delta(\tau) \rightarrow \infty$  from negative values

$$\begin{aligned} \lim_{\tau \rightarrow \infty} C_{v_i v_i}^L(\tau) &= \lim_{\Delta \rightarrow \infty} \frac{1}{2} \exp\left(-\frac{\pi^2 \Delta}{2}\right) \times \\ &\times \left( I_0\left(\frac{\pi^2 \Delta}{2}\right) (1 - \pi^2 \Delta) + I_1\left(\frac{\pi^2 \Delta}{2}\right) \pi^2 \Delta \right) = \\ &= \lim_{\Delta \rightarrow \infty} \frac{-1}{4\pi^3 \Delta \sqrt{\pi \Delta}} = 0, \end{aligned} \quad (49)$$

where we have used the asymptotics of the modified Bessel function with  $|\arg z| < \pi/2, |z| \rightarrow \infty, \mu = 4\nu^2$

$$I_\nu(z) \sim \frac{\exp(z)}{\sqrt{2\pi z}} \left( 1 - \frac{\mu - 1}{8z} + \frac{(\mu - 1)(\mu - 9)}{2! (8z)^2} - \dots \right).$$

The diffusion coefficient obtained by the integration of this correlation function also tends to zero, as  $\tau \rightarrow \infty$ ,  $\Delta(\tau) \rightarrow \infty$ . Using the first integral of motion, we have

$$\begin{aligned} \lim_{\tau \rightarrow \infty} D(\tau) &= \lim_{\Delta \rightarrow \infty} \left[ \Delta \exp\left(-\frac{\pi^2 \Delta}{2}\right) \right]^{1/2} \times \\ &\times \left[ I_0\left(\frac{\pi^2 \Delta}{2}\right) - I_1\left(\frac{\pi^2 \Delta}{2}\right) \right]^{1/2} = \\ &= \lim_{\Delta \rightarrow \infty} \frac{1}{(4\pi^7 \Delta)^{1/4}} = 0, \end{aligned} \quad (50)$$

where the mentioned asymptotics for the modified Bessel is used.

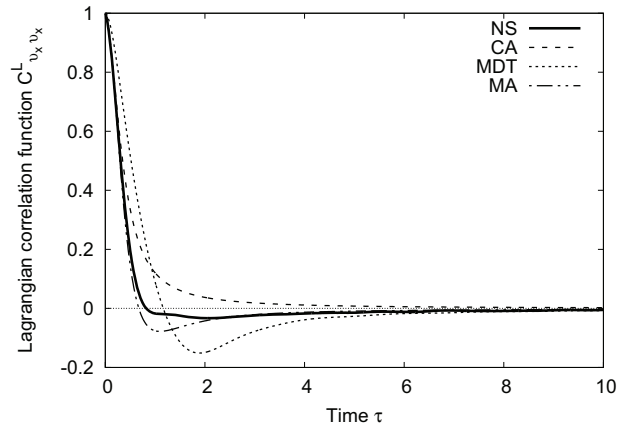
The properties of decorrelation trajectories are studied in Section 9.

### 8. Results of Numerical Simulations

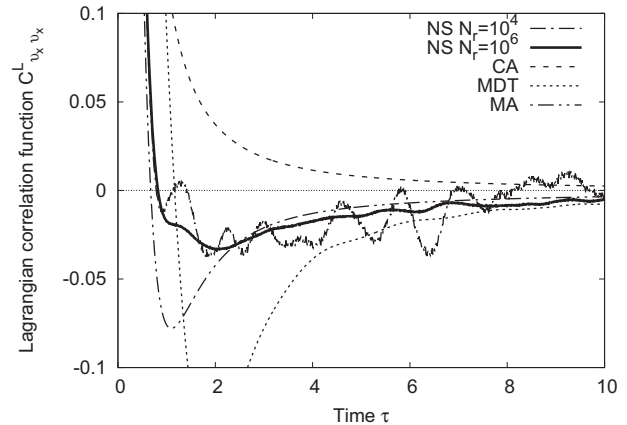
Let us present here the results of numerical simulations for the Lagrangian correlation function of velocity components and the mean square displacement. The Lagrangian correlation function of velocity components for small times obtained by numerical simulations and by three analytic approximations is presented in Fig. 1. The Corrsin approximation cannot reproduce the negative values of Lagrangian correlation function, while the method of decorrelation trajectories and the moment approximation qualitatively demonstrate this feature. Moreover, the moment approximation gives a more accurate result.

The behavior of the correlation function obtained by numerical simulations is influenced by fluctuations, which are caused by a finite number of realizations and a finite number of waves. The more the number of realizations, the less the fluctuations obtained, which is demonstrated in Fig. 2. There are also such important characteristics of the diffusion process as the diffusion coefficient and the mean square displacement. They defined as integrals of the correlation function over the time; thus, they are much less influenced by fluctuations.

The time evolution of the mean square displacement is presented in Fig. 3 and gives a possibility to study the quantitative difference between the results of the numerical simulation and analytic approximations for larger times. The Corrsin approximation gives the incorrect time evolution for the mean square



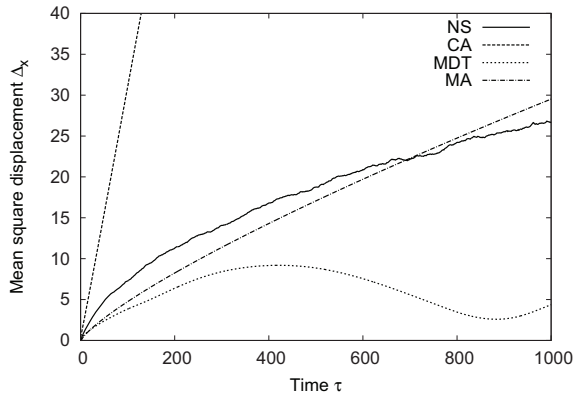
**Fig. 1.** Lagrangian correlation function of velocity components obtained by numerical simulations (NS) (number of realizations  $N_r = 2 \times 10^6$ ), Corrsin approximation (CA), method of decorrelation trajectories (MDT) (number of subensembles  $N_s = 1728 \times 10^3$ ), and moment approximation (MA). The region of negative values of correlation function represents the effect of particles' trapping



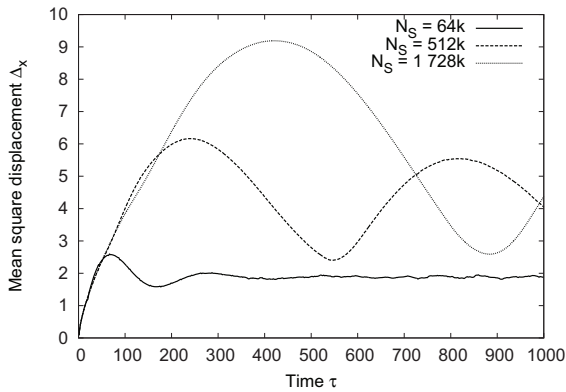
**Fig. 2.** Scaled part of Fig. 1 of the Lagrangian correlation function of velocity components. It demonstrates that much more realizations are needed in numerical simulations (raised up from  $10^4$  to  $10^6$ ) to obtain a bit more persistent behavior of the tail of the correlatioin function in the negative region

displacement: it grows faster, than a linear function in time. The moment approximation qualitatively reproduces the behavior obtained in the numerical simulation – there is a subdiffusion process.

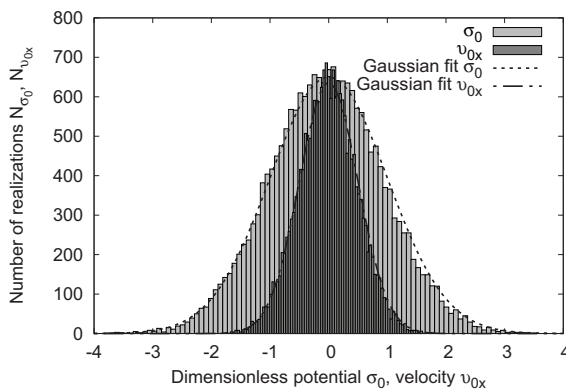
The method of decorrelation trajectories gives a significant fluctuation of the mean square displacement caused by a finite number of subensembles. Fig. 4 demonstrates the time evolution of the mean square displacement obtained by the method



**Fig. 3.** Mean square displacement obtained by a numerical simulation (NS) (number of realizations  $N_r = 2 \times 10^4$ ), the Corrsin approximation (CA), method of decorrelation trajectories (MDT) (number of subensembles  $N_s = 1728 \times 10^3$ ), and moment approximation (MA).



**Fig. 4.** Mean square displacement obtained by the method of decorrelation trajectories for different numbers of subensembles  $N_s = 64 \times 10^3$ ,  $512 \times 10^3$ ,  $1728 \times 10^3$



**Fig. 5.** Distribution over the initial values of potential  $\sigma_0$  and velocity  $v_{0x,y}$  using Eqs. (41) and (40) for a  $N_r = 2 \times 10^4$  realizations: it is close to the Gaussian one

of decorrelation trajectories for different numbers of subensembles  $N_s$ . Such a dependence on the number of subensembles can be less obvious for a nonstationary turbulence [8] because of the additional exponential dumping of correlations caused by it.

The dependence of the results of the method of decorrelation trajectories on the non-physical parameter of the number of subensembles demands for a detailed analysis. It is useful to check the distribution of subensembles in a direct numerical simulation. We are also interested in the formulation of a condition for a subensemble for a direct numerical simulation and in the testing of the uniqueness of particle's dynamics for different subensembles. Further, we will analyze the properties of decorrelation trajectories, their dependence on subensemble's parameters, and their contribution to the ensemble average.

### 9. Subensembles

The method of decorrelation trajectories is based on the assumption of the possibility of a splitting of the ensemble of realizations into subensembles by the set of the initial values of random fields (14). The probability of their realization is of Gaussian type (22). According to the assumption of the method of decorrelation trajectories, each subensemble describes a process of dumping of correlations in the corresponding group of particles (14) in terms of the partial diffusion coefficient.

Let us look at the probability distribution for the initial values of random potential and velocity components in a numerical simulation, whose results were presented in the previous section. As shown in Fig. 5, the distribution isn't identical to a continuous Gaussian distribution from the method of decorrelation trajectories. But we can consider that the discrete distribution obtained by a numerical simulation tends to the Gaussian one for the infinite number of realizations.

It is important to verify the assumption of unique dynamics for different subensembles within the method of decorrelation trajectories by a numerical simulation. We can calculate the partial diffusion coefficients or correspondent mean square displacements for different subensembles and relate them to one another. According to the method of decorrelation trajectories, the difference in the dynamics of these particles, which are described by the partial dif-

fusion coefficient (31) and defined by the initial values of random fields, can be sufficiently high:

$$\frac{\Delta_x(\tau; \sigma_0, \mathbf{v}_0)}{\Delta_x(\tau; \alpha\sigma_0, \alpha\mathbf{v}_0)} = \frac{\Delta_x(\tau; \sigma_0, \mathbf{v}_0)}{\Delta_x(\alpha\tau; \sigma_0, \mathbf{v}_0)}. \quad (51)$$

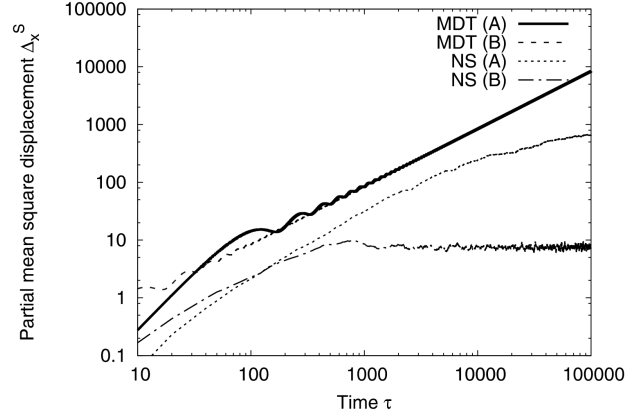
The results of numerical simulations for the mean square displacement for two different subensembles  $\{\sigma_0, v_{0x}, v_{0y}\}$ : (A) $\{0.0032, 0.01, 0.01\}$  and (B) $\{0.032, 0.1, 0.1\}$  are presented in Fig. 6. Equations (31) and (32) give the mean square displacement growing in time with oscillations for the subensembles. It can be seen in Fig. 6 that the mean square displacements for the subensembles tend to infinity in both subensembles and have a similar tangent. The time dependences differs by the period and the amplitude of oscillations only. At the same time, the mean square displacement for the different subensembles obtained by numerical simulations grows much slowly, and there is no oscillations. Moreover, the time dependences of the mean square displacement for different subensembles differ significantly: for the subensemble (B) $\{0.032, 0.1, 0.1\}$ , it tends to some constant with fluctuations at the time  $\tau = 1000$ . For the subensemble (A) $\{0.0032, 0.01, 0.01\}$ , it is growing during all the time of the simulation.

The given results of the numerical simulation lead us to the conclusion that the mean square displacement is different for different subensembles. At the same time, the results obtained by the method of decorrelation trajectories and found in the numerical simulation differ drastically.

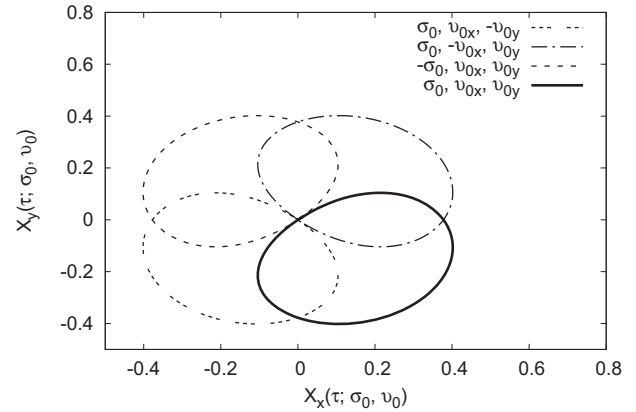
So, the subensembles with different particle's dynamics actually exist, but the method of decorrelation trajectories is not sufficient to describe such dynamics correctly. The reason for this can be different: this can be caused by involving only the second cumulants during the averaging (21) or in the closure of the equations using the partial diffusion coefficient (31). Now, we are going to study the closure procedure for decorrelation trajectories.

## 10. Decorrelation Trajectories

Before we study the closure procedure, we need analyze in depth the dependence of partial diffusion coefficients on subensemble's parameters. In particular, the analysis of Eq. (47) gives that we have three different classes of trajectories according to subensembles' parameters: 1) periodic ones:  $\{\sigma_0 \neq 0, v_{0x,y} \neq$



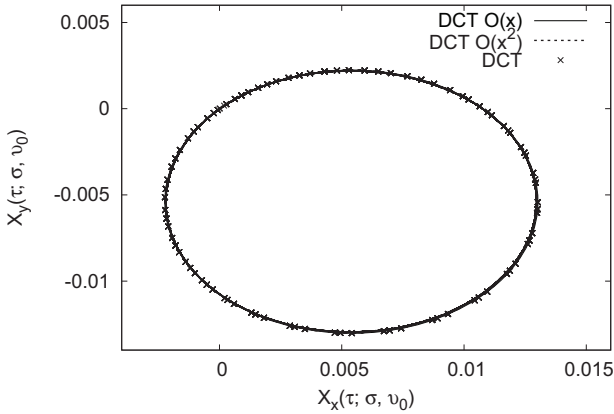
**Fig. 6.** Mean square displacement in subensembles by  $\{\sigma_0, v_{0x}, v_{0y}\}$ : (A) $\{0.0032, 0.01, 0.01\}$  and (B) $\{0.032, 0.1, 0.1\}$  obtained by a numerical simulation (NS) (number of realizations  $N_r = 10^3$ ) and the method of decorrelation trajectories (MDT) for large times



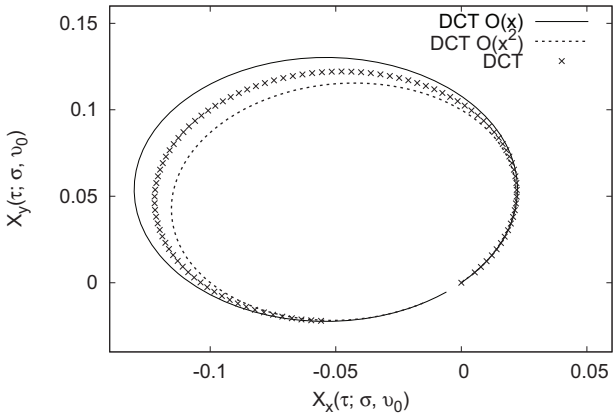
**Fig. 7.** Symmetries of decorrelation trajectories, when the sign of parameters of subensembles is changed  $\{\sigma_0, v_{0x}, v_{0y}\}$

$\neq 0\}$ , with the subensemble density  $\sim 1$ ; 2) open ones:  $\{\sigma_0 = 0, v_{0x,y} \neq 0\}$ , with the subensemble density  $\sim N_s^{-1/3}$ ; 3) zero ones:  $\{\sigma_0 \neq 0, v_{0x,y} = 0\}$ , with the subensemble density  $\sim N_s^{-2/3}$ . The first type of trajectories giving a leading contribution to the diffusion coefficient are periodic trajectories with different traveling velocities along them. That's why we will further consider decorrelation trajectories as periodic trajectories only.

We present four decorrelation trajectories for the different parameters of subensembles  $\sigma_0, \mathbf{v}_0$  in Fig. 7. Changing the sign of the initial velocity  $v_{0x,y}$  along a decorrelation trajectory reflects on the correspondent axis, while the form of the trajectory is constant. So



**Fig. 8.** Exact (DCT) and approximated (DCTO(X)), (DCTO(X<sup>2</sup>)) solutions of the equation for decorrelation trajectories when  $|\mathbf{X}| \sim 0.01$



**Fig. 9.** Exact (DCT) and approximated (DCTO(X)), (DCTO(X<sup>2</sup>)) solutions of the equation for decorrelation trajectories when  $|\mathbf{X}| \sim 0.1$

we can suppose that a decorrelation trajectory is an odd function of the initial velocity. In the same way, we see that the change of a sign of the initial potential leads to the inversion of a trajectory. When we rescale the subensemble parameters  $\sigma_0, \mathbf{v}_0$ , the trajectory doesn't change its shape. Instead, the speed of bypass is changed, since it is determined by the initial potential  $\sigma_0$ . Exactly in this way, the partial dispersion coefficients are related to each other in Fig. 5. We note that the size of the closed trajectory area is determined by the relation of the initial velocity to the initial potential.

Because of the closed form of decorrelation trajectories and the fact that their sizes are determined by the relation  $\mathbf{v}_0/\sigma_0$ , we can approximate the right-

hand side of the equation of a decorrelation trajectory (31), as an expansion of the first order in  $X_{x,y}$ . The approximated equation can be solved, and we obtain the decorrelation trajectory of the first order in  $X_{x,y}$ :

$$X_i = \frac{v_{0i}}{\pi^2 \sigma_0} \sin(\pi^2 \sigma_0 t) + \epsilon_{ij} \frac{v_{0j}}{\pi^2 \sigma_0} (1 - \cos(\pi^2 \sigma_0 t)). \quad (52)$$

In the same way, we can obtain an approximated decorrelation trajectory in the second order in  $X_{x,y}$ . We demonstrate the approximated and exact decorrelation trajectories for two different subensembles in Figs. 8 and 9. When  $|\mathbf{X}| \sim 0.01$ , we have a good convergence of the approximated and exact solutions with the relative error  $\epsilon_{0.01} \approx 0.06\%$ . For  $|\mathbf{X}| \sim 0.1$ , we have the larger relative error  $\epsilon_{0.1} \approx 7.6\%$ . This deviation between the exact and approximated solutions grows faster than a linear function in time. It is worth to note that the first-order approximation is covers the exact solution, and the second-order one is covered by exact solution. The obtained approximated decorrelation trajectories have all the properties of the exact solution: the dependence of the size of a trajectory on the relation  $\mathbf{v}_0/\sigma_0$ , the speed of bypass of a trajectory depends on  $\sigma_0$  only, and, what is most important, the odd dependence on subensembles' parameters  $\sigma_0, \mathbf{v}_0$ .

The above consideration leads us to the idea of that not all of the terms in Eq. (31) are important. Some part of them will be eliminated during the averaging over the subensembles (48). We have no possibility to perform an continuous averaging over a subensemble in Eq. (48) and obtain the time evolution of the diffusion coefficient in a limit of the infinite number of subensembles  $N_s \rightarrow \infty$ , because we have no exact analytical solution of Eq. (31) for a decorrelation trajectory. That is why in order to purge the dependence of the mean square displacement on the number of subensembles in the method of decorrelation trajectories, we perform the averaging over the subensembles in Eq. (48) before the closure of the equations using the diffusion coefficient. At the same time, we suppose that the symmetry of decorrelation trajectories shown in Fig. 7 is still present for all trajectories. In view of the assumption about the independence of averaged trajectories in a subensemble on the initial values of potential  $\sigma_0$  and velocity  $\mathbf{v}_0$ , we have

$$C_{v_i v_i}^L(\tau) = \langle v_i(0)v_i(\tau) \rangle. \quad (53)$$

We obtain the approximated Lagrangian correlation function in a similar way to Eqs. (28)–(31) as

$$C_{v_i v_i}^L(\tau) \approx C_{v_i v_i}^E(\tilde{X}(\tau)), \quad (54)$$

and we close our equation, by using the full diffusion coefficient

$$\tilde{X}_i = \frac{D_i}{\sqrt{C_{v_i v_i}^L(0)}}. \quad (55)$$

We chose a Lagrangian correlation function of velocity components as a normalizing value for the diffusion coefficient. This leads us to the final equation that determines the diffusion coefficient

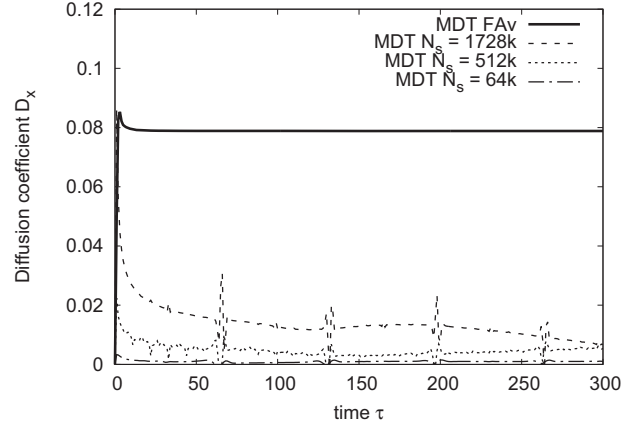
$$\begin{aligned} \frac{d}{d\tau} D_i &= \exp\left(-\frac{(\pi D)^2}{2}\right) I_0\left(\frac{(\pi D)^2}{2}\right) (1 - 2\pi^2 D_j^2) + \\ &+ \exp\left(-\frac{(\pi D)^2}{2}\right) I_1\left(\frac{(\pi D)^2}{2}\right) \left(2\pi^2 D_j^2 + \frac{2D_j^2}{D^2} - 1\right), \\ D^2 &= \sum_{i=x,y} D_i^2. \end{aligned} \quad (56)$$

The full diffusion coefficients obtained by the method of decorrelation trajectories for the different numbers of subensembles and by a modified one based on Eq. (56) are presented in Fig. 10. In the terms of the used assumptions, the diffusion coefficient obtained from Eq. (56) is a limit for the method of decorrelation trajectories, when  $N_s \rightarrow \infty$ . This solution demonstrates (56) that the diffusion coefficient doesn't asymptotically tend to zero. This shows that the way to close up the equations, by using the partial (31) or full (55) diffusion coefficient, is not enough for the frozen turbulence. It isn't an expected result that the method of decorrelation trajectories with a finite number of subensembles gives a slightly better result, when the diffusion coefficient fluctuates around zero value at large times.

At finite times, the method of decorrelation trajectories gives a satisfactory reproduction of the time evolution of the mean square displacement, while the number of subensembles is enough. That is why in the non-frozen case where the asymptotics is not important, it can give a satisfactory result.

## 11. Discussion and Conclusions

We compare three approximations of the Lagrangian correlation function in this work: the Corrsin approximation, method of decorrelation trajectories, and



**Fig. 10.** Diffusion coefficient obtained by a modified decorrelation method (MDT FAv) and by the decorrelation method (MDT  $N_s$ ) with different numbers of subensembles

moment approximation. They are used to calculate the diffusion of magnetized particles in a frozen electrostatic turbulent field. The results of these methods are verified using a direct numerical simulation. In contrast to the Corrsin approximation, the method of decorrelation trajectories gives a satisfactory evolution at the beginning, but it doesn't work for the asymptotics. The moment approximation reproduces the evolution of the Lagrangian correlation function of velocity components and the evolution of the mean square displacement obtained by a numerical simulation in a better way and for larger times.

The direct numerical simulation demonstrates that the idea of subensembles by the initial values of random fields is valid. But the description of the behavior of particles provided by the method of decorrelation trajectories doesn't correspond to the results of a numerical simulation. Moreover, the diffusion coefficient obtained by the method of decorrelation trajectories depends on the number of subensembles.

In order to validate the way to close up the equations of the method of decorrelation trajectories separately of the assumption about subensembles, we payed attention to the types and shapes of decorrelation trajectories. It is shown that some terms in Eq. (47) are not important and disappear during the averaging because of symmetry. The method of decorrelation trajectories is reformulated for the infinite number of subensembles considering the facts above. This method accounts only the important values and gives a possibility to see the influence of the

closure procedure solely. As a result, we saw that the closure by the diffusion coefficient in the method of decorrelation trajectories doesn't provide the evolution of the Lagrangian correlation function of velocity components that was obtained by the numerical simulation. Moreover, the asymptotic diffusion coefficient obtained by this method is nonzero.

So, the comparison of two approximations with a numerical simulation shows that the closure by the mean square displacement in the moment approximation is more adequate, than the closure by the diffusion coefficient in the method of decorrelation trajectories. It is also important to note that the moment approximation doesn't have any free choices or free parameters contrary to the method of decorrelation trajectories.

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ДИФУЗІЯ ЧАСТИНОК  
У ДВОВИМІРНОМУ ВИПАДКОВОМУ  
ПОЛІ ШВИДКОСТІ

Резюме

На основі детального аналізу порівняно два підходи до опису дифузії частинок у двовимірному випадковому полі швидкості, а саме метод декорельованих траєкторій та наближення моментами. Розглянуто заморожену турбулентність, оскільки вона є найбільш складним тестом для перевірки статистичних теорій. Результати аналітичних наближень порівняно з даними числового моделювання.