YU.A. LASHKO,[1] G.F. FILIPPOV,[1] L. CANTON[2]

[1] Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine
(14b, Metrolohichna Str., Kyiv 03680, Ukraine; e-mail: ylashko@gmail.com)
[2] Istituto Nazionale di Fisica Nucleare, Sezione di Padova
(Via Marzolo 8, 35131 Padova, Italy)

SCATTERING OF ^{6}He ON α-PARTICLE: MICROSCOPIC GUIDANCE FOR ORTHOGONALIZING PSEUDOPOTENTIALS

PACS 21.60.Gx; 21.60.-n

Within a microscopic two-cluster model, we discuss the influence of the Pauli exclusion principle on the scattering of ^{6}He on α-particle. The structure of the Pauli-forbidden and the Pauli-allowed states is analyzed in detail. The influence of the Pauli-allowed states with eigenvalues other than unity on the kinetic energy of the relative motion of ^{6}He and α-particle results in the effective interaction between these nuclei. This effect can be simulated to some extent with finite values of orthogonalizing pseudopotential strength. We estimate the strength and the range of such interaction within a microscopic model and provide a guidance for choosing the parameters of the orthogonalizing pseudopotential.

Keywords: microscopic cluster model, Pauli exclusion principle, resonating group method.

1. Introduction

Mainly, different cluster models fall into two groups – macroscopic and microscopic. In macroscopic models, clusters are considered to be structureless particles, and cluster-cluster interactions are approximated by some local potentials, which are fitted to reproduce relevant data on the cluster-cluster systems. As for the Pauli exclusion principle, it is usually simulated either with an additional repulsive potential between clusters or with an orthogonalizing pseudopotential (OPP) containing the operators of projection onto the forbidden states. In fact, the elimination of the Pauli-forbidden states with the OPP technique is achieved by the infinite strength of the pseudopotential.

However, as is evident from microscopic studies, the elimination of the Pauli-forbidden states in the wave function does not exhaust all exchange effects. In particular, in [1] and [2], it has been shown that the essential part of such effects is directly related to the eigenvalues of the antisymmetrization operator. Such eigenvalues are not exactly identical to 1 for all non-forbidden states and characterize the probability of the formation of the corresponding Pauli-allowed basis states in the wave function of the cluster system. The involvement of the eigenvalues of the antisymmetrization operator in the Schrödinger equation leads to changes in the relative kinetic energy, while the clusters approach each other. Consequently, the clusters are affected by an effective repulsion or attraction induced by the Pauli principle to the intercluster kinetic energy operator [1]. Such an effective interaction substantially affects the dynamics of the cluster-cluster interaction and can, on occasion, produce the resonance behavior in the scattering phase-shift or even a bound state in a compound nuclear system [2].

Whereas the eigenvalues of the antisymmetrization operator in nucleon-nucleon systems can take only the value 0 or 1, the eigenvalues of Pauli-allowed states in two-cluster systems tend to unity only in the limit of a large distance between clusters. Because of the exchange of nucleons belonging to different clusters, at small intracluster distances the eigenvalues of the antisymmetrization operator are not equal to unity. The eigenvalues, which are less than unity, correspond to the partly forbidden states and result in the effective repulsion of clusters. Alternatively, the eigenvalues, which exceed unity, correspond to the superallowed states and result in the effective attraction of clusters [1].

The evidence of the effective repulsion resulting from partly forbidden states was also observed in a

macroscopic calculation of the nucleon-nucleus scattering within the multichannel algebraic scattering method [3]. Namely, the authors noticed that the use of some finite values (of the order of a few to a few tens of MeV) of pseudopotential strength for some Pauli-allowed states results in a better agreement with experimental data. In Refs. [3,4], the finite values of pseudopotential strength have been taken as phenomenological parameters. To link the latter parameters to the eigenvalues of the antisymmetrization operator is a challenging task.

In this paper, we consider ^{10}Be nucleus, which has recently become the object of numerous experimental and theoretical investigations [5–9]. Since the threshold of the ^{10}Be decay into alpha-particle and ^{6}He is located only 1.2 MeV above the second 0^+ state of ^{10}Be, we can treat ^{10}Be in a vicinity of this threshold as the $^6\mathrm{He}+{}^4\mathrm{He}$ cluster system. We have noted that, in the systems containing clusters with an open p-shell, the range of an effective cluster-cluster potential generated by the Pauli principle can be rather large [1, 10]. Furthermore, such systems are multichannel systems, where both effective attraction and repulsion could emerge. Finally, the analysis of the $^6\mathrm{He}+{}^4\mathrm{He}$ cluster system allows an understanding of the role and the structure of the Pauli-forbidden states in multichannel systems, where the orbital momenta of relative motion of the clusters are not the integrals of motion.

The paper is organized as follows. In Section 2, we formulate our approach and explain the nature of the effective interaction between clusters due to a modification of the kinetic energy operator of relative motion of clusters by the Pauli exclusion principle. Section 3 introduces the norm kernel of the $^6\mathrm{He}+\alpha$ system, which provides a complete basis of the eigenfunctions and the eigenvalues of the antisymmetrization operator of the system under study. In Section 4, we discuss the structure of the Pauli-forbidden states and give recommendations on how these states can be related to the scattering channels characterized by the orbital momentum of ^{6}He nucleus and the momentum of relative motion of ^{6}He and α-particle. In Section 5, we present the effective $^6\mathrm{He}+\alpha$ interaction generated by partly forbidden and superallowed states and analyze the impact of this interaction on the phase shifts of the elastic scattering of ^{6}He nucleus on α-particle. Conclusions are given in Section 6.

2. Formulation of the Approach

The most complete description of nucleon systems comprised of light nuclei is given by microscopic models capable of taking the Pauli exclusion principle into account. Approaches based on the ideas of the resonating-group method (RGM) [11] meet this requirement. The most straightforward way to consider the Pauli exclusion principle is to build a complete basis of allowed states of a harmonic oscillator and their eigenvalues, which makes it possible to express the wave functions of a nuclear system in the form of a linear superposition of allowed states and to reduce the problem of their determination to solving a set of linear algebraic equations for expansion coefficients. It is also useful, in order to check a number of statements concerning the elements of the S-matrix, to have a complete basis of forbidden states, whose eigenvalues equal zero by definition.

Following RGM, it will be supposed hereafter that the considered nuclear systems consist of two clusters. An RGM wave function is built in the form of an antisymmetrized product of cluster internal wave functions and a wave function of their relative motion. The internal wave functions of the clusters are fixed [1], and the wave function of relative motion of the clusters, which depends only on the Jacobi vector of the considered two-cluster system, is found by solving an integro-differential equation. The latter is obtained by the substitution of the RGM wave function into the Schrödinger equation followed by the integration with respect to single-particle coordinates. The integro-differential equation can be transformed into a set of linear equations by expanding the wave function of the cluster relative motion into the compete basis of the Pauli-allowed harmonic oscillator states, as the algebraic version of RGM (AVRGM) suggests [12].

First and foremost, the AVRGM calls for the construction of the complete basis of Pauli-allowed harmonic oscillator states and their classification. This is accomplished by solving the eigenvalue and eigenfunction problem for the norm kernel I, i.e., the overlap integral of the two Slater determinants composed of the single-particle orbitals. Here, the integration is performed over single-particle variables to give the expression, which depends only on the distance be-

[1] We assume the intrinsic cluster wave functions to be the simplest functions of a translation-invariant shell model.

tween clusters, provided that the clusters are closed-shell nuclei. The method of construction of the norm kernels of two-cluster systems in the Fock–Bargmann space, as well as the detailed analysis of the fundamental properties of the norm kernels, are given in [13].

The norm kernel can be represented in the form of a sum of degenerate orthogonal kernels, with a certain number of quanta n corresponding to each of them:

$$I = \sum_{n=n_{\min}}^{\infty} \sum_{(\lambda,\mu)} \sum_{\alpha} \Lambda_{n,(\lambda,\mu)_\alpha} \sum_{L,M} \psi^{L,M,\alpha_L}_{n,(\lambda,\mu)_\alpha} \tilde{\psi}^{L,M,\alpha_L}_{n,(\lambda,\mu)_\alpha}.$$

For a complete description of the eigenfunctions $\psi^{L,M,\alpha_L}_{n,(\lambda,\mu)_\alpha}$ and the eigenvalues $\Lambda_{n,(\lambda,\mu)_\alpha}$ of the norm kernel, it is necessary to define, along with the number of oscillator quanta n, the indices (λ,μ) of their SU(3) symmetry, the additional quantum number $\alpha_{(\lambda,\mu)}$, if there are several differing (λ,μ) multiplets, the total orbital momentum L and its projection M, and one more additional quantum number α_L, if the multiplet (λ,μ) has several states with the same values of L. Hereinafter, we shall omit the indices α, α_L, and M, because they are redundant in the problem considered.

In the discrete representation, the Schrödinger equation is reduced to a set of linear equations for the expansion coefficients of the wave functions of discrete states with the energy $E_\kappa = -\kappa^2/2 < 0$, and of continuum states with the energy $E > 0$:

$$\Psi_{\kappa\,(E)}(\mathbf{r}) = \sum_n C^{\kappa\,(E)}_n \Psi_n(\mathbf{r}),$$
$$\sum_{\tilde{n}} \frac{\langle n|\hat{H}|\tilde{n}\rangle}{\sqrt{\Lambda_n \Lambda_{\tilde{n}}}} C_{\tilde{n}} - EC_n = 0.$$

To understand the results of action of the antisymmetrization operator, first, let us discuss a set of the algebraic equations where only the operator of kinetic energy of the relative motion of clusters (in the c.o.m. frame) is retained:

$$\sum_{\tilde{n}} \frac{\langle n|\hat{T}|\tilde{n}\rangle}{\sqrt{\Lambda_n \Lambda_{\tilde{n}}}} C_{\tilde{n}} - EC_n = 0. \tag{1}$$

In this section, we consider the simplest case of a two-cluster system composed of two closed shell or (0s)-shell clusters. In this case, the problem is single-channel, and the orbital momentum l (of relative motion of clusters) is a good quantum number, and the basis functions differ only by the number of oscillator quanta $n = 2k$.

The matrix of the kinetic energy is tridiagonal in the harmonic-oscillator representation:

$$\langle l, 2k+2|\hat{T}|l, 2k\rangle =$$
$$= -\sqrt{\frac{\Lambda_{2k}}{\Lambda_{2k+2}}} \frac{1}{4} \sqrt{(2k-l+2)(2k+l+3)},$$
$$\langle l, 2k-2|\hat{T}|l, 2k\rangle =$$
$$= -\sqrt{\frac{\Lambda_{2k-2}}{\Lambda_{2k}}} \frac{1}{4} \sqrt{(2k-l)(2k+l+1)},$$
$$\langle l, 2k|\hat{T}|l, 2k\rangle = \frac{1}{2}\left(2k+\frac{3}{2}\right).$$

The equations of set (1) for the collision of clusters in the state with angular momentum l can be written in the form of the finite-difference equations

$$-\frac{1}{2}\left\{\left(1+\frac{\Lambda_{n-2}}{\Lambda_n}\right)\left(n+\frac{3}{2}-\frac{(2l+1)^2}{8n}\right)+1-\frac{\Lambda_{n-2}}{\Lambda_n}\right\}\times$$
$$\times\frac{1}{4}\left(C_{n+2}-2C_n+C_{n-2}\right)-\frac{1}{4}\left(C_{n+2}-C_{n-2}\right)\times$$
$$\times\frac{1}{2}\left\{1+\frac{\Lambda_{n-2}}{\Lambda_n}+\left(1-\frac{\Lambda_{n-2}}{\Lambda_n}\right)\left(n+\frac{3}{2}-\frac{(2l+1)^2}{8n}\right)\right\}+$$
$$+\left\{\left(1+\frac{\Lambda_{n-2}}{\Lambda_n}\right)\frac{(2l+1)^2}{32n}+\frac{1}{4}\left(1-\frac{\Lambda_{n-2}}{\Lambda_n}\right)\left(n+\frac{1}{2}\right)\right\}C_n =$$
$$= \frac{mr_0^2}{\hbar^2} EC_n. \tag{2}$$

The equation is transformed into the Bessel differential equation in the limit $n \gg 1$, when the eigenvalues Λ_n can be set to unity:

$$\left(\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}-\frac{(2l+1)^2}{4}\frac{1}{y^2}+\frac{mr_0^2}{\hbar^2}\,2E\right)C(y)=0;$$
$$y=\sqrt{2n+3}.$$

The diagonal matrix $||U^{\text{Pauli}}_{n,\tilde{n}}\delta_{n,\tilde{n}}||$ entering Eq. (2),

$$U^{\text{Pauli}}_{n,n} = \frac{1}{4}\left(1-\frac{\Lambda_{n-2}}{\Lambda_n}\right)\left(n+\frac{1}{2}\right), \tag{3}$$

can be considered as the matrix of the operator of effective cluster-cluster interaction generated due to the Pauli exclusion principle for the kinetic energy. Obviously, this interaction vanishes for $\Lambda_n = 1$ and appears as a result of the exchange of nucleons belonging to different clusters. In the case of

the scatterring of particles, which do not have internal structure, Eq. (2) reduces to the finite-difference Bessel equation. Neglecting the Pauli principle in two-cluster systems also gives the finite-difference Bessel equation without any additional interaction.

The absolute value of its intensity decreases if the difference $\Lambda_{n-2} - \Lambda_n$ tends to zero. If the latter remains negative, as n increases (the eigenvalues monotonically tend to unity from above), then the effective Pauli potential is attractive. If, under the same alteration of n, the difference in eigenvalues stays positive (the eigenvalues monotonically tend to unity from below), then the effective potential of antisymmetrization is repulsive. Evidently, the radius of this interaction depends on how rapidly the eigenvalues tend to unity, as n increases. The main properties of such effective potentials have been detailed in [2].

In this paper, we do not consider the potential between the clusters generated by the bare nucleon-nucleon forces. This is because the basic features of the antisymmetrization effects on the nucleus-nucleus interaction may be learned by studying specifically the exchange effects on the kinetic-energy operator. This supports the statement that the range of the influence of the Pauli principle on the kinetic energy appears to be significantly larger than that of the cluster-cluster interaction generated by the nucleon-nucleon potential (see [1]).

3. Norm Kernel of ^{6}He + α

In this section, we consider the norm kernel of the ^{6}He + α system. As we already mentioned, we assume that the intrinsic wave functions of both clusters are described by the lowest functions of a translation-invariant shell model. Within this approximation, the α-cluster has a closed shell and can be only in its 0^+ ground state, while ^{6}He has two neutrons in the p-shell and can be either in the 0^+ ground state or in the 2^+ excited state, which is a quite narrow resonance state located at $E = 1.8$ MeV above the threshold for the ^{6}He breakup into $\alpha + n + n$.

The norm kernel of the ^{6}He + α system is written as

$$I = \sum_n \{\Lambda_{(n+2,0)} \langle (n+2,0)|(n+2,0)\rangle + \\ + \Lambda_{(n,1)} \langle (n,1)|(n,1)\rangle + \Lambda_{(n-2,2)} \langle (n-2,2)|(n-2,2)\rangle\},$$

The values of $\Lambda_{(n+2,\,0)}$ are all zeros at $n < 6$. The eigenvalues $\Lambda_{(n,1)}$ equal to zero, if $n < 5$, and, finally, $\Lambda_{(n-2,\,2)}$ vanishes for $n < 4$. In other words, the states belonging to the SU(3) irreducible representation $(n-2, 2)$ become allowed if $n \geq 4$, the minimal number of quanta for the $(\lambda, \mu) = (n, 1)$ states is 5, and finally, the states with $(\lambda, \mu) = (n+2, 0)$ are allowed only if $n \geq 6$ (Table). These eigenvalues have been obtained in [13]. Explicit expressions for the SU(3) invariants $\langle (n+2,0)|(n+2,0)\rangle$, $\langle (n,1)|(n,1)\rangle$ and $\langle (n-2,2)|(n-2,2)\rangle$ have been also given in [13]. The SU(3) invariants for the ^{6}He + α system coincide with those for the ^{6}He $+p$ system.

As the data of Table suggest, two SU(3)-branches of positive parity and one SU(3)-branch of negative parity are characterized by the eigenvalues, which exceed unity. Hence, the states belonging to $(2k+2, 0)$, $(2k-2, 2)$, and $(2k+1, 1)$ SU(3)-branches can be assigned to the superallowed states, while states $(2k, 1)$, $(2k+3, 0)$, and $(2k-1, 2)$ are partly forbidden. As follows from Eq. (3), we should expect the effective attraction of ^{6}He and α-particle in the former case and the effective repulsion in the latter case.

Working with the SU(3) basis, one faces a problem of formulation of asymptotic conditions for the wave function expansion coefficients. Therefore, the introduction of functions of the angular-momentum coupled basis $\Phi_n^{(\mathcal{L},\,l,\,L)}$, where $\mathcal{L}$ and l are partial angular momenta of ^{6}He cluster and the relative motion of the clusters, respectively, appears to be useful. Furthermore, the authors simulating the Pauli principle with the OPP also use the partial angular momenta of clusters and their relative motion to label different reaction channels.

Eigenvalues $\Lambda_{(\lambda,\mu)}$ of the norm kernel of ^{6}He + α

k	States with $n = 2k$			States with $n = 2k+1$		
	$(n+2,0)$	$(n,1)$	$(n-2,2)$	$(n+2,0)$	$(n,1)$	$(n-2,2)$
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	1.2056	0	1.0549	0.4521
3	0.9419	0.2721	1.1587	0.1831	1.2192	0.7650
4	1.2922	0.5698	1.0834	0.4045	1.1795	0.9011
5	1.3264	0.7645	1.0408	0.5983	1.1160	0.9581
6	1.2566	0.8760	1.0194	0.7448	1.0676	0.9821
7	1.1743	0.9363	1.0090	0.8454	1.0371	0.9923
8	1.1090	0.9678	1.0046	0.9097	1.0196	0.9967
9	1.0645	0.9840	1.0019	0.9489	1.0101	0.9985
10	1.0367	0.9921	1.0009	0.9718	1.0051	0.9994

It is important to note that the norm kernel has a diagonal form only in the SU(3) classification of the eigenvalues $\Lambda_{(\lambda,\mu)}$ and the eigenfunctions $\Psi_{(\lambda,\mu)}$. The SU(3)-basis and angular-momentum coupled basis functions are related by the matrix of a unitary transformation. Any unitary transformation of the basis functions $\Psi_{(\lambda,\mu)}$ will break the diagonal form of the kernel I, since the eigenvalues $\Lambda_{(\lambda,\mu)}$ differ from unity in a certain region of oscillator quanta. Angular-momentum coupled basis functions are not the eigenfunctions of the antisymmetrization operator at a small number of quanta, and the orbital momentum of relative motion of clusters cannot serve as a good quantum number for a system composed of nonspherical clusters.

The basis states $\Phi_n^{(\mathcal{L},\,l,\,L)}(\mathbf{u},\mathbf{r})$ of the angular-momentum coupled basis is a convolution of oscillator functions, one of which describes the internal structure of a nonspherical cluster ^{6}He, while another one is the wave function of relative motion of ^{6}He and α-particle. For an even number of quanta $n = 2k$, the functions $\Phi_n^{(\mathcal{L},\,l,\,L)}(\mathbf{u},\mathbf{r})$ are determined by the expression

$$\Phi_k^{(\mathcal{L},\,l,\,L)}(\mathbf{u},\mathbf{r}) = \phi^{\mathcal{L}}_{1-\mathcal{L}/2}\left(\frac{u}{r_0}\right)\phi^{l}_{k-l/2}\left(\frac{r}{r_0}\right)\times$$
$$\times\,\{Y_{\mathcal{L}}(\hat{u})\otimes Y_l(\hat{r})\}_{LM};$$
$$\phi^l_\nu(r) = (-1)^\nu\sqrt{\frac{2\Gamma(\nu+1)}{\Gamma(\nu+l+3/2)}}\times$$
$$\times\, r^l L_\nu^{l+1/2}(r^2)\exp\{-r^2/2\}.$$

For an odd number of quanta $n = 2k+1$, these functions are defined as

$$\Phi_k^{(\mathcal{L},\,l,\,L)}(\mathbf{u},\mathbf{r}) = \phi^{\mathcal{L}}_{1-\mathcal{L}/2}\left(\frac{u}{r_0}\right)\phi^{l}_{k+(1-l)/2}\left(\frac{r}{r_0}\right)\times$$
$$\times\,\{Y_{\mathcal{L}}(\hat{u})\otimes Y_l(\hat{r})\}_{LM}.$$

Here, r_0 is an oscillator length, which is chosen to be equal to 1.37 fm in our calculation to reproduce the root-mean-square radius of alpha-particle.

The vector $\mathbf{r}$ denotes the relative distance between α-particle and ^{6}He cluster, while the vector $\mathbf{u}$ is required to describe the orientation of an anisotropic ^{6}He cluster in space. The latter vector indicates the distance from the α-core to the pair of valence neutrons in ^{6}He.

4. Pauli-Forbidden States

4.1. Positive parity

All states of positive parity correspond to even numbers quanta $n = 2k$. There are six Pauli-forbidden states belonging to the $SU(3)$ representations (2,0), (4,0), (0,2), (6,0), (2,1), and (4,1). Four of these states contain $L = 0$ as the lowest possible orbital momentum:

$$\Psi^{L=0}_{(2,0)} = \Phi^{(0,0,0)}_{k=0};$$
$$\Psi^{L=0}_{(4,0)} = \sqrt{\frac{5}{9}}\Phi^{(0,0,0)}_{k=1} + \sqrt{\frac{4}{9}}\Phi^{(2,2,0)}_{k=1};$$
$$\Psi^{L=0}_{(0,2)} = \sqrt{\frac{4}{9}}\Phi^{(0,0,0)}_{k=1} - \sqrt{\frac{5}{9}}\Phi^{(2,2,0)}_{k=1};$$
$$\Psi^{L=0}_{(6,0)} = \sqrt{\frac{7}{15}}\Phi^{(0,0,0)}_{k=2} + \sqrt{\frac{8}{15}}\Phi^{(2,2,0)}_{k=2},$$

while the last two representations contain $L = 1$:

$$\Psi^{L=1}_{(2,1)} = \Phi^{(2,2,1)}_{k=1};\quad \Psi^{L=1}_{(4,1)} = \Phi^{(2,2,1)}_{k=2}.$$

As will be observed, the Pauli principle mixes the states with different partial momenta of the ^{6}He cluster $\mathcal{L}$ and the relative motion of the clusters l. Hence, strictly speaking, we cannot associate any of the Pauli-forbidden states with a certain function of the angular-momentum coupled basis $\Phi_n^{(\mathcal{L},\,l,\,L)}$. However, the Pauli-forbidden states of the ^{6}He + α system with $L^\pi = 1^+$ contain only the component $\mathcal{L} = l = 2$, and the first Pauli-forbidden state $\Psi^{L=0}_{(2,0)}$ involves solely the component $\mathcal{L} = l = 0$. Therefore, we can state with assurance that there are two Pauli-forbidden states in the $\mathcal{L} = l = 2$ channel, provided that $L^\pi = 1^+$, and at least one Pauli-forbidden state in the $\mathcal{L} = l = 0$ channel in the case $L^\pi = 0^+$.

As to the remaining three Pauli-forbidden states corresponding to the $L^\pi = 0^+$, they are the superpositions of the two components $\mathcal{L} = l = 2$ and $\mathcal{L} = l = 0$. Analyzing the weights of these two components in the Pauli-forbidden states, we can assume that the forbidden state $\Psi^{L=0}_{(4,0)}$ can be associated with the channel $\mathcal{L} = l = 0$, while the forbidden states $\Psi^{L=0}_{(0,2)}$ and $\Psi^{L=0}_{(6,0)}$ can be attributed to the $\mathcal{L} = l = 2$ channel. Summarizing, we have two forbidden states for $L^\pi = 1^+$ and four forbidden states for $L^\pi = 0^+$. In the former case, both of the forbidden states belong to the $\mathcal{L} = l = 2$ channel, because the total orbital momentum $L = 1$ cannot be realized with

$\mathcal{L} = l = 0$. Among four forbidden states characterized by $L^\pi = 0^+$, two forbidden states belong to the $\mathcal{L} = l = 0$ channels, while two others should reveal themselves in the $\mathcal{L} = l = 2$ channel.

The behavior of the phase shifts for the elastic ^{6}He + α scattering in the $L^\pi = 0^+$ state of the compound nucleus ^{10}Be confirms the existence of two forbidden states in the channel $^6\mathrm{He}(0^+) + \alpha \to {}^6\mathrm{He}(0^+) + \alpha$ and two forbidden states in the channel $^6\mathrm{He}(2^+) + \alpha \to {}^6\mathrm{He}(2^+) + \alpha$.

Figure 1 presents the phase shifts for the elastic ^{6}He + α scattering provided that all eigenvalues of the Pauli-allowed states are equal to unity and that $L^\pi = 0^+$. Hence, these phase shifts are created only by the Pauli-forbidden states. As Fig. 1 suggests, the amplitudes of both phase shifts $\delta_{l=0}(E)$ and $\delta_{l=2}(E)$ equal 2π. The dependence of the scattering phases on the energy, when the latter varies from zero to infinity, is regulated by the generalized Levinson's theorem. According to this theorem, the phase variation amplitude is proportional to the number of bound states and forbidden states of the system. Since there are no bound states in this calculation, we can conclude that two forbidden states show up indeed in either of the channels.

4.2. Negative parity

There are six Pauli-forbidden SU(3) irreducible representations for the states of negative parity $n = 2k+1$, and all these SU(3) representations contain $L = 1$ as the lowest possible orbital momentum:

$$\Psi^{L=1}_{(3,0)} = \sqrt{\frac{5}{9}}\Phi^{(0,1,1)}_{k=0} + \sqrt{\frac{4}{9}}\Phi^{(2,1,1)}_{k=0};$$

$$\Psi^{L=1}_{(1,1)} = \sqrt{\frac{4}{9}}\Phi^{(0,1,1)}_{k=0} - \sqrt{\frac{5}{9}}\Phi^{(2,1,1)}_{k=0};$$

$$\Psi^{L=1}_{(5,0)} = \sqrt{\frac{7}{15}}\Phi^{(0,1,1)}_{k=1} + \frac{2}{5}\sqrt{\frac{7}{3}}\Phi^{(2,1,1)}_{k=1} + \frac{2}{5}\Phi^{(2,3,1)}_{k=1};$$

$$\Psi^{L=1}_{(3,1)} = \frac{2}{3\sqrt{5}}\Phi^{(0,1,1)}_{k=1} - \frac{11}{15}\Phi^{(2,1,1)}_{k=1} + \frac{2}{5}\sqrt{\frac{7}{3}}\Phi^{(2,3,1)}_{k=1};$$

$$\Psi^{L=1}_{(1,2)} = \frac{2}{3}\Phi^{(0,1,1)}_{k=1} - \frac{2}{3\sqrt{5}}\Phi^{(2,1,1)}_{k=1} + \sqrt{\frac{7}{15}}\Phi^{(2,3,1)}_{k=1};$$

$$\Psi^{L=1}_{(7,0)} = \sqrt{\frac{3}{7}}\Phi^{(0,1,1)}_{k=2} + 2\sqrt{\frac{3}{35}}\Phi^{(2,1,1)}_{k=2} + 2\sqrt{\frac{2}{35}}\Phi^{(2,3,1)}_{k=2}.$$

The basis function $\Phi^{(0,1,1)}$ dominates in three forbidden states: $\Psi_{(3,0)}$, $\Psi_{(5,0)}$ and $\Psi_{(7,0)}$. The function $\Phi^{(2,1,1)}$ prevails in the forbidden states $\Psi_{(1,1)}$ and $\Psi_{(3,1)}$, while the component $\Phi^{(2,3,1)}$ has maximum weight only in the $\Psi_{(1,2)}$ state. Consequently, one would associate three forbidden states with the $\mathcal{L} = 0$, $l = 1$ channel, two forbidden states with the $\mathcal{L} = 2$, $l = 1$ channel and a single forbidden state with the $\mathcal{L} = 2$, $l = 3$ channel.

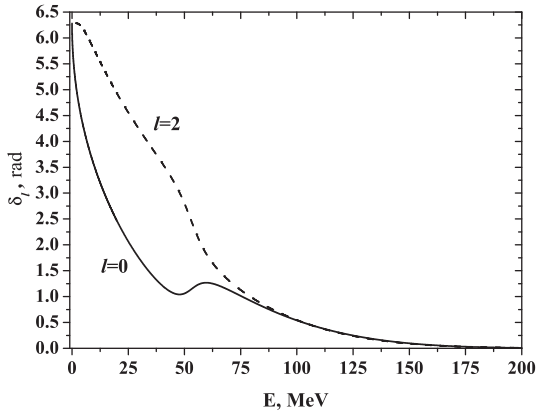

Fig. 1. Phase shifts $\delta_l(E)$ for the ^{6}He + α system that are generated by the kinetic-energy operator, provided that all eigenvalues of the Pauli-allowed states $\Lambda_n = 1$. The values of the orbital angular momentum l of the relative motion are indicated on the curves

5. Pauli-Allowed States

Although the eigenvalues of the antisymmetrization operator do not depend on the total orbital momentum L, the structure of the Pauli-allowed states does. Namely, the weights of different partial momenta differ for different values of L. Moreover, SU(3) irreducible representations do not all contain a given value of total orbital momentum L. Hence, the information on the partial composition of the Pauli-allowed states can be useful for the estimation of parameters of OPP.

5.1. Positive parity

First, let us consider the terms of the norm kernel containing the basis states with orbital momentum $L = 0$ and positive parity. They belong to two $SU(3)$ representations, $(2k+2, 0)$ and $(2k-2, 2)$. All of them have $n = 2k$.

Each of the states with $L = 0$ is a superposition of the basis functions $\Phi_k^{(\mathcal{L}=0,\, l=0,\, L=0)}$ and $\Phi_k^{(\mathcal{L}=2,\, l=2,\, L=0)}$

$$\Psi^{L=0}_{(2k+2,0)} = \sqrt{\frac{2k+3}{3(2k+1)}}\Phi^{(0,0,0)}_k + \sqrt{\frac{4k}{3(2k+1)}}\Phi^{(2,2,0)}_k;$$

$$\Psi^{L=0}_{(2k-2,2)} = \sqrt{\frac{4k}{3(2k+1)}}\Phi^{(0,0,0)}_k - \sqrt{\frac{2k+3}{3(2k+1)}}\Phi^{(2,2,0)}_k.$$

If $L = 1$, there is only one Pauli-allowed state, which can be written both in the SU(3)-basis and the angular-momentum coupled basis:

$$\Psi^{L=1}_{(2k,1)} = \Phi^{(2,2,1)}_k.$$

This means that we have a two-channel problem for the states with $L = 0$ and a single-channel problem for the states with $L = 1$.

The component $\mathcal{L} = l = 0$ has maximum weight in the $\Psi_{(2k-2,2)}$ Pauli-allowed state, while $\mathcal{L} = l = 2$ channel dominates in the $\Psi_{(2k+2,0)}$ state.

5.2. Negative parity

All states of negative parity correspond to odd numbers of quanta $n = 2k + 1$. The lowest value of total orbital momentum L with negative parity is $L = 1$. Each Pauli-allowed basis function with orbital momentum $L = 1$ is a superposition of three angular-momentum coupled basis states:

$$\Psi^{L=1}_{(2k+3,0)} = \sqrt{\frac{2k+5}{3(2k+3)}}\Phi^{(0,1,1)}_k +$$
$$+ 2\sqrt{\frac{2k+5}{15(2k+3)}}\Phi^{(2,1,1)}_k + 2\sqrt{\frac{k}{5(2k+3)}}\Phi^{(2,3,1)}_k,$$
$$\Psi^{L=1}_{(2k+1,1)} = \sqrt{\frac{4}{3(2k+1)(2k+3)}}\Phi^{(0,1,1)}_k -$$
$$- \frac{(6k+5)}{\sqrt{15(2k+1)(2k+3)}}\Phi^{(2,1,1)}_k +$$
$$+ \sqrt{\frac{4k(2k+5)}{5(2k+1)(2k+3)}}\Phi^{(2,3,1)}_k,$$
$$\Psi^{L=1}_{(2k-1,2)} = 2\sqrt{\frac{k}{3(2k+1)}}\Phi^{(0,1,1)}_k -$$
$$- 2\sqrt{\frac{k}{15(2k+1)}}\Phi^{(2,1,1)}_k + \sqrt{\frac{2k+5}{5(2k+1)}}\Phi^{(2,3,1)}_k.$$

All angular-momentum coupled basis functions $\Phi^{(\mathcal{L},\,l,\,L)}_k$ are presented in the $\Psi^{L=1}_{(2k+3,0)}$ state on equal footing. At the same time, the weight of $\mathcal{L} = 0$, $l = 1$ component amounts up to 60% in the $\Psi^{L=1}_{(2k-1,1)}$ allowed state and practically vanishes in the $\Psi^{L=1}_{(2k+1,2)}$ state.

5.3. Pauli potential

The ^{6}He + α system is somewhat more complicated, than a single-channel case considered in this section, because the dynamics of the α-cluster is influenced by the presence of neutrons in the p-shell of ^{6}He. Due to this fact, the problem becomes multichannel, and different SU(3)-branches are coupled by the kinetic-energy operator. Nevertheless, we can make a rough estimate of the effective Pauli potential by analyzing the diagonal part (with respect to SU(3)-indices) of the Pauli term given in Eq. (3).

In Fig. 2, the dependence of the Pauli potential

$$U^{\rm eff}_n(r_n) = \frac{\hbar^2}{mr_0^2}\,U^{\rm Pauli}_{n,n},\ r_n = \sqrt{\frac{10}{6\times 4}}\,r_0\sqrt{2n+3},$$

on the distance r_n between the centers of mass of the ^{6}He and α clusters is shown both for the states of positive and negative parity. As evident from the left panel of Fig. 2, the branch $(2k, 1)$ is repulsive, the branch $(2k-2, 2)$ is attractive, while the branch $(2k+2, 0)$ has a mixture of repulsion and attraction. The right panel of Fig. 2 suggests that, for negative parity states, two branches $(2k + 3, 0)$ and $(2k - 1, 2)$ are repulsive, while some attraction is present only in the $(2k + 1, 1)$ branch.

All the Pauli-allowed states, except $(2k, 1)$ state, are superpositions of the angular-momentum coupled basis functions. Hence, we can make definite conclusions only about the repulsion in the d-wave of relative motion of α-particle scattered on ^{6}He in the first 2^+ state. In other cases, we should consider also the weights of different partial momenta in the SU(3)-basis functions.

5.4. Phase shifts of the elastic ^{6}He + α scattering, $L^\pi = 0^+$

Finally, we calculated the phase shifts of the elastic ^{6}He + α scattering for the states of positive parity and $L = 0$ generated by the exchange terms of the kinetic energy operator.

The equations for the expansion coefficients in the SU(3) basis remain coupled even in the limit of the large number of excitation quanta, whereas the set of corresponding equations in the angular momentum coupled basis is unlinked in this asymptotic region. In AVRGM, the asymptotic behavior of the expansion coefficients $C^{(\mathcal{L},\,l,\,L)}_E(k)$ in the angular momentum coupled basis is expressed in terms of

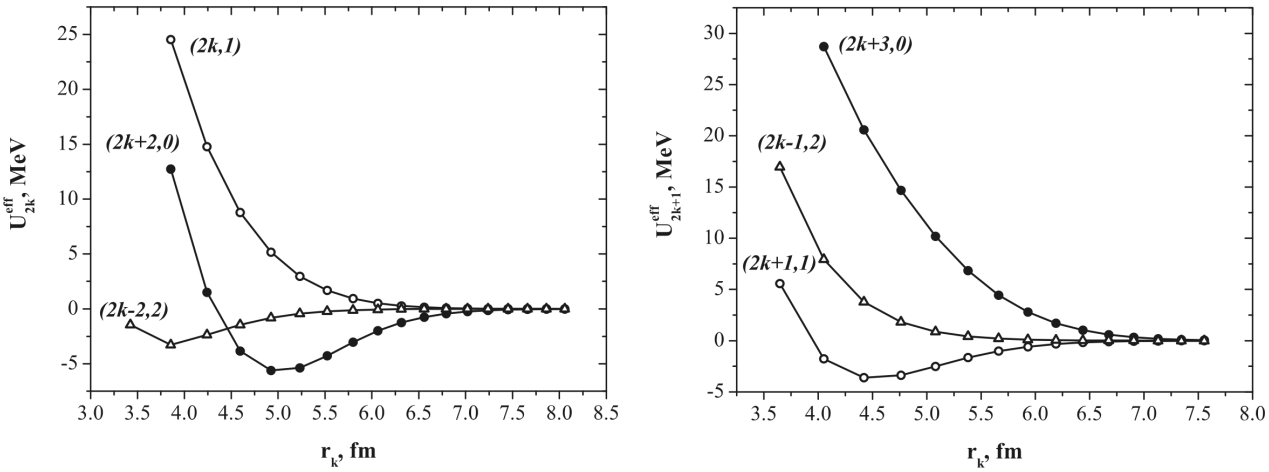

Fig. 2. Effective ^{6}He + α interaction generated by the kinetic energy exchange terms. Indices of SU(3) symmetry are indicated on the curves

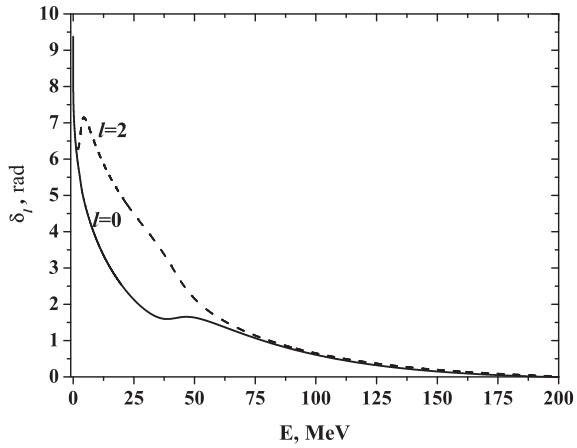

Fig. 3. Phase shifts $\delta_l(E)$ for the ^{6}He + α system that are generated by the modified kinetic-energy operator. The values of orbital angular momentum l of the relative motion of a cluster are indicated on the curves

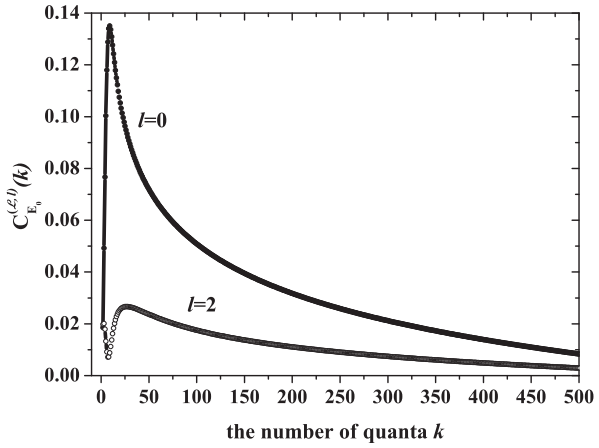

Fig. 4. Coefficients $C^{(\mathcal{L},\,l)}_{E_0}(k)$ in the expansion of the wave function for the $L^\pi = 0^+$ ground state of ^{10}Be $=^6$ He + α nucleus with the energy $E_0 \simeq -10$ keV in the angular-momentum coupled basis, provided that only the modified kinetic energy operator is taken into account. The values of the orbital angular momentum $l = \mathcal{L}$ of the relative motion of a cluster are indicated on the curves

the Hankel functions of the first and second kinds and the scattering S-matrix elements. As soon as all the eigenvalues Λ_n approach unity, a unitary transformation from the basis SU(3) to the angular momentum coupled basis becomes possible. Hence, we can define the asymptotic behavior of the coefficients of the expansion of the wave function in the SU(3) basis with the matrix of the unitary transformation between the two bases [13],[14]. We would like to note that the coupling between the SU(3) channels through the kinetic energy operator, provided that the eigenvalues of the different SU(3) branches are not identical, immediately results in the off-diagonal elements of the S-matrix and inelastic processes during the collision of two clusters.

The phase-shifts for the ^{6}He + α scattering calculated with regard for the eigenvalues of the Pauli-allowed states are shown in Fig. 3. Unlike the phase shifts in Fig. 1, which are produced merely by the Pauli-forbidden states, the phase shifts in Fig. 3 are

generated both by the Pauli-forbidden and Pauli-allowed states.

Since both Pauli-allowed states $\Psi^{L=0}_{(2k+2,0)}$ and $\Psi^{L=0}_{(2k-2,2)}$ are superallowed states, the influence of the Pauli principle on the kinetic energy of the relative motion of ^{6}He and α-particle results in the attraction. This attraction is strong enough to produce a bound state with the very small binding energy $E_0 \simeq -10$ keV below the $^{10}\text{Be} \to {}^6\text{He} + \alpha$ decay threshold, even without considering the nucleon-nucleon interaction between the nucleons of different clusters. The coefficients of the expansion of the wave function of this state in the basis of a harmonic oscillator are shown in Fig. 4.

As evident from Fig. 2, the SU(3) branch $(2k-2, 2)$ is purely attractive, whereas $(2k+2, 0)$ branch is repulsive at small distances and attractive at large distances between clusters. Since the $^6\text{He}(0^+) + \alpha$ channel prevails in the $\Psi^{L=0}_{(2k-2,2)}$ Pauli-allowed state, it is just this channel that dominates in the ground-state wave function.

Since the s-wave has a dominant contribution to this state, the phase shift $\delta_{l=0}$ at zero energy is equal to 3π instead of 2π. As observed in Fig. 3, the phase shift $\delta_{l=2}$ also exhibit the resonance behavior. Comparing Fig. 3 and Fig. 1, we can conclude that the overall effective interaction generated by the superallowed states $(2k+2, 0)$ and $(2k-2, 2)$ is attractive for the $L^\pi = 0^+$ states of the $^6\text{He} + \alpha$ system.

6. Conclusion

Within a microscopic two-cluster model, we have analyzed the influence of the Pauli principle on the kinetic energy of relative motion of ^{6}He nucleus and α-particle. The eigenvalues and the eigenfunctions of the antisymmetrization operator have been examined in detail. We have constructed the Pauli-forbidden states of the $^6\text{He} + \alpha$ system in the explicit form and discussed their structure. Although the Pauli-forbidden states are generally superpositions of the angular-momentum coupled basis functions, each of the latter functions can be related to a particular Pauli-forbidden state based on the partial composition of this state. These are just the angular-momentum coupled basis functions that enter a given Pauli-forbidden state with the maximum weight. The behavior of the phase shifts of the elastic scattering of α-particle on ^{6}He nucleus generated merely by the Pauli-forbidden states support our conclusions.

The above analysis can be useful in projecting out the Pauli-forbidden states with orthogonalizing pseudopotentials in two-cluster systems involving open-shell clusters.

The structure of the Pauli-allowed states has been extensively discussed. The effective $^6\text{He} - \alpha$ interactions generated due to the effect of the partly-forbidden and superallowed states on the kinetic energy of relative motion of clusters have been constructed. We have shown that the partly forbidden states cause the effective repulsion of clusters, while the superallowed states generate an effective attraction of clusters. We have considered the scattering of ^{6}He nucleus on α-particle in the states of positive parity and zero orbital momentum of the compound ^{10}Be nucleus. This is precisely the case where the effective attraction produced by the two branches of the superallowed states is the most prominent. The strength of this attraction appeared to be high enough to ensure the formation of a near-threshold state of ^{10}Be nucleus in a vicinity of the $^{10}\text{Be} \to {}^6\text{He} + \alpha$ decay threshold, even if there was no nucleon-nucleon interaction between nucleons belonging to different clusters.

It should be noted that the above-mentioned effective interactions resulted from a modification of the kinetic energy of the relative motion of clusters by the Pauli principle are inherent in all two-cluster models. The character of this interaction depends only on the intrinsic cluster wave functions. As soon as we fixed the latter functions, we can expect some effective intracluster interaction, whose main features are determined by the behavior of the eigenvalues of the antisymmetrization operator of the system under study.

Summarizing, our results could provide guidance for choosing the sign and the value of the orthogonalizing pseudopotential strength to simulate the effect of partly-forbidden and superallowed states on the scattering of light neutron-rich nuclei.

Y.L. acknowledges the hospitality of the Padova Section of INFN, where this work was commenced. This work was supported in part by the Program of Fundamental Research of the Physics and Astronomy Division of the National Academy of Sciences of Ukraine.

1. G. Filippov and Yu. Lashko, Phys. Rev. C **70**, 064001 (2004).
2. G.F. Filippov and Yu.A. Lashko, Phys. Part. Nucl. **36**, 714 (2005).
3. L. Canton, J.P. Svenne, K. Amos, and S. Karataglidis, Phys. Rev. Lett. **96**, 072502 (2006).
4. K. Amos, L. Canton, P.R. Fraser, S. Karataglidis, J.P. Svenne, and D. van der Kniff, Nucl. Phys. A **912**, 7 (2013).
5. M. Freer *et al.*, Phys. Rev. Lett. **96**, 042501 (2006).
6. M. Ito, K. Kato, and K. Ikeda, Phys. Lett. B **588**, 43 (2004).
7. D. Suzuki *et al.*, Phys. Rev. C **87**, 054301 (2013).
8. K. Arai, Phys. Rev. C **69**, 014309 (2004).
9. P. Descouvemont, Nucl. Phys. A **699**, 463 (2002).
10. Yu.A. Lashko and G.F. Filippov, Phys. Atom. Nucl. **70**, 1017 (2007).
11. J.A. Wheeler, Phys. Rev. **32**, 1083 (1937).
12. G.F. Filippov, Sov. J. Nucl. Phys. **33**, 488 (1981).
13. G.F. Filippov, Yu.A. Lashko, S.V. Korennov, and K. Katō, Few-Body Syst. **33**, 173 (2003).
14. G.F. Filippov, Yu.A. Lashko, S.V. Korennov, and K. Katō, Few-Body Syst. **34**, 209 (2004).

Received 20.01.15

Ю.А. Лашко, Г.Ф. Філіппов, Л. Кантон

РОЗСІЯННЯ ^{6}He НА α-ЧАСТИНЦІ: ВИБІР ПАРАМЕТРІВ ОРТОГОНАЛІЗУЮЧОГО ПСЕВДОПОТЕНЦІАЛУ НА ОСНОВІ МІКРОСКОПІЧНОГО ПІДХОДУ

Р е з ю м е

У рамках двокластерної мікроскопічної моделі обговорено вплив принципу Паулі на реакцію розсіяння ядра ^{6}He на α-частинці. Детально проаналізовано структуру заборонених і дозволених принципом Паулі станів. Вплив дозволених принципом Паулі станів з відмінними від одиниці власними значеннями на кінетичну енергію відносного руху ^{6}He і α-частинки приводить до ефективної взаємодії між цими ядрами. Це явище можна до певної міри змоделювати за допомогою ортогоналізуючого потенціалу скінченої інтенсивності. Ми оцінили інтенсивність і радіус такої взаємодії на основі мікроскопічної моделі, щоб надати рекомендації до вибору параметрів ортогоналізуючого потенціалу.