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PROPAGATION OF MEASUREMENT ERRORS AND MEASURED MEANS OF A PHYSICAL QUANTITY FOR THE ELEMENTARY FUNCTIONS cos x AND arccos x

New exact rules have been obtained for the propagation of the error and the mean value for a measured physical quantity onto another one with a functional relation of the cos x *or* arccos x *type between those quantities. The obtained formulas are shown to provide an accurate result, if being applied to a set of data obtained in a real experiment. This is a consequence of the fact that the distribution of experimental data is inherently based on the Gaussian weight scheme. An analytical form used to present the mentioned rules ("analytical propagation rules") and the exact character of the latter allow the processing and the analysis of experimental data to be simplified and accelerated.*

K e y w o r d s: propagation of error, propagation of uncertainty.

1. Introduction

It is often impossible to measure the value of a certain physical quantity y directly. Instead, this value has to be determined with the help of another quantity x by using the functional relation $y = h(x)$ between them. The measured x-values, x_i, form a set of random numbers, i.e. a statistical set $\{x_i\}$. The latter is described by two parameters: the mean value (or, simply, the mean) $\langle x\rangle$ and the mean error $|\Delta x|$, which is related with the mean-square deviation $\langle\Delta x^2\rangle$. Those means determine the physical quantity x.

For the given function $y = h(x)$, we can calculate a set of values $\{y_i = h(x_i)\}$. This set also has a statistical character, being described by two parameters: the mean $\langle y\rangle$ and the "error" $|\Delta y|$, which determine, in turn, the calculated physical quantity y. Sometimes, however, we cannot construct the set $\{y_i\}$ and use it to determine $\langle y\rangle$ and $|\Delta y|$. Therefore, in this case, we have to look for the relations $\langle x\rangle \to \langle y\rangle$ and $|\Delta x| \to |\Delta y|$, by using the properties of the functional relation $y = h(x)$. This is the essence of the propagation of the error of the physical quantity x on a new physical quantity $y = h(x)$ and the calculation of its "shifted mean value" after processing a set of physical measurements $\{x_i\}$. This problem is rather challenging.

For example, when carrying out X-ray diffraction measurements, we are not interested, generally speaking, in the values and the measurement accuracy of X-ray scattering angles from a crystal. Our goal is the unit cell parameters and their "propagated" accuracy. In the simplest case of the Bragg–Wulf equation,

$$2d\sin\theta = n\lambda,$$

it looks like the error propagation $\Delta\theta \to \Delta d$. In such a simple case, the error Δd can be roughly estimated by differentiating this equation, i.e.

$$\Delta d = -\cot\theta\,\Delta\theta.$$

However, in more complicated cases, this procedure is not so simple and may produce wrong results. For instance, in practice, the same parameters of a unit cell are determined from an overdetermined system of quadratic-type equations (50–100 equations) of the Bragg–Wulf type,

$$\lambda^2\big(h^2a_*^2+k^2b_*^2+l^2c_*^2+2hka_*b_*\cos\gamma_*+$$

$$+2hla_*c_*\cos\beta_*+2klb_*c_*\cos\alpha_*\big)=4\sin^2\theta_i,$$

where the right-hand sides contain the known, i.e. experimentally measured, values. From this system, using statistical methods, six means and six deviations are obtained for six unknown quantities: a_*^2, b_*^2, c_*^2, $a_*b_*\cos\gamma_*$, $a_*c_*\cos\beta_*$, and $b_*c_*\cos\alpha_*$. Then the mean values have to be calculated for the reciprocal lattice parameters a_*, b_*, c_*, α_*, β_*, and γ_*, and the deviations have to be propagated on them. At the next stage, we have to obtain six means for the direct lattice parameters ($a_*\to a$, $b_*\to b$, $c_*\to c$, $\alpha_*\to\alpha$, $\beta_*\to\beta$, $\gamma_*\to\gamma$) and propagate six variances on them, by using an involved system of relations (7 equations) of the type

$$\cos\alpha=\frac{\cos\beta_*\cos\gamma_*-\cos\alpha_*}{\sin\beta_*\sin\gamma_*}.$$

The calculation procedure for the means and deviations also becomes complicated and works badly if the function $H(\cos x,\arccos x)$ is a chain of functions $\cos x$ and $\arccos x$ or any other combination of those functions, because the whole function H has to be differentiated with respect to x. The expansion in series [1] at the point $x_0=\langle x\rangle$,

$$H(x)-H(x_0)=\frac{dH}{dx_0}(x-x_0)+\frac{1}{2}\frac{d^2H}{d^2x_0}(x-x_0)^2+...,$$

can give more exact results, if higher-order terms in the expansion are taken into account. However, the calculations become more cumbersome in this case.

Analytical formulas for the propagation of error and the shifted mean would greatly simplify the required calculations. However, till now, they were known only for the linear function $y=kx$ [1]. It should be noted that the propagation of errors with the help of the expansion in a Taylor series ("differentiation"), if it is regarded as a method, has a more general character, because it is applicable to any continuous function. On the contrary, the "analytical" approach reduces its usage to specific functions (in this work, these are $\cos x$ and $\arccos x$). Therefore, in all modern theoretical and practical applications, methods, and considerations of the error propagation, this procedure is built exclusively on the basis of the differentiation operation [3–11]. The best review of the problems associated with the "analytical" propagation of errors was made in work [1].

2. New Rules for the Calculation of Mean and Propagation of Error in the Case of Elementary Functions cos x and arccos x

In order to obtain the analytical rules for two chosen functions, cos x and arccos x, the mean $\langle x\rangle$ and the "error" $k\,\langle\Delta x\rangle^2$ were related (formalized) to the basic concepts of mathematical statistics: the mathematical expectation E_x and the variance D_x of the measured quantity x,

$$\langle x\rangle\approx E_x,\quad k\,\langle\Delta x\rangle^2\approx D_x.$$

In the framework of this formalization, the individual values x_i of a measured physical quantity x are assumed to appear in accordance with a certain function $f(x)$, which describes the probability distribution for the appearance of x_i. Of course, this function depends on the measurement conditions (it implicitly depends on the measurement device, chosen technique, and so on). As usual, the function $f(x)$ is normalized, and, if the physical quantity x has a continuous distribution, it is called the probability density function for the appearance of x [1]:

$$\int\limits_{-\infty}^{\infty}f(x)dx=1. \tag{1}$$

In this case, the true value of x, which is called the mathematical expectation, can be calculated if the function $f(x)$ is known:

$$\mu=E_x=\int\limits_{-\infty}^{\infty}xf(x)dx \tag{2}$$

Equation (2) is the definition of mathematical expectation E_x [1]. Simultaneously, the function $f(x)$ determines the dispersion of the physical quantity x [1], i.e. the spread of its values at measurements:

$$D_x=\int\limits_{-\infty}^{\infty}(x-E_x)^2f(x)dx=$$

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$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx; \qquad \mu = E_x. \tag{3}$$

Among the distributions $f(x)$, the so-called normal (Gaussian) probability distribution is considered to be the most important [1]:

$$f(x) = \frac{p}{\sqrt{\pi}} \exp[-p^2(x-\mu)^2], \tag{4}$$

where

$$p^2 = \frac{1}{2D_x}, \quad \mu = E_x.$$

In the case where the quantities x and y are related by the functional dependence $y = h(x)$, the mathematical expectation and the variance for the function $h(x)$ equal [1]

$$\chi = E_h = \int_{-\infty}^{\infty} h(x)f(x)dx, \tag{5}$$

$$D_h = \int_{-\infty}^{\infty} [h(x) - E_h]^2 f(x)dx = \int_{-\infty}^{\infty} [h(x) - \chi]^2 f(x)dx. \tag{6}$$

Expression (6) can be rewritten in a more convenient form [1],

$$D_h = \int_{-\infty}^{\infty} [h^2(x) - 2h(x)E_h + E_h^2]f(x)dx =$$
$$= \int_{-\infty}^{\infty} h^2(x)f(x)dx - E_h^2. \tag{7}$$

In Eqs. (4)–(7), the quantities $\mu = E_x$ and D_x enter $f(x)$ as parameters. Therefore, strictly speaking, $f(x) = f(x, E_x, D_x)$, and

$$E_h = \int_{-\infty}^{\infty} h(x)\, f(x, E_x, D_x)dx, \tag{8}$$

$$D_h + E_h^2 = \int_{-\infty}^{\infty} h^2(x)\, f(x, E_x, D_x)dx. \tag{9}$$

It is easy to see that Eqs. (8) and (9) are integral equations. Having solved them, we could obtain the desired analytical relations, on the one hand, between E_h and D_h (they are analogs of the means for the function $h(x)$) and, on the other hand, those between E_x and D_x (analogs of the measured means).

In the case of two elementary functions, cos x and arccos x, it turned out that the tabulated integrals [2] similar to Eqs. (8) and (9) can be chosen, which makes the problem resolved (see Appendix). For the function cos x, those relations look like

$$E_{\cos} = \exp\left(-\frac{D_x}{2}\right) \cos E_x;$$
$$D_{\cos} = \frac{1}{2}[1 - \exp(-D_x)][1 - \exp(-D_x) \cos 2E_x], \tag{10}$$

where E_x and D_x are the mean and the error, respectively, for measured data, whereas $E_{\cos}$ and $D_{\cos}$ are the corresponding quantities for the propagation of the results using the function cos x.

For the function arccos x, the corresponding relations read

$$E_{\arccos} = \arccos \frac{E_x}{\pm\sqrt{E_x^2 + \sqrt{(1-E_x^2)^2 - 2D_x}}};$$
$$D_{\arccos} = \ln \frac{1}{E_x^2 + \sqrt{(1-E_x^2)^2 - 2D_x}}, \tag{11}$$

where E_x and D_x are the mean and the error, respectively, for measured data, whereas $E_{\arccos}$ and $D_{\arccos}$ are the corresponding quantities for the propagation of the results using the function arccos x.

Hence, we obtained the desired rules for the propagation of error and the calculation of a shifted mean of the type $E_h = E_h(E_x, D_x)$ and $D_h = D_h(E_x, D_x)$ for the functions $h(x) = \cos x$ and arcos x.

3. Application of New Rules to Experimental Data

The set of experimental data is a collection of separate random values x_i measured for a physical quantity x; this is the so-called "sample" $\{x_i\}$. The distribution of the quantity x can be continuous [1], i.e. $\{x_i\}$ is a set of values randomly "chosen" by a measurement device from a continuous set.

Let us consider how the obtained relations work in the case of samples. For this purpose, let us calculate the means for four samples: selected from two sets of experimental data $\{x_i\}$ and from two sets of calculated functions cos x and arccos x. First, let us calculate them in the standard way (it will be considered as a reference). The obtained result will be compared with the results calculated, by using relations (10) and (11), and with the results obtained by the series expansion method (differentiation) [1].

3.1. Example for $\cos x$

As an example, let the sample $\{x_i\}$ contain 20 measurements for an angle of the unit cell (hereafter, the presented samples were constructed on the basis of the measurement data obtained on a three-circle diffractometer [12, 13]):

$\{x_i\}$ = 70.5, 70.58, 70.66, 70.74, 70.82, 70.9, 70.98, 71.06, 71.14, 71.22, 70.5, 70.42, 70.34, 70.26, 70.18, 70.1, 70.02, 69.94, 69.86, and 69.78 (deg).

The arithmetic means calculated for this sample with the constant probability $w_i = 1/20$ give us the following values:

$$E_n = 70.5; \quad D_n = 0.1824, \quad \Delta_n = 0.42708.$$

Using them as the first approximation, we calculate the Gaussian means (this routine takes 2 to 3 iterations) with the help of the Gaussian weight scheme:

$$E_x = \frac{\sum x_i w_i}{\sum w_i}, \quad D_x = \frac{\sum (x_i - E_x)^2 w_i}{\sum w_i}, \quad \Delta_x = \sqrt{D_x}, \tag{12}$$

where

$$w_i = \frac{p}{\sqrt{\pi}} \exp[-p^2 (x_i - \mu)^2], \quad p^2 = \frac{1}{2D_x}. \tag{13}$$

Then we obtain

$$E_x = 70.5, \quad D_x = 0.11736, \quad \Delta_x = 0.34258.$$

In other words, for this sample, we have $E_x = 70.5 \pm \pm 0.3$ deg.

For the calculation of means for the function cos x to be correct, it is necessary to construct a new statistical sample $\{\cos x_i\}$ and calculate the means for it. The new sample looks like

$\{\cos x_i\}$ = 0.33381, 0.33249, 0.33117, 0.32986, 0.32854, 0.32722, 0.3259, 0.32458, 0.32326, 0.32194, 0.33381, 0.33512, 0.33644, 0.33775, 0.33907, 0.34038, 0.341690, 0.343, 0.34432, and 0.34563.

Using the values of E_x, D_x, and Δ_x, as well as formulas (12) and (13), we obtain the sought means for the function cos x in the standard way (the reference):

$$E_{\cos} = 0.3338, \quad D_{\cos} = 3.17649 \times 10^{-5},$$

$$\Delta_{\cos} = 0.00564.$$

The use of the analytical relations (10) for the propagation of errors gives the values:

$$E_{\cos} = 0.33381, \quad D_{\cos} = 3.18104 \times 10^{-5},$$

$$\Delta_{\cos} = 0.00564.$$

We intentionally left more digits than required (two digits for D_x, and only one for Δ_x) in order to trace all calculations in more details. The results demonstrate that, in the case of function cos x, the standard deviations $\Delta_{\cos x}$ completely coincide. In other words, the propagation of errors for the function cos x according to relations (10) is correct and gives good results for samples.

3.2. Example for arccos x

The other example will be considered for the function arccos x. The following sample $\{\cos x_i\}$ for the measured angle α of the unit cell is used [12, 13]:

$\{y_i\} = \{\cos x_i\}$ = 0.18224, 0.17674, 0.17399, 0.16436, 0.16436, 0.16298, 0.16298, 0.1616, 0.1616, 0.16023, 0.18224, 0.18772, 0.19047, 0.20005, 0.20005, 0.20142, 0.20142, 0.20279, 0.20279, and 0.20415.

We repeat the standard procedure for the calculation of sample means:

(i) we calculate the arithmetic means (the probability $w_i = 1/20$):

$$E_n = 0.18221, \quad D_n = 2.80422 \times 10^{-4}, \quad \Delta_n = 0.01675;$$

(ii) then, we calculate the Gaussian means using weight scheme (13):

$$E_y = 0.18222, \quad D_y = 1.9731 \times 10^{-4}, \quad \Delta_y = 0.00444;$$

(iii) finally, we form an array (sample) in accordance with the function arccos y:

$\{\arccos y_i\}$ = 79.5, 79.82, 79.98, 80.54, 80.54, 80.62, 80.62, 80.7, 80.7, 80.78, 79.5, 79.18, 79.02, 78.46, 78.46, 78.38, 78.38, 78.3, 78.3, and 78.22 (deg). The standard statistical processing of this sample with the use of the values of E_y, D_y, and Δ_y results in

$$E_{\text{arccos}} = 79.5, \quad D_{\text{arccos}} = 0.67007,$$

$$\Delta_{\text{arccos}} = 0.81858.$$

The calculations by relations (11) give the following values (attention should be paid that the second equation in (11) gives values for D_{arccos} in radians, which we transform into degrees according to work [1]):

$E_{\text{arccos}} = 79.4998, \quad D_{\text{arccos}} = 0.66995,$

$\Delta_{\text{arccos}} = 0.81851.$

One can see that the coincidence in this case is almost ideal again. In other words, for the function arccos x, the propagation of errors using relations (10) and (11) is also correct and gives good results for samples.

The propagation of errors using the series expansion (differentiation) gives the following values:

$E_{\text{arccos}} = 79.5009, \quad D_{\text{arccos}} = 0.066939,$

$\Delta_{\text{arccos}} = 0.25872.$

The numerical results for all three methods can be compared easily.

4. Some Common Properties of the Obtained Relations

The analytical form obtained for the propagation rules allows the features of corresponding relations to be easily distinguished and even the relevant dependences to be plotted graphically, which is very useful for planning and analyzing the physical experiment.

It should be noted that the quantities E_h, D_h, E_x, and D_x are interrelated. In addition, E_h and D_h are functions of two variables rather than one:

$$E_h = E_h(E_x, D_x), \quad D_h = D_h(E_x, D_x).$$

Sometimes, this fact may be difficult to get used to, as, e.g., the fact that the errors $\Delta_{\cos}$ and Δ_{arccos} of the function $h(x)$ depend on the measured mean value $\langle x \rangle$. All that is well illustrated in Figs. 1 and 2, where the dependences of the variances $D_h = D_{\cos}(E_x, D_x)$ and $D_h = D_{\text{arccos}}(E_x, D_x)$ on the values of measured "mean" arguments E_x (D_x is a parameter) are depicted. In addition, the possibility to plot the obtained relations allows the character of future measurements to be discussed and planned.

It becomes clear why the radicand in the second equation in (11) is always positive, i.e. $(1 - E_x^2)^2 \geqslant 2D_x$.

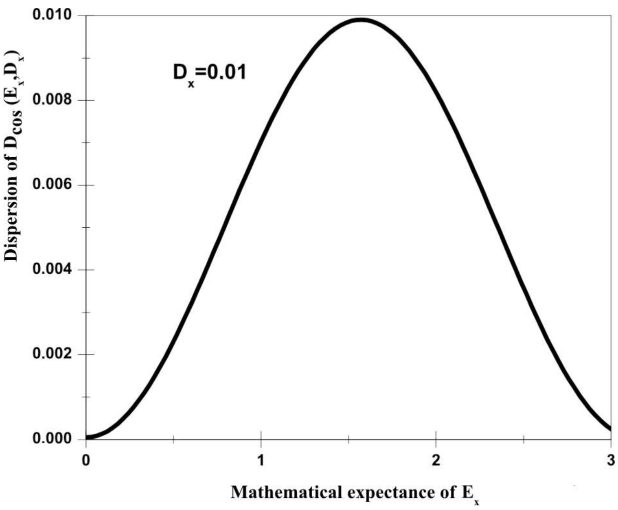

Fig. 1. Dependence of the $D_{\cos}(E_x, D_x)$ function variance on E_x at $D_x = 0.01$

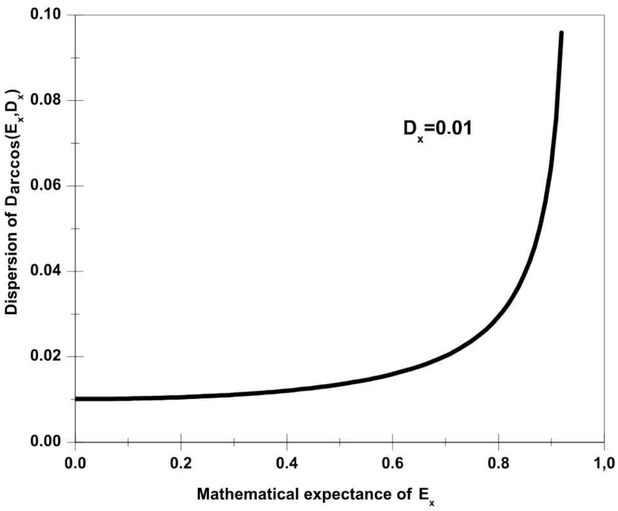

Fig. 2. The same as in Fig. 1, but for the $D_{\text{arccos}}(E_x, D_x)$ function

Note that, in the limiting case $D_x = 0$,

$E_h = E_{\cos} = \cos E_x; \quad D_h = D_{\cos} = 0;$

$E_h = E_{\text{arccos}} = \arccos \pm E_x, \quad D_h = D_{\text{arccos}} = 0.$

Therefore, the "usual" propagation rules

$E_{\cos} = \cos E_x, \quad E_{\text{arccos}} = \arccos \pm E_x$

can be applied. In other cases where the D_x-values are rather considerable, expressions (10) and (11 for $E_{\cos}$ and E_{arccos} give more adequate values.

5. Conclusions

Relations (10) and (11) provide a correct result for samples and can be widely used to considerably reduce and to simplify computational procedures in the case of the functions cos x and arccos x. In the case where the initial array of experimental data is absent, the method of error propagation may turn out a unique simple correct way to calculate E_h and D_h, as well as the errors σ, for the indicated functions. Since D_h and the errors σ for both examined functions practically coincide with the corresponding real values, the exact propagation of errors is possible for a chain of functions of the type cos(arccos(cos(arccos... (x)))) or any other sequence of indicated functions.

Therefore, on the basis of the obtained analytical relations, two simple universal algorithms for the calculation of pairs of the separate values $(E_{\cos}, D_{\cos})$ and $(E_{\arccos}, D_{\arccos})$ can be constructed. Those algorithms can be inserted as separate modules (subroutines) into any software program. The algorithms remain transparent (easy for reading) at that. This is essentially impossible for other propagation methods, because the latter demand that the superposition of functions should be expanded in series (or differentiated) as a whole. Therefore, a separate procedure has to be built for every problem.

The magnitude of function error can be predicted, and its dependence in the planned region of measurements of a physical quantity can be plotted.

Interesting is the possibility to obtain an exact mean shift for $E_{\cos}$ and $E_{\arccos}$. In the presented examples, this shift does not affect the mean values and does not play any role. However, in some applications, it does exist, and its value can be used.

Since the analytical expressions for the means $(E_{\cos}; D_{\cos})$ and $(E_{\arccos}; D_{\arccos})$ are inherently connected with the Gaussian distribution, the calculated value allows them to be compared with the values of the same quantities calculated for different distributions. The minimum of $D_{\cos}$ or $D_{\arccos}$ is a criterion to decide, which of them is better.

APPENDIX

In this Appendix, the validity of the relations obtained for two functions, $E_{\cos}$ and $E_{\arccos}$, i.e. the reduction of integral equations (8) and (9) to tabulated integrals and the reduction of the obtained relations to the convenient forms (10) and (11), is proved mathematically.

1. Mathematical expectation E_h for the function $h(x) = \cos x$

Making allowance for the Gaussian distribution $f(x)$ (see Eq. (4)) in Eq. (8) and substituting $y = x - \mu$, we obtain

$$\chi = E_h = \frac{p}{\sqrt{\pi}} \int_{-\infty}^{\infty} \cos(x)\, \exp[-p^2(x-\mu)^2]dx =$$

$$= \frac{p}{\sqrt{\pi}} \int_{-\infty}^{\infty} \cos(y+\mu) \exp[-p^2y^2]dy = \frac{p}{\sqrt{\pi}}\, J. \quad (14)$$

The integral J is nothing else but the tabulated integral T_2 (3 896.2) from work [2]. As $p \leftrightarrow q$, it looks like

$$T_2 = \int_{-\infty}^{\infty} \cos q(y+\lambda) \exp[-p^2y^2]dy = \frac{\sqrt{\pi}}{p}\, \exp\left(-\frac{q^2}{4p^2}\right) \cos\lambda.$$

It is evident that $J = T_2$, if $q = 1$ and $\lambda = \mu$. Then we immediately obtain

$$J = \frac{\sqrt{\pi}}{p} \exp\left(-\frac{q^2}{4p^2}\right) \cos\lambda = \frac{\sqrt{\pi}}{p} \exp\left(-\frac{1}{4p^2}\right) \cos\mu. \quad (15)$$

Recalling that $\mu = E_x$ and substituting this value into expression (14), we obtain a final relation between the integral $E_{\cos}$ and the integrals E_x and D. Taking into account that $p^2 = \frac{1}{2D_x}$, this relation looks like

$$\chi = E_{\cos} = \exp\left(-\frac{D_x}{2}\right) \cos E_x. \quad (16)$$

This is the sought result. At small D_x, there is a small shift induced by the factor $\exp(-\frac{D_x}{2}) \approx 1$; so it can be ignored under certain conditions. However, Eq. (16) is an exact working formula for $h(x) = \cos x$.

2. Variance D_h for the function $h(x) = \cos x$

From Eq. (7), we obtain the error propagation

$$D_{\cos} = \int_{-\infty}^{\infty} \cos^2(x)\, f(x)dx - E_h^2 = J_0 - E_h^2. \quad (17)$$

Let us transform J_0 to the tabulated form:

$$J_0 = \int_{-\infty}^{\infty} \cos^2(x)\, f(x)dx = \int_{-\infty}^{\infty} (1+\cos 2x)/2\, f(x)dx =$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)dx + \int_{-\infty}^{\infty} \cos 2x\, f(x)dx\right] = \frac{1}{2} + \frac{1}{2}\, J_{01}.$$

For $y = x - \mu$, we obtain the expression for J_{01}:

$$J_{01} = \frac{p}{\sqrt{\pi}} \int_{-\infty}^{\infty} \cos 2x\, \exp[-p^2(x-\mu)^2]dx =$$

$$= \frac{p}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \cos 2(y+\mu)\, \exp[-p^2y^2]dx = \frac{p}{\sqrt{\pi}}\, J_{02}.$$

The integral J_{02} for $q = 2$ and $\lambda = \mu$ coincides with the tabulated integral T_2 (3 896.2) in work [2], which, as $p \leftrightarrow q$, looks like

$$T_2 = \int\limits_{-\infty}^{\infty} \cos q(y+\lambda)\, \exp[-p^2y^2]dy = \frac{\sqrt{\pi}}{p}\, \exp\left(-\frac{q^2}{4p^2}\right) \cos q\lambda.$$

Therefore, using the substitutions $\mu = E_x$ and $p^2 = \frac{1}{2D_x}$ again, we finally obtain

$$J_0 = \frac{1}{2} + \frac{1}{2}\, J_{01} = \frac{1}{2} + \frac{1}{2}\, \frac{p}{\sqrt{\pi}}\, J_{02} =$$
$$= \frac{1}{2} + \frac{1}{2}\, \frac{p}{\sqrt{\pi}} \frac{\sqrt{\pi}}{p}\, \exp\left(-\frac{1}{p^2}\right) \cos 2\mu =$$
$$= \frac{1}{2} + \frac{1}{2}\, \exp(-2D_x)\, \cos 2E_x.$$

Substituting J_0 into Eq. (17), we obtain the following "crude" expression for $D_{\cos}$, because it contains $E^2_{\cos}$:

$$D_{\cos} = \frac{1}{2} + \frac{1}{2}\, \exp(-2D_x)\, \cos 2E_x - E^2_{\cos}.$$

Substituting $E^2_{\cos}$ with the help of Eq. (16) into this formula, we obtain the explicit dependence $D_{\cos}(E_x, D_x)$:

$$D_{\cos} = \frac{1}{2} + \frac{1}{2}\, \exp(-2D_x)\, \cos 2E_x - \exp(-D_x)\, \cos^2 E_x. \quad (18)$$

This formula is the rule of "error propagation" for $h(x) = \cos x$. Expression (11) can be rewritten in a more homogeneous form:

$$D_{\cos} = \frac{1}{2}\, [1 - \exp(-D_x)][1 - \exp(-D_x)\, \cos 2E_x], \quad (19)$$

if the relation

$$\cos^2 E_x = \frac{1}{2}[1 + \cos 2E_x]. \quad (20)$$

is taken into account.

It should be noted that all mathematical procedures performed above (the presentation of an integral as a sum of integrals, factorization, and so on) are correct operations from the viewpoint of statistics rules [1].

3. The mean E_h and the variance D_h for the function $h(x) = \arccos x$

The direct way to calculate $E(\arccos x)$ and $D(\arccos x)$ using tabulated integrals is rather a problematic task. The desired relations can be obtained, if the function $\arccos x$ is considered as the inverse function to $\cos x$, and relations (10) are applied. Really, Eqs. (10) give us explicit relations between four integrals or, roughly speaking, four values: $E_{\cos}$, $D_{\cos}$, E_x, and D_x:

$$E_{\cos} = E_{\cos}(E_x, D_x);\ D_{\cos} = D_{\cos}(E_x, D_x). \quad (21)$$

The inverse functions $E_x = E_x(E_{\cos}, D_{\cos})$ and $D_x = D_x(E_{\cos}, D_{\cos})$ obtained from Eqs. (10) and (21) must also correctly describe the mathematical relations between four integrals E_x, D_x, $E_{\cos}$, and $D_{\cos}$. However, if $E_{\cos}$ and $D_{\cos}$ are obtained in any other way (e.g., if they are measured) and have the same numerical values as those calculated by Eq. (10), they will satisfy the constraint equations (10) for four integrals if and only if the quantities E_x and D_x have the same values as in Eq. (10).

In other words, if $y = \cos x$ and, accordingly, $x = \arccos y$, then relations of the type $E_x = E_x(E_y, D_y)$ and $D_x = D_x(E_y, D_y)$, which are inverse to Eqs. (10) and (21), give us true values for the integral expressions of the mathematical expectation E_x and variance D_x that were determined using Eqs. (8) and (9) for a random variable function y, which is connected with the variable x by means of the law $y = \cos x$ (or $x = \arccos y$). Therefore, by solving Eq. (10) with respect to x, we can simply calculate the values of E_x and D_x on the basis of E_y- and D_y-values, which are the means for the measured random variable y, if the latter is connected with x by the relation $x = \arccos y$.

Let us solve Eq. (10) with respect to E_y and D_y. For this purpose, let us rewrite those equations in the form

$$E_y = \exp\left(-\frac{D_x}{2}\right) \cos E_x, \quad (22)$$

$$D_y = \frac{1}{2}\, [1 - \exp(-D_x)][1 - \exp(-D_x)\, \cos 2E_x]. \quad (23)$$

bearing in mind that the integrals E_y and D_y are coupled with the function $y = \cos x$, and the integrals E_x and D_x with the function $x = \arccos y$. Let us solve those equations with respect to the integrals E_x and D_x, i.e. let us obtain the equations inverse to Eqs. (10), (22), and (23). From Eq. (20), we have

$$\cos 2E_x = 2\cos^2 E_x - 1.$$

From Eq. (22), we obtain the equation, whose both terms are denoted as f:

$$\frac{E_y^2}{\cos^2 E_x} = \exp(-D_x) = f; \quad Z = \frac{1}{\cos^2 E_x}; \quad f = Z\, E_y^2. \quad (24)$$

Then Eq. (23) reads

$$2D_y = 1 - f^2 + 2f^2\, \cos^2 E_x - 2f\, \cos^2 E_x^2.$$

Substituting notations (24) into this equation and carrying out simple transformations, we obtain a quadratic equation for $Z = Z(E_x)$,

$$Z^2 - 2Z + B = 0, \quad (25)$$

where

$$B = \frac{[2E_y^2 + 2D_y - 1]}{E_y^4} = B(E_y, D_y).$$

The solution of this equation brings us to

$$\cos^2 E_x = \frac{E_y^2}{E_y^2 \pm \sqrt{(1 - E_y^2)^2 - 2D_y}}. \quad (26)$$

Since $\cos^2 E_x \leqslant 1$, we have to select the plus sign in front of the root sign. Ultimately, we have

$$E_x = \arccos \frac{E_y}{\pm\sqrt{E_y^2 + \sqrt{(1 - E_y)^2 - 2D_y}}}. \quad (27)$$

The sign plus or minus is selected, by depending on the "common sense", i.e. on the expected value of E_x.

The solution for D_x is found from Eqs. (22) and (26) as

$$D_x = \ln \frac{\cos^2 E_x}{E_y^2} = \ln \frac{1}{E_y^2 + \sqrt{(1-E_y^2)^2 - 2D_y}}. \quad (28)$$

Since $D_x \geqslant 0$, there must be

$$E_y^2 + \sqrt{(1-E_y^2)^2 - 2D_y} \leqslant 1.$$

This inequality is satisfied, because $E_y \leqslant 1$ on all occasions. As a consequence, the following chain of inequalities has to be obeyed:

$$(1-E_y^2)^2 - 2D_y \leqslant (1-E_y^2)^2 \to$$

$$\to \sqrt{(1-E_y^2)^2 - 2D_y} \leqslant 1 - E_y^2 \to$$

$$\to E_y^2 + \sqrt{(1-E_y^2)^2 - 2D_y} \leqslant 1.$$

The radicand in Eq. (27) must be positive. This assertion can be understood from the following consideration. The quantity E_y is, in essence, the function cos x, i.e. $E_y \leqslant 1$ and separate measurements give $E_i \leqslant 1$ as well. For the confidence interval $\sigma\sqrt{2}$, the average deviation $E_y + \sigma\sqrt{2}$ has to satisfy the inequality $E_y + \sigma\sqrt{2} \leqslant 1$. Accordingly, $\sigma\sqrt{2} \leqslant 1 - E_y \leqslant 1 - E_y^2$, so that $2D_y \leqslant (1-E_y^2)^2$ and, finally, $(1-E_y^2)^2 - 2D_y \geqslant 0$.

Now, let us rewrite the obtained relations (10), (26), and (27) in a clearer symbolic form, by using the notation x for the measured physical quantity (argument) and the notation h for the corresponding function (cos x or arccos x):

$$E_h = E_{\cos} = \exp\left(-\frac{D_x}{2}\right)\cos E_x;$$
$$D_h = D_{\cos} = \frac{1}{2}[1-\exp(-D_x)][1-\exp(-D_x)\cos 2E_x]; \quad (29)$$

$$E_h = E_{\arccos} = \arccos \frac{E_x}{\pm\sqrt{E_x^2 + \sqrt{(1-E_x^2)^2 - 2\,D_x}}};$$
$$D_h = D_{\arccos} = \ln\left(\frac{1}{E_x^2 + \sqrt{(1-E_x^2)^2 - 2\,D_x}}\right). \quad (30)$$

In view of the formalization

$$x \approx E_x; \quad k(\Delta x)_2 \approx D_x,$$

the obtained relations correspond to the desired "propagation rules" for the means and errors of the cosine and arccosine functions:

$$X \to H;\ |\Delta X| \to |\Delta H|.$$

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ПЕРЕНОС ПОХИБОК ТА СЕРЕДНІХ ВИМІРІВ ФІЗИЧНОЇ ВЕЛИЧИНИ ДЛЯ ЕЛЕМЕНТАРНИХ ФУНКЦІЙ cos(x) ТА arccos(x)

Р е з ю м е

Отримані нові точні "правила переносу похибки та середнього" однієї вимірюваної фізичної величини на іншу, що пов'язана з нею функційним зв'язком типу cos(x) або arccos(x). Показано, що добуті співвідношення ідеально працюють при обробці набору даних реального фізичного дослідження. Це пов'язано з тим, що по природі в них неявно вже закладена вагова схема Гауса. Аналітична форма, в якій наведені згадані правила ("аналітичні правила переносу"), а також точний характер їх дозволяє спростити і прискорити процедуру обробки й аналізу експериментальних даних.