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COMMENT ON PERIHELION ADVANCE DUE TO COSMOLOGICAL CONSTANT

We comment on the recent paper "Note on the perihelion/periastron advance due to cosmological constant" by H. Arakida (Int. J. Theor. Phys. ***52****, 1408 (2013)) and provide simple derivations of both the main result of this paper and of the Adkins–McDonnell's precession formula, on which this main result is based.*

Keywords: celestial mechanics, gravitation, cosmological constant, dark energy.

1. Introduction

Recently Hideyoshi Arakida in the interesting paper [1] clarified some confusion existing in the literature concerning the eccentricity dependence of the perihelion/periastron advance of celestial bodies due to the cosmological constant Λ. He showed that the correct expression for the perihelion/periastron shift per period is

$$\Delta\Theta_p = \frac{\pi c^2 \Lambda a^3}{GM}\sqrt{1-e^2}, \tag{1}$$

where a is the semimajor axis of the orbit, and e is the eccentricity. This result was obtained in [1] with the help of the general formula

$$\Delta\Theta_p = -\frac{2p}{\alpha e^2}\int\limits_{-1}^{1}\frac{dV(z)}{dz}\,\frac{z}{\sqrt{1-z^2}}\,dz \tag{2}$$

for the perihelion/periastron shift per period due to a small central-force perturbation

$$V(z) = V(r(z)) = V\left(\frac{p}{1+ez}\right), \quad p = a\left(1-e^2\right),$$

to the Newtonian potential $V_0(r) = -\alpha/r$, $\alpha = GMm$. Formula (2) was first obtained in [2]. Note that, contrary to [1], but in accord with [2], our $V(r)$ is the perturbation potential energy, not the perturbation potential as in [1]. Therefore, it includes the mass of the orbiting particle m.

We now demonstrate that suitably modified Landau and Lifshitz's approach [3] allows the simple derivation of both (1) and (2).

2. Landau and Lifshitz's Approach

Landau and Lifshitz provided [3] the following expression for $\Delta\Theta_p$ (see the solution of Problem 3 in §15):

$$\Delta\Theta_p = \frac{\partial}{\partial L}\int\limits_{r_{\min}}^{r_{\max}}\frac{2mV(r)}{\sqrt{2m\left(E+\frac{\alpha}{r}\right)-\frac{L^2}{r^2}}}\,dr, \tag{3}$$

where L is the angular momentum, E is the total energy, and the integration is over the unperturbed Keplerian orbit. A simple derivation of (3) can be found in [3]. Using

$$\frac{dr}{\sqrt{2m\left(E+\frac{\alpha}{r}\right)-\frac{L^2}{r^2}}} = \frac{1}{m}\,dt,$$

and extending the integration over the whole orbital period T, it is convenient to rewrite (3) in the form [4]

$$\Delta\Theta_p = \frac{\partial}{\partial L}\int\limits_0^T V(r(t))\,dt = \frac{\partial}{\partial L}\left(T\langle V\rangle\right), \tag{4}$$

where

$$\langle V\rangle = \frac{1}{T}\int\limits_0^T V(r(t))\,dt$$

is the time-average value of the perturbation potential energy over the unperturbed orbit. Note that this time-averaged value is a function of L and E, and it is the total energy E that is to be kept constant when taking the partial derivative $\partial/\partial L$ in (4).

To apply (4) to the problem considered in [1] with the perturbation potential energy

$$V(r) = -\frac{1}{6}\Lambda mc^2r^2, \tag{5}$$

let us use the following parametrization of the unperturbed motion on the Keplerian ellipse [3]:

$$t = \sqrt{\frac{ma^3}{\alpha}}(\xi - e\,\sin\xi), \quad r = a(1 - e\,\cos\xi), \tag{6}$$

where the parameter ξ changes from 0 to 2π. As a result, we get

$$\Delta\Theta_p = -\frac{1}{6}\Lambda mc^2a^2\sqrt{\frac{ma^3}{\alpha}}\frac{\partial}{\partial L}\int\limits_0^{2\pi}(1 - e\,\cos\xi)^3\,d\xi. \tag{7}$$

After the elementary evaluation of the integral, we have

$$\Delta\Theta_p = -\frac{\pi}{3}\Lambda mc^2a^2\sqrt{\frac{ma^3}{\alpha}}\frac{\partial}{\partial L}\left(1 + \frac{3}{2}e^2\right). \tag{8}$$

But

$$e^2 = 1 + \frac{2EL^2}{m\alpha^2} = 1 - \frac{L^2}{m\alpha a},$$

and, therefore,

$$\frac{\partial e^2}{\partial L} = -\frac{2L}{m\alpha a} = -2\sqrt{\frac{1 - e^2}{m\alpha a}}, \tag{9}$$

which together with (8) yield the validity of (1):

$$\Delta\Theta_p = \frac{\pi\Lambda mc^2a^3}{\alpha}\,\sqrt{1 - e^2} = \frac{\pi\Lambda c^2a^3}{GM}\,\sqrt{1 - e^2}. \tag{10}$$

It remains to clarify how the Adkins–McDonnell precession formula (2) can be obtained from (4). Using again parametrization (6), we can write (with $r(\xi) = a(1 - e\,\cos\xi)$)

$$\Delta\Theta_p = \sqrt{\frac{ma^3}{\alpha}}\frac{\partial}{\partial L}\int\limits_0^{2\pi} V(r(\xi))\,(1 - e\,\cos\xi)\,d\xi. \tag{11}$$

Because $\cos(2\pi - \xi) = \cos\xi$, this can be rewritten in the form

$$\Delta\Theta_p = 2\sqrt{\frac{ma^3}{\alpha}}\frac{\partial}{\partial L}\int\limits_0^{\pi} V(r(\xi))\,(1 - e\,\cos\xi)\,d\xi. \tag{12}$$

Now, let us apply the Leibniz integral rule (differentiation under the integral sign) to get

$$\Delta\Theta_p = 2\sqrt{\frac{ma^3}{\alpha}}\,\frac{\partial e}{\partial L}\times$$
$$\times\left[\int\limits_0^{\pi} V'(r(\xi))\,(-a\cos\xi)(1 - e\,\cos\xi)\,d\xi + I\right], \tag{13}$$

where

$$I = -\int\limits_0^{\pi} V(r(\xi))\,\cos\xi\,d\xi = -\int\limits_0^{\pi} V(r(\xi))\,d(\sin\xi), \tag{14}$$

and $V'(r) = \frac{dV(r)}{dr}$. The integration by parts, along with

$$\frac{dV(r(\xi))}{d\xi} = ea\,V'(r(\xi))\,\sin\xi,$$

allows us to rewrite (14) in the form

$$I = ea\int\limits_0^{\pi} V'(r(\xi))\,\sin^2\xi\,d\xi. \tag{15}$$

Substituting (15) into (13), we get

$$\Delta\Theta_p = 2\sqrt{\frac{ma^3}{\alpha}}\,\frac{\partial e}{\partial L}\,a\int\limits_0^{\pi}(e - \cos\xi)\,V'(r(\xi))\,d\xi. \tag{16}$$

At this stage, let us introduce a new integration variable z:

$$z = \frac{\cos\xi - e}{1 - e\,\cos\xi}, \quad \cos\xi = \frac{z + e}{1 + e\,z}. \tag{17}$$

It follows from (17) that

$$e - \cos\xi = -\frac{(1-e^2)z}{1+e\,z}, \quad 1 - e\,\cos\xi = \frac{1-e^2}{1+e\,z},$$
$$\sin^2\xi = \frac{(1-z^2)(1-e^2)}{(1+e\,z)^2}, \tag{18}$$

and

$$d\xi = -\frac{1-e^2}{(1+e\,z)^2}\,\frac{1}{\sin\xi}\,dz = -\frac{\sqrt{1-e^2}}{(1+e\,z)\sqrt{1-z^2}}\,dz. \tag{19}$$

Since

$$\frac{d}{dz}V\left(\frac{a(1-e^2)}{1+e\,z}\right) = V'(r(z))\left(-\frac{a\,e\,(1-e^2)}{(1+e\,z)^2}\right), \tag{20}$$

we can express $V'(r(\xi)) = V'(a(1-e\,\cos\xi)) = V'(r(z))$ in terms of $\frac{dV(r(z))}{dz}$. After taking all these relations into account, (16) becomes

$$\Delta\Theta_p = 2\sqrt{\frac{ma^3}{\alpha}}\,\frac{1}{e}\frac{\partial e}{\partial L}\,\sqrt{1-e^2}\int\limits_{-1}^{1}\frac{dV(r(z))}{dz}\frac{z\,dz}{\sqrt{1-z^2}}, \tag{21}$$

and this relation coincides with the Adkins–McDonnell precession formula (2), because, due to (9),

$$2\sqrt{\frac{ma^3}{\alpha}}\,\frac{1}{e}\,\frac{\partial e}{\partial L}\,\sqrt{1-e^2} = -2\frac{a(1-e^2)}{\alpha\,e^2} = -\frac{2p}{\alpha\,e^2}.$$

3. Concluding Remarks

Various approaches to account for the influence of the cosmological constant on the celestial dynamics can be found in references cited in [1]. Kotkin and Serbo's variant (4) of the Landau and Lifshitz's precession formula and parametrization (6) of the unperturbed motion on the Keplerian ellipse provide, probably, the simplest way to calculate the perihelion/periastron advance of celestial bodies due to the cosmological constant in the framework of the Schwarzschild–de Sitter (Kottler) space-time. This approach also allows a simple derivation of the Adkins–McDonnell precession formula (2) (another simple derivation of this formula, based on the precession of Hamilton's vector, was given in [5]).

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ЗАУВАЖЕННЯ
ПРО ЗМІЩЕННЯ ПЕРИГЕЛІЮ
ПІД ВПЛИВОМ КОСМОЛОГІЧНОЇ СТАЛОЇ

Р е з ю м е

Ми коментуємо нещодавню роботу Х. Аракіда (H. Arakida) “Примітка про зміщення перигелію/периастра через космологічну сталу” (Int. J. Phys. Теор. **52**, 1408–1414 (2013)) і наводимо простий висновок основного результату цієї роботи, а також формули прецесії Адкінса–Макдоннелла, на якому цей результат заснований.