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PARAMETRIC EXCITATION OF SURFACE MAGNETOSTATIC MODES IN AN AXIALLY MAGNETIZED ELLIPTIC CYLINDER UNDER LONGITUDINAL PUMPING

A rigorous analytical theory of parametric excitation under the longitudinal pumping has been developed for the surface magnetostatic modes of a long elliptic ferrite cylinder magnetized along its axis with regard for the boundary conditions at the surface of the cylinder. It is shown that a pair of frequency-degenerated counter-propagating surface modes at half the pumping frequency can be parametrically excited, and the expressions for the corresponding parametric excitation threshold have been derived. The threshold demonstrates a strong dependence on the mode number and elliptic cylinder's aspect ratio and tends from above for the large aspect ratio to the value deduced on the basis of the plane-wave analysis. The simple analytical relation between the ratio of axes of the high-frequency magnetization polarization ellipse of excited surface magnetostatic oscillations and the parametric excitation threshold is obtained, discussed, and graphically illustrated.

Keywords: parametric processes, surface magnetostatic oscillations, elliptic cylinder, yttrium-iron garnet, film ferrite resonator.

1. Introduction

The parametric excitation of spin waves due to the intrinsic nonlinear properties of ferromagnetic materials plays a key role in practical applications. While the magnetization oscillations with small amplitudes can be safely analyzed in the linear approximation [1], the nonlinear properties of ferrite for relatively large amplitudes of the high-frequency magnetization lead to various nonlinear effects [2], including the parametric excitation (PE) of spin waves [3]. On the one hand, such phenomenon restricts the dynamic range of the input RF power of magnetostatic resonators. On the other hand, a number of nonlinear devices, such as a power limiter and a signal-to-noise enhancer are based on this effect [4]. Therefore, the careful examination of parametric excitation with regard for the specific features of a ferrite resonator and excitation conditions is of importance for the applied research.

Suhl [3] developed the basics of the PE theory, as applied to a transversely pumped isotropic ferromagnet. Subsequently, the theoretical model was improved to account for the arbitrary orientation and polarization of a microwave pumping field [5]. Finally, it was generalized for the first and second bands under the dual pumping of ferrite materials with either uniaxial or cubic magnetocrystalline anisotropy, by using arbitrary polarized and oriented RF fields [6].

However, Suhl's theory utilizes the expansion of the high-frequency magnetization in uniform plane spin waves, which is justified only when the wave number k of excited spin waves is much larger than the inverse dimensions of a sample. But, for thin ferrimagnetic films with thickness of the order of a few to a few tens of microns, the typical wave numbers of resonator eigen-excitations – surface magnetostatic oscillations (SMSO) – are much less than the inverse thickness. In this case, one ought to expand the magnetization vector $\mathbf{m}$ and the RF magnetic field in problem's normal modes $\mathbf{m}_n$: $\mathbf{m} = \sum_n (A_n\mathbf{m}_n + \text{c.c.})$ [7, 8], instead of plane spin waves.

In this paper, the parametric excitation of SMSO in a longitudinally magnetized yttrium-iron garnet (YIG) film resonator with elliptic cross-section under the longitudinal pumping will be considered, by taking the actual boundary conditions at the resonator surface into account. The ferrite anisotropy is neglected, by assuming the external static magnetic field to be much larger than typical YIG cubic and uniaxial anisotropy fields (≈50 Oe).

2. General Theory

The exact analytical theory of SMSO in infinitely long isotropic ferrite resonator magnetized along its axis with elliptic cross-section (Fig. 1) in the nonexchange limit is presented in [9].

It was shown [9, 10], that the eigenfrequency of the SMSO n-th mode of such resonator is given by:

$$\omega_n^2 = (\omega_H + \omega_M/2)^2 - 1/4\omega_M^2((a-b)/(a+b))^{2|n|}. \quad (1)$$

The magnetostatic modes of the infinitely long elliptic resonator can be characterized by three indices [7], namely, the number of nodes in the circumferential direction n, index r of a solution of the characteristic equation, and wavenumber β corresponding to the propagation along the cylinder axis. For the axially uniform oscillations, $\beta = 0$, and the surface modes are labeled by $r = 0$, according to [7]. Hereafter, we will designate each mode with the single subscript n instead of all three indices $(n, 0, 0)$, for the sake of brevity.

Since $\omega_n = \omega_{-n}$ (1), the external pumping RF magnetic field $\mathbf{h}$ (applied in parallel to the DC field $\mathbf{H}_0$) with the frequency $\omega_p = 2\omega_n$ can parametrically excite two frequency-degenerated counterpropagating surface magnetostatic modes with the indices n and $-n$.

The magnetostatic potential Ψ for the SMSO n-th mode in an elliptic ferrite cylinder with semiaxes a and b can be expressed in the modified elliptic coordinate system (ρ, ϕ, z) as [9]

$$\Psi_n(\rho,\phi) = B_n(R_n^+(\rho)\cos(n\phi) - $$
$$- i\,\mathrm{sgn}(n)\frac{\mu + 1/A}{\mu_a}R_n^-(\rho)\sin(n\phi)), \quad (2)$$

where B_n is the mode amplitude, $R_n^+(\rho) = (\rho^{|n|} + (c/2)^{2|n|}\rho^{-|n|})$, $R_n^-(\rho) = (\rho^{|n|} - (c/2)^{2|n|}\rho^{-|n|})$, $A = (1-(c/(a+b))^{2|n|})/(1+(c/(a+b))^{2|n|}$, $c = \sqrt{a^2-b^2}$, $\mu = (\omega^2-\omega_1^2)/(\omega^2-\omega_H^2)$, $\mu_a = \omega\omega_M/(\omega^2-\omega_H^2)$, $\omega_1^2 = \omega_H(\omega_H+\omega_M)$, $\omega_M = \gamma 4\pi M_0$, $\omega_H = \gamma H_0$, γ is the gyromagnetic ratio, and M_0 is the saturation magnetization. In (2), one should treat ω as the eigenfrequency ω_n of the n-th mode at a given magnetic field H_0 (see (1)).

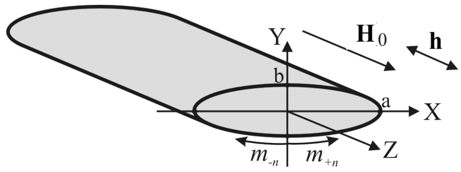

Fig. 1. Longitudinally magnetized ferrite elliptic cylinder under parallel pumping

In view of the standard expressions

$$\mathbf{h}_n = \mathrm{grad}\,\Psi_n, \quad \mathbf{m}_n = \hat{\chi}\mathbf{h}_n,$$

$$\hat{\chi} = \frac{\omega_M}{\omega_H^2 - \omega^2}\begin{pmatrix}\omega_H & i\omega \\ -i\omega & \omega_H\end{pmatrix},$$

relations (2) yield the explicit formulae for the components of the high-frequency magnetization vector $\mathbf{m}_n$ for each SMSO mode. Then we verified,by straightforward calculations, the orthogonality relation $\int(m_{\rho n}m_{\varphi m} - m_{\varphi n}m_{\rho m})\,dV = 0$ and calculated the eigenmode normalization constant $D_n = -i\int(m_{\rho n}m_{\varphi n}^* - m_{\rho n}^*m_{\varphi n})dV$ [7]. It was found, that $D_n = D_{-n} = \frac{2\pi C_n^2}{|n|}E_n$, where C_n is some expression depending on the frequency, magnetic field, saturation magnetization, and geometric parameters of a sample, and $E_n = 2\omega_n(2\omega_H + \omega_M(1 + ((a-b)/(a+b))^{|n|}))^{-1}$. Note that, in modified elliptical cylindrical coordinates [9], we have $dV \equiv \rho h_\rho^2 d\rho d\varphi dz$, $h_\rho = \sqrt{(1 - c^2/4\rho^2)^2 + (c^2/4\rho^2)\sin^2\varphi}$.

In [7], the general expression for the experimentally observed parametric excitation threshold RF field h_c was found to be

$$(\gamma h_c)^2 = \frac{4\omega_{rn}\omega_{rm}}{\lambda_{n,m}\lambda_{m,n}^*}, \quad (3)$$

where $\lambda_{n,m} = (1/D_n)\int(\mathbf{m}_n^*\mathbf{m}_m^*)\,dV$, and ω_{rn} is the relaxation frequency of the proper mode.

Using (2), we obtain

$$\int\left(m_{\rho n}^* m_{\rho m}^* + m_{\varphi n}^* m_{\varphi m}^*\right)dV = \frac{\pi C_n^2}{|n|}(1 - E_n^2),$$

when $|n| = |m|$, and is equal to zero otherwise (here, C_n and E_n are exactly the same as in the expression for D_n). Therefore,

$$\lambda_{n,m} = \frac{(1-E_n^2)}{2E_n}\delta_{|n|,|m|} = \lambda_{m,n}^*.$$

This means that the PE process like $\omega_p = \omega_n + \omega_m$, $|n| \neq |m|$, which could be allowed by the energy conservation law, would have, nevertheless, the infinite threshold due to the zero overlapping integral $\lambda_{n,m}$. Thus, only the parametric excitation of two SMSO modes having opposite azimuthal indices $n = -m$ is allowed and would be considered further. By substituting all the previously calculated expressions into (3), we obtain

$$h_c = \left(\frac{2\omega_{rn}}{\gamma}\right) \frac{2E_n}{(1 - E_n^2)}.$$

After some cumbersome calculations, the final expression takes the form

$$h_c = \left(\frac{2\omega_{rn}}{\gamma}\right) \frac{2\omega_n}{\omega_M \left(\dfrac{a-b}{a+b}\right)^{|n|}} = \frac{\eta\omega_p}{\omega_M \left(\dfrac{a-b}{a+b}\right)^{|n|}} \Delta H_k, \quad (4)$$

where $\eta = \partial\omega_n/\partial\omega_H = (\omega_H + \omega_M/2)/\omega_n$ is the ellipticity factor [11], and ΔH_k is the ferromagnetic resonance linewidth.

Apparently, the excitation threshold for SMSO under the longitudinal pumping strongly depends on the geometric parameters of a sample (e.g., the aspect ratio a/b). But otherwise, expression (4) is similar to that deduced on the basis of the plane-wave analysis [1].

As it was pointed out earlier [8], the PE process efficiency strongly correlates with the polarization of excited spin waves. Next, we will elucidate this statement for the problem under consideration and express it in the strict mathematical form.

Using the explicit expressions for $\mathbf{m}_n$, our calculations show that the ratio of axes of the high-frequency magnetization polarization ellipse is given by the formula

$$m_{\min}/m_{\max} = \tan(1/2 \arcsin(2E_n/(1 + E_n^2))).$$

It is worth noting that $m_{\min}/m_{\max}$ does not depend on coordinates, i.e., it is spatially uniform.

After some simplifications, we obtain a formula that explicitly expresses the ratio of axes of the eigenmode polarization ellipse in terms of the magnetic and geometric parameters of the SMSO resonator:

$$\frac{m_{\min}}{m_{\max}} = \tan\left(1/2 \arcsin \frac{\omega_n}{\omega_H + \frac{\omega_M}{2}}\right). \quad (5)$$

Considering the expression for the ellipticity factor and relation (1), one can see that magnetization's polarization is defined by the ellipticity factor η only, according to $m_{\min}/m_{\max} = \tan(1/2 \arcsin(1/\eta))$.

Since the critical field h_c and the polarization state depend on the same coefficient E_n, we can express one physical quantity directly via another one. Thus, we have

$$h_c = \left(\frac{2\omega_{rn}}{\gamma}\right) \frac{m_{\min}/m_{\max}}{1 - (m_{\min}/m_{\max})^2}. \quad (6)$$

The physical origin of such correlation is clear: for the circular precession of the magnetization ($m_{\min}/m_{\max} = 1$), the longitudinal component of the magnetization m_z is absent, and no coupling with the pumping field is possible ($h_c \to \infty$). For a more elliptic precession, m_z becomes correspondently larger. Hence, the interaction is stronger, and the threshold is lower [8].

3. Discussion

Let us consider two limiting cases of (4): a circular ferrite rod ($a = b$) and a very elongated elliptic cylinder ($a \gg b$). In the first case, $h_c \to \infty$, since the characteristic equation admits a solution only for $n > 0$, and a pair of counterpropagating surface modes required for the PE process is absent, as it was correctly pointed out in [7]. As for the second case, let us use the previously published expression for the PE of traveling surface magnetostatic waves with the wavevectors $\pm k$ in a thin magnetic film with thickness d [12]. In that situation, the threshold is equal to $h_c = (2\omega_{rn}/\gamma)(\omega_p/\omega_M)\exp(|k|d)$, which for $kd \ll 1$ reduces to $h_c = (2\omega_{rn}/\gamma)(\omega_p/\omega_M)(1 + |k|/d)$. On the other hand, expression (4) for $b/a \ll 1$ reduces to $h_c = (2\omega_{rn}/\gamma)(\omega_p/\omega_M)(1 + 2b|n|/a)$. Those two formulae would be identical, if we make the natural replacement $2b \to d$ and assume that an "equivalent" wavevector $|k| = |n|/a$ can be assigned to each eigenmode with index n. Since $1/a \to 0$, the discrete set of mode indices smoothly transforms into a continuous manifold of k. Thus, expression (4) gives the correct results in both limiting cases.

The dependence of the threshold on the cylinder shape is illustrated in Fig. 2, where the normalized microwave threshold field $h_c/\Delta H_k$ for a few lowest-order SMSO modes is depicted as a function of the aspect ratio a/b. In calculations, we used $H_0 = 1$ kOe and the value of $4\pi M_0 = 1.75$ kG typical of YIG. The dash-dotted line shows the normalized threshold for infinite isotropic media (that is equal to ω_p/ω_M), by assuming that ΔH_k in both cases are identical. One can see that the parametric excitation threshold drastically increases for cylinder's shape close to the circular one. But, for elliptic cylinders with a large aspect ratio (for example, thin-film resonators), it is approaching the value for plane spin waves. Specifically, for the $n = 1$ mode, the difference from the "bulk" value is less than 15% for $a/b > 20$. Moreover, the threshold noticeably increases with the mode number.

Thus, the calculations presented here allow one to evaluate the PE threshold for any given mode of an elliptic resonator and give the more flexibility to an SMSO resonator designer in choosing the dynamic power range of a device. For example, if the operation at a larger input power is required, the resonator, according to (4) and Fig. 2, should work on higher modes with large h_c or must be manufactured as a circular cylinder. On the other hand, for the applications like a power limiter, one can precisely set the desired resonator's threshold power, by simply selecting the appropriate axis ratio (see Fig. 2).

The analysis of expression (5) demonstrates that, for a circular cylinder ($a = b$), all modes without exception have circular polarization. However, when cylinder's aspect ratio a/b increases, the $m_{\min}/m_{\max}$ ratio start decreasing, and the modes with larger index n are always being more "circular". In addition, the polarization ellipse aspect ratio increases with H_0, tending to 1 for the large bias (see Fig. 3).

The threshold vs. polarization dependence, as described by expression (6), is illustrated in Fig. 4. It is clearly seen that the more elliptic precession of the magnetization (smaller $m_{\min}/m_{\max}$) facilitates, indeed, PE under the longitudinal pumping, as was pointed out earlier. Note that the very elliptic (close to linear) precession is beyond the scope of the current theory, since the assumption $m_z \ll m_x, m_y$ (m_i being the dynamical (high-frequency) components of the magnetization) used when deriving the expressions for tensor magnetic permeability is no longer valid in this case.

In order to define the limits of current theory's applicability, the investigation of the longitudinal and transversal high-frequency components of the magnetization vector, assuming $|\mathbf{M}| = \text{const}$, was done,

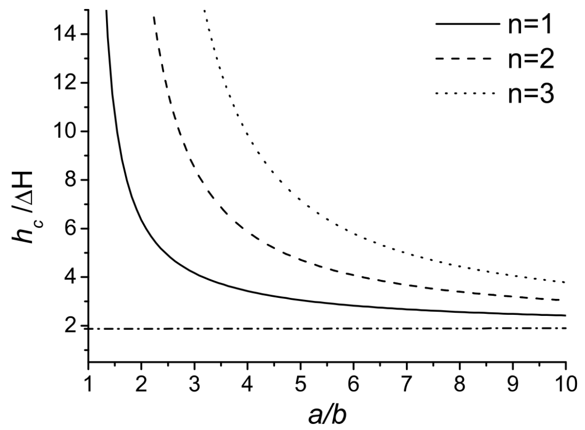

Fig. 2. Normalized microwave threshold field as a function of the elliptic cylinder aspect ratio

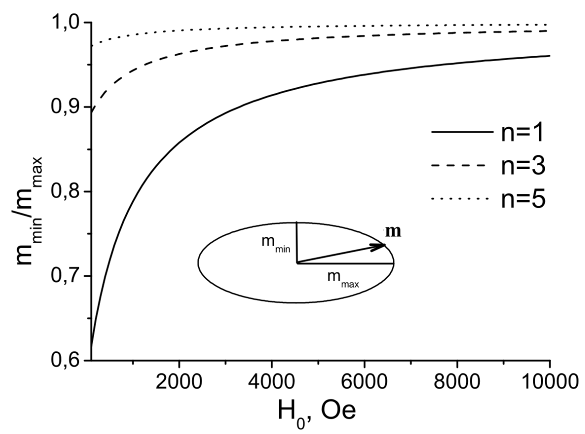

Fig. 3. Modes magnetization ellipse axis ratio as a function of the bias magnetic field (cylinder's aspect ratio $a/b = 3$)

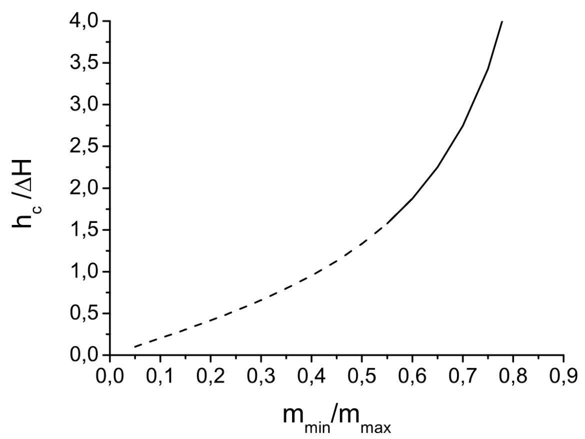

Fig. 4. Parametric excitation threshold vs. the polarization axis ratio of excited magnetostatic oscillations

resulting in the following formula:

$$\frac{m_{\min}}{m_{\max}} = \sqrt{4\left(\frac{\alpha}{m_{x0}}\right)^2 + 1} - 2\frac{\alpha}{m_{x0}}. \qquad (7)$$

Expression (7) defines the polarization ellipse aspect ratio $m_{\min}/m_{\max}$, for which m_z reaches a value equal to $\alpha\, m_y$, $0 < \alpha < 1$, for the given polarization ellipse normalized major semiaxis $m_{x0} = m_x/M_0$. The parameter $\alpha = m_z/m_y$ determines the smallness of the longitudinal component m_z relative to the transversal component m_y. If $\alpha \ll 1$, the standard expressions for tensor magnetic permeability are entirely valid. Otherwise, those expressions are no longer applicable, and all the theoretical results presented here are doubtful. Formula (7) allows one to estimate the range of $m_{\min}/m_{\max}$, for which our theoretical model remains correct. For example, for fixed $m_x/M_0 = 0.1$, α is equal 0.05 for $m_{\min}/m_{\max} = 0.41$, and $\alpha = 0.1$ for $m_{\min}/m_{\max} = 0.24$. Thus, the safe interval is roughly $0.5 < m_{\min}/m_{\max} < 1$. For smaller m_x/M_0 we will always get lesser values of the lower boundary of $m_{\min}/m_{\max}$. Therefore, it would be safe to assume that, for relatively small m_x/M_0 (which is typical of the parametric excitation processes under the parallel pumping), the curve in Fig. 4 is trustworthy for $m_{\min}/m_{\max}$ above approximately 0.5.

4. Conclusions

Analytical calculations by the theory of parametric excitation of magnetostatic surface oscillations in longitudinally magnetized elliptic cylinders under the longitudinal pumping have been conducted. The final expressions are obtained in the simple convenient form suitable for the further analysis.

The parametric excitation threshold for various mode numbers and cylinder aspect ratios has been derived and analyzed. It is shown that the parametric excitation threshold for SMSO in a thin ferrite film is of the same order of magnitude with that calculated within the classical theory for plane spin waves (SW). The interpretation of the experimental results and the thorough analysis of both possible mechanisms of parametric excitation are carried out. Indeed, we have the relation $h_c^{\rm SMSO}/h_c^{\rm SW} = \left(\Delta H_k^{\rm SMSO}/\Delta H_k^{\rm SW}\right)(a+b)^{|n|}/(a-b)^{|n|}$. For example, for the pumping field frequency $\omega_p = 10$ GHz, the parametric SMSO excited at $\omega_p/2$ in an YIG resonator biased with $H_0 = 1000$ Oe will have low k and $\Delta H_{k\to 0}^{\rm SMSO} \approx 0.25$ Oe [13]. At the same time, the plane spin waves with equal frequency would have $k \approx 10^5$ cm^{-1} and much larger $\Delta {\rm H}_k^{\rm SW} \approx 0.7$ Oe due to the additional contribution from the dipolar 3-magnon confluence process [14]. In this situation, the SMSO main mode ($n = 1$) in a resonator with aspect ratio $a/b > 2$ will have a lower parametric excitation threshold than plane spin waves.

The analytical expression for the ratio of axes of the high-frequency magnetization polarization ellipse is obtained, and the correspondence between the polarization state and the PE threshold is investigated. The expression directly connecting the ratio of axes and the PE threshold is found and graphically illustrated, and the bounds of its applicability are indicated.

Earlier [9], it was shown that it the nonexchange limit SMSO spectrum of a long YIG longitudinally biased resonator with rectangular cross-section can be calculated with the use of the geometric approximation of the resonator cross-section with inscribed ellipse. Thus, the presented theory, though being derived for an elliptic resonator, can be potentially applied to the widely used film ferrite resonators with rectangular shape.

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ПАРАМЕТРИЧНЕ ЗБУДЖЕННЯ
ПАРАЛЕЛЬНОЮ НАКАЧКОЮ ПОВЕРХНЕВИХ
МАГНІТОСТАТИЧНИХ КОЛИВАНЬ
В ПОЗДОВЖНЬО НАМАГНІЧЕНОМУ
ЕЛІПТИЧНОМУ ЦИЛІНДРІ

Р е з ю м е

Розроблено аналітичну теорію параметричного збудження паралельною накачкою поверхневих магнітостатичних коливань нескінченно довгого еліптичного феромагнітного циліндра, намагніченого вздовж осі, з урахуванням граничних умов на поверхні феромагнетика. Показано можливість параметричного збудження пари вироджених мод з протилежними напрямками поширення, з частотами, що дорівнюють половині частоти накачки, та отримано вирази для порога цього процесу. Знайдено, що порогова амплітуда поля накачки сильно залежить від номера моди та відношення великої та малої півосі еліптичного циліндра і при великому значенні цього відношення прямує зверху до величини, що розрахована на основі моделі плоских хвиль. Було отримано, проаналізовано та графічно проілюстровано просте аналітичне співвідношення між еліптичністю поляризації високочастотної намагніченості збуджених поверхневих магнітостатичних коливань та порогом їх параметричного збудження.