doi: 10.15407/ujpe62.11.1007

V.O. PELYKH, Y.V. TAISTRA
Ya.S. Pidstryhach Institute for Applied Problems of Mechanics and Mathematics,
Nat. Acad. of Sci. of Ukraine
(3b, Naukova Str., Lviv 79060, Ukraine; e-mail: pelykh@iapmm.lviv.ua, ythelloworld@gmail.com)

PACS 04.20.Jb, 95.30.Sf

NULL ONE-WAY FIELDS IN THE KERR SPACETIME

Analytical solutions of the equations for massless fields with arbitrary spins have been obtained in the Kerr metric in the null one-way form, i.e. in the form of ingoing or outgoing, according to Chandrasekhar, fields propagating to or from a black hole, respectively. On the basis of the Newman–Penrose approach in the spinor formulation, the null one-way fields in the Petrov-type D spacetime are considered. A general analytical solution and an analytical solution with separated variables are found for the generalized equations of those fields in the Kerr metric. In the partial case of electromagnetic field, the Maxwell tensor and the energy-momentum tensor for the outgoing and ingoing one-way fields are calculated.

Keywords: massless field, null one-way field, Maxwell spinor, Kerr spacetime, separation of variables.

1. Introduction

The research of the gravitational field influence on classical physical fields (scalar, Dirac–Weyl, Maxwell, and Rarita–Schwinger ones) and on gravitational perturbations is a challenging task of modern mathematical and theoretical physics, and astrophysics. In order to simplify the problem, the influence of those fields on the gravitational one is neglected, by considering them as test fields or perturbations. The study of the behavior of those fields in the gravitational fields created by black holes – in the Schwarzschild, Kerr, and Kerr–Newman metrics – is especially important and interesting.

The main difficulty in studying the fields with nonzero spins is the interdependence of the systems of equations that describe them. Therefore, if no restrictions are imposed on the space generality, none of the gauge (in the case of electromagnetism) or coordinate (in the case of gravitation) conditions can decouple those equations. Teukolsky [1], making use of the Newman–Penrose formalism, partially decoupled the equations for the gravitational, electromagnetic, and neutrino fields in the Petrov-type D spacetime. As a result, he obtained two separate equations for two "extreme" field components. When considering the equations in the Kinnersley tetrad, they were generalized to the Teukolsky master equation (TME), which describes extreme components of the fields with all integer and half-integer spins in the Kerr metric. Using the ansatz $\psi = e^{-i\omega t}e^{im\phi}R(r)S(\theta)$, Teukolsky obtained two ordinary differential equations (ODEs), which are known as the Teukolsky angular equation (TAE) and Teukolsky radial equation (TRE).

Further important results in this direction were obtained, in particular, in works [2–6]. However, the obtaining of solutions for the Maxwell field (as well as other fields, except for scalar ones) in a curved spacetime, which would be rather general or suitable for an effective analysis, as well as researches of their properties, remain to be a complicated task [7]. The difficulty consists in the non-linear character of the eigenvalue problem, because the separation constant ω enters the equation through the parameter

$E_l^m = E_l^m(a\omega)$ (see p. 653 in work [2]). For simplification, special cases of the Maxwell field are considered, which makes it possible to obtain exact solutions for corresponding equations.

In our works [8–10], we separately considered the cases of fields propagating to black holes (according to Chandrasekar's terminology, ingoing fields) and from black hole's vicinity (outgoing fields), i.e. null one-way (NOW) fields (NOWFs), which correspond to two orientations of the electromagnetic principal null direction with respect to the gravitational principal null direction. We obtained a general solution that is expressed in terms of an arbitrary function of integrals of a system of partial differential equations (PDEs) of the first order.

In work [9], we noted that, unlike Teukolsky, we do not neglect the solutions with a singularity on the rotation axis $\theta = 0$, $\theta = \pi$ for the following reasons. The singularity on the semiaxis $\theta = 0$ – both in the Kerr solution and in the solutions of field equations against the Kerr spacetime background – is a consequence of the application of the Boyer–Lindquist coordinate system, which generalizes a spherical coordinate system with its singularity on the semiaxis $\theta = 0$ (here, the determinant of the metric tensor equals zero) onto the Kerr spacetime. Since the metric in the Kerr solution does not cease to be determined at $r = 0$, the values $r < 0$ are also allowed. Therefore, an additional specific semiaxis $\theta = \pi$ arises in the equations for all fields.

However, the singularities on the rotation axis of either the metric tensor of the Kerr spacetime or the solutions of field (e.g., the electromagnetic one) equations against the Kerr spacetime background are not invariant. A single invariant in the Kerr geometry and the invariants of the electromagnetic field have no singularities on the rotation axis, and the metric quadratic form can be analytically continued to it (except for points on the horizon). This approach was proposed in work [9], and it will be applied, when considering the fields with other spins. Accordingly, the domain of definition of such physically meaningful solutions will be restricted by the condition $0 < \theta < \pi$. Such coordinate-singular solutions, owing to their simple form, can be effectively applied to describe processes in a vicinity of the Kerr black hole, which will be dealt with in the next paper. It is expected that the invariant characteristics of fields and processes would also have no singularities at $\theta = 0$ and $\theta = \pi$. In a flat spacetime, in special cases, the solution in the Kerr field obtained by us describes a circularly polarized plane wave and an electromagnetic field, which is similar of a null field in the form of knots and links, and arises from the Hopf fibration [11, 12].

The behavior of algebraically special fields was studied in works [13–16] in detail. In particular, Torres [13] obtained a general solution for an algebraically special Maxwell field in a flat spacetime. In Chandrasekhar's work [14], a gravitational case of the algebraically special field in the Kerr metric was considered, and a solution with separated variables, which contains terms with the $1/r$-, $1/r^2$-, $1/r^3$-, and $1/r^4$-asymptotics, was obtained [see Eqs. (9) and (14) in the cited work]. This result differs from the ours: our solution with the separated variables has only the $1/r$-asymptotic.

The method developed by us for solving a system of equations describing NOW Maxwell fields can also be generalized to the case of NOWFs with arbitrary spin values. Such a generalization and the solution of the system of equations that describes fields of all spin values identically are the aim of this work.

Besides the derivation of the analytical general solution for the generalized system of equations, we will also obtain a solution, by using the variable separation method, which allows one to describe some properties of physical fields in more details (see, e.g., work [17]). We also compared our results with Teukolsky's ones and indicated their further application. In addition, using the Maxwell field as an example, we will construct wave-like solutions in the form of NOWFs. For each solution, we calculate the Maxwell and energy-momentum tensors. Finally, we determine conditions in the coordinate form that are specific to NOWFs.

All equations below are presented in the geometrized system of units, in which $c = G = 1$. Furthermore, we assume all functions to be smooth enough, which does not restrict the physical generality of consideration.

2. Test Zero-Rest-Mass One-Way Free Fields with the Spin l in the Vacuum Type D Spacetime

Let us consider test zero-rest-mass free fields with the spin $l = |s|$, where $s = \pm 1/2, \pm 1, \pm 3/2, \pm 2, ...$ are the spin weight values. The fields are given by a symmetric spinor $\varphi_{ABC\dots KL}$ with $2l$ indices. The evolution equation for such fields looks like [18]

$$\nabla^{AA'}\varphi_{ABC\dots KL} = 0. \quad (1)$$

Let us extend the approach proposed by us in work [9], while considering null electromagnetic fields, onto fields with other spins. For this purpose, let us select the spin basis so that the principal spinors of the Weyl spinor, which are multiple in pairs, because the Kerr spacetime belongs to the type D according to Petrov, would be proportional to the basis ones, i.e. $\Psi_{ABCD} = \gamma_{(A}\gamma_B\delta_C\delta_{D)}$, where $\gamma_A = \gamma_1 o_A$, $\delta_A = -\delta_0\iota_A$, o_A, and ι_A are basis spinors. As a result, we obtain $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and, in accordance with the Goldberg–Sachs theorem, $\kappa = \sigma = \nu = \lambda = 0$.

Below, we consider algebraically special physical fields. We assume that all principal spinors α_A, β_B, ..., λ_L of the spinor $\varphi_{ABC\dots KL} = \alpha_{(A}\beta_B\dots\lambda_{L)}$ are multiple of a multiple of the principal spinor γ_A of the Weyl spinor, i.e. $\alpha_A \sim \gamma_A$, $\beta_B \sim \gamma_B$, ..., $\lambda_L \sim \gamma_L$. As a result, the expansion of the field spinor in the spin basis looks like

$$\varphi_{ABC\dots KL} = \varphi_{2l}\underbrace{o_A o_B \dots o_L}_{2l}, \quad (2)$$

where $\varphi_{2l} = \varphi_{ABC\dots KL}\iota^A\iota^B\iota^C\dots\iota^K\iota^L$. The field $\varphi_{ABC\dots KL}$ is null under this choice [18]. Following Chandrasekhar, we will call it "outgoing". In the case of gravitational field, condition (2) distinguishes wave-type fields according to Lichnerowicz.

Definition 1. A field given by a spinor of form (2) is called the *outgoing null one-way field.*

The components of Eq. (1) for the outgoing NOWF (2) in the vacuum type D spacetime look like

$$\begin{cases} D\varphi_{2l} + (2l\epsilon - \rho)\varphi_{2l} = 0,\\ \delta\varphi_{2l} + (2l\beta - \tau)\varphi_{2l} = 0, \end{cases} \quad (3)$$

where $D = l^a\nabla_a$, $\delta = m^a\nabla_a$, $\Delta = n^a\nabla_a$, and $\bar\delta = \bar m^a\nabla_a$ are derivatives along the directions of the Newman–Penrose null tetrad; and α, β, γ, ϵ, κ, σ, ρ, τ, ν, λ, μ, and π are Newman–Penrose scalars.

Analogously, the "ingoing" NOWF is obtained by selecting all principal spinors α_A, β_B, ..., λ_L of the spinor $\varphi_{ABC\dots KL} = \alpha_{(A}\beta_B\dots\lambda_{L)}$ to be multiple of a multiple of the principal spinor δ_A of the Weyl spinor, i.e. $\alpha_A \sim \delta_A$, $\beta_B \sim \delta_B$, ..., $\lambda_L \sim \delta_L$. In this case, the expansion of the field spinor in the spin basis looks like

$$\varphi_{ABC\dots KL} = \varphi_0\iota_A\iota_B\dots\iota_K, \quad (4)$$

where $\varphi_0 = \varphi_{ABC\dots KL}o^Ao^Bo^C\dots o^L$.

Definition 2. A field given by a spinor of form (4) is called the *ingoing null one-way field.*

The components of Eq. (1) for the ingoing NOWF in the vacuum type D spacetime look like

$$\begin{cases} \Delta\varphi_0 + (\mu - 2l\gamma)\varphi_0 = 0,\\ \bar\delta\varphi_0 + (\pi - 2l\alpha)\varphi_0 = 0. \end{cases} \quad (5)$$

Note that NOWFs (2) and (4) are algebraically special fields of the type N, i.e. all principal spinors of such fields are multiple.

3. General Solution of a Generalized Equation Describing One-Way Fields with the Spin *l* in the Kerr Metric

Let us consider the systems of equations (3) for an outgoing NOWF and (5) for an ingoing one in the Kerr metric in the Boyer–Lindquist coordinates,

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right)dt^2 + \frac{4Mra\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2, \quad (6)$$

where $M > 0$ is the black hole mass, a the specific angular momentum ($0 < a < M$), $\Sigma = r^2 + a^2\cos^2\theta$, and $\Delta = r^2 - 2Mr + a^2$ [1]. The roots of the equation $\Delta = 0$, namely, $r_+ = M + \sqrt{M^2 - a^2}$ and $r_- = M - \sqrt{M^2 - a^2}$, determine the event and Cauchy horizons, respectively. The Newman–Penrose null tetrad is chosen as the Kinnersley tetrad [19]:

$$\begin{aligned} l^a &= \left(\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right),\\ n^a &= \frac{1}{2\Sigma}\left(r^2 + a^2, -\Delta, 0, a\right),\\ m^a &= \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right),\\ \bar m^a &= \frac{1}{\sqrt{2}(r - ia\cos\theta)}\left(-ia\sin\theta, 0, 1, \frac{-i}{\sin\theta}\right). \end{aligned} \quad (7)$$

[1] The application of the same notation Δ for different quantities is traditional for the Newman–Penrose formalism, when describing the Kerr spacetime, and does not lead to a misunderstanding.

The systems of equations for the outgoing and ingoing NOWFs with arbitrary spins l have similar forms in the Boyer–Lindquist coordinates, if the corresponding change of functions is performed. Therefore, let us construct the generalized system of equations,

$$\begin{cases} \dfrac{r^2+a^2}{\Delta}\dfrac{\partial\psi}{\partial t} - k\dfrac{\partial\psi}{\partial r} + \dfrac{a}{\Delta}\dfrac{\partial\psi}{\partial\phi} = 0, \\ ia\sin\theta\dfrac{\partial\psi}{\partial t} - k\dfrac{\partial\psi}{\partial\theta} + \dfrac{i}{\sin\theta}\dfrac{\partial\psi}{\partial\phi} = 0; \end{cases} \tag{8}$$

where $k = \operatorname{sgn} s$, and

$$\psi = \begin{cases} \varphi_{2l}(r - ia\cos\theta)\sin^l\theta, & k=-1; \\ \varphi_0\dfrac{\Delta^l\sin^l\theta}{2(r-ia\cos\theta)^{2l-1}}, & k=1. \end{cases} \tag{9}$$

The general solution of system (8) can be found, by sequentially integrating the partial differential equations of the first order. We obtain

$$\psi = e^{F(\zeta_1,\zeta_2)}, \tag{10}$$

where F is an arbitrary function of the complex integrals of system (8):

$$\zeta_1 = t + k\Bigg(r + M\ln\Delta + \\ + \frac{M^2}{\sqrt{M^2-a^2}}\ln\left|\frac{r-r_+}{r-r_-}\right| - ia\cos\theta\Bigg), \tag{11}$$

$$\zeta_2 = \phi + k\left(\frac{a}{2\sqrt{M^2-a^2}}\ln\left|\frac{r-r_+}{r-r_-}\right| + i\ln\left|\frac{1-\cos\theta}{\sin\theta}\right|\right). \tag{12}$$

In the case of electromagnetic field ($s=\pm1$), the general solution (10) was obtained in our previous work [9]. In the partial case of flat spacetime, this solution is reduced to that by Torres [13]. In this work, the exact solution for a field with an arbitrary spin in the Kerr field was obtained for the first time.

4. Separation of Variables in the System of Equations for the NOWF

The application of the variable separation method for finding regular solutions of the master Teukolsky equation made it possible to reveal the main properties of perturbations and predict bright physical effects in the Kerr field [1, 4]. Bearing all that in mind and due to a necessity to compare the NOWF approach with the others, let us apply this method to the NOWFs.

Let us seek the solution of the system of equations (8) in the form

$$\psi(t,r,\theta,\phi) = T(t)R(r)S(\theta)\Phi(\phi). \tag{13}$$

For the unknown functions, we obtain a system of four ODEs:

$$\begin{cases} T'(t) - \lambda T(t) = 0, \\ \Phi'(\phi) - \nu\Phi(\phi) = 0, \\ R'(r) - k\left(\dfrac{\lambda(r^2+a^2)}{\Delta} + \dfrac{\nu a}{\Delta}\right)R(r) = 0, \\ S'(\theta) - k\left(ia\lambda\sin\theta + \nu\dfrac{i}{\sin\theta}\right)S(\theta) = 0, \end{cases} \tag{14}$$

where $\lambda\in\mathbb{C}$ and $\nu\in\mathbb{C}$ are separation constants. Having solved those equations, we obtain a solution of system (8),

$$\psi = Ce^{\lambda\xi_1 + \nu\xi_2 - iak\lambda\cos\theta + i\nu k\ln\left|\frac{1-\cos\theta}{\sin\theta}\right|}, \tag{15}$$

where

$$\xi_1 = t + k\left(r + M\ln\Delta + \frac{M^2}{\sqrt{M^2-a^2}}\ln\left|\frac{r-r_+}{r-r_-}\right|\right), \tag{16}$$

$$\xi_2 = \phi + k\frac{a}{2\sqrt{M^2-a^2}}\ln\left|\frac{r-r_+}{r-r_-}\right|, \tag{17}$$

and C is a complex constant.

Hence, the equations for NOWFs have solutions with separated variables of form (15), where the function ψ is defined by relations (9). Solution (15) is partial. It is obtained from the general solution (10) by choosing $F(\zeta_1,\zeta_2) = \lambda\zeta_1 + \nu\zeta_2$.

Note that the separation of variables for the system of first-order equations for NOWFs differs from the separation of variables in the Teukolsky approach: the function ψ in this work [see Eq. (9)] is defined differently from the function ψ in work [1].

5. Solutions with Separated Variables in the Cases of Outgoing and Ingoing NOW Maxwell Fields

As an example, let us consider solutions with separated variables in the case of zero-rest-mass free NOW Maxwell fields with $s=\pm1$. The case $s=-1$

describes an outgoing NOW Maxwell field, whereas $s = 1$ an ingoing one [9, 10].

The equation for a free Maxwell field looks like

$$\nabla^{AA'}\varphi_{AB} = 0, \tag{18}$$

where

$$\varphi_{AB} = \varphi_2 o_A o_B - \varphi_1(o_A\iota_B + \iota_A o_B) + \varphi_0\iota_A\iota_B \tag{19}$$

is the spinor of the electromagnetic field (the Maxwell spinor); and $\varphi_2 : \varphi_2 \mapsto \mathbb{C}$, $\varphi_1 : \varphi_1 \mapsto \mathbb{C}$, and $\varphi_0 : \varphi_0 \mapsto \mathbb{C}$ are the components of the spinor φ_{AB} in the spin basis.

In the case of outgoing NOWF, the Maxwell spinor looks like $\varphi_{AB} = \varphi_2 o_A o_B$. The solution with separated variables, φ_2, can be written, by using Eqs. (15) and (9) taken at $k = -1$ and $l = 1$:

$$\varphi_2 = C\frac{e^{\lambda\eta_1+\nu\eta_2+ia\lambda\cos\theta-i\nu\ln\left|\frac{1-\cos\theta}{\sin\theta}\right|}}{\sin\theta\,(r-ia\cos\theta)}, \tag{20}$$

where

$$\eta_1 = t - r - M\ln\Delta - \frac{M^2}{\sqrt{M^2-a^2}}\ln\left|\frac{r-r_+}{r-r_-}\right|, \tag{21}$$

$$\eta_2 = \phi - \frac{a}{2\sqrt{M^2-a^2}}\ln\left|\frac{r-r_+}{r-r_-}\right|. \tag{22}$$

When considering the first and second ODEs in system (14), the following requirements are imposed on their solutions. First, the function $T(t)$ must be finite at $t \to \infty$. As a result, we obtain that the separation constant λ has to be imaginary: $\lambda = i\omega$, $\omega \in \mathbb{R}$. In so doing, we exclude quasinormal solutions from consideration.

The second requirement consists in that the function $\Phi(\phi)$ has to be 2π-periodic, i.e. $\Phi(\phi) = \Phi(\phi+2\pi)$ for any argument value ϕ. Whence, we obtain that $\nu = im$, $m \in \mathbb{Z}$. Then a solution with separated variables, which is finite in time and 2π-periodic in the azimuthal argument, has the form [10]

$$\varphi_2 = C\frac{e^{i\omega\eta_1+im\eta_2-a\omega\cos\theta}}{\sin\theta\,(r-ia\cos\theta)}\left(\frac{1-\cos\theta}{\sin\theta}\right)^m. \tag{23}$$

The solution $S(\theta)$ of system (14) at $k = -1$ has a singularity at the point $\theta = 0$ or $\theta = \pi$, depending on the value of separation constant m. The solution $R(r)$ is determined everywhere, except the points $r = r_+$ and $r = r_-$. Below, we will consider solution (23) in the domain $(0 < \theta < \pi, r > r_+)$, where, as was marked above, it is physically meaningful.

The Maxwell tensor $F_{ab} = 2\varphi_2 l_{[a}m_{b]} + 2\bar{\varphi}_2 l_{[a}\bar{m}_{b]}$ corresponding to solution (23) was calculated with the help of the software package GRTensor2 [20]. As a result, we obtained

$$F_{ab} = \sqrt{2}\begin{pmatrix} 0 & -\frac{a}{\Delta}P & -\frac{1}{\sin\theta}Q & P \\ \frac{a}{\Delta}P & 0 & \frac{\Sigma}{\sin\theta\,\Delta}Q & -\frac{r^2+a^2}{\Delta}P \\ \frac{1}{\sin\theta}Q & -\frac{\Sigma}{\sin\theta\,\Delta}Q & 0 & -a\sin\theta Q \\ -P & \frac{r^2+a^2}{\Delta}P & a\sin\theta\,Q & 0 \end{pmatrix}, \tag{24}$$

where $P = c_1\sin(\omega\eta_1 + m\eta_2) + c_2\cos(\omega\eta_1 + m\eta_2) \times \times\, e^{-a\omega\cos\theta}\left(\frac{1-\cos\theta}{\sin\theta}\right)^m$, $Q = c_1\cos(\omega\eta_1 + m\eta_2) - - c_2\sin(\omega\eta_1 + m\eta_2)e^{-a\omega\cos\theta}\left(\frac{1-\cos\theta}{\sin\theta}\right)^m$, $C = c_1 + ic_2$.

The NOWF condition (2) in the coordinate form looks like

$$\begin{cases} (r^2+a^2)F_{tr} - aF_{r\phi} = 0, \\ a\sin^2\theta F_{t\theta} - F_{\theta\phi} = 0, \\ \Sigma F_{t\theta} + \Delta F_{r\theta} = 0, \\ F_{tr} + \frac{a}{\Delta}F_{t\phi} = 0. \end{cases} \tag{25}$$

Now, let us calculate the energy-momentum tensor $T_{ab} = (1/2\pi) \times |\varphi_2|^2 l_a l_b$ corresponding to solution (23):

$$T_{ab} = \frac{|\varphi_2|^2}{2\pi}\begin{pmatrix} 1 & -\frac{\Sigma}{\Delta} & 0 & -a\sin^2\theta \\ -\frac{\Sigma}{\Delta} & \frac{\Sigma^2}{\Delta^2} & 0 & a\sin^2\theta\frac{\Sigma}{\Delta} \\ 0 & 0 & 0 & 0 \\ -a\sin^2\theta & a\sin^2\theta\frac{\Sigma}{\Delta} & 0 & a^2\sin^4\theta \end{pmatrix}, \tag{26}$$

$$|\varphi_2|^2 = \frac{|C|^2e^{-2a\omega\cos\theta}}{\sin^2\theta\,\Sigma}\left(\frac{1-\cos\theta}{\sin\theta}\right)^{2m}. \tag{27}$$

Let us also consider a solution with separated variables for the ingoing NOWF, when the Maxwell spinor equals $\varphi_{AB} = \varphi_0\iota_A\iota_B$. This solution, φ_0, can be written, by using Eqs. (15) and (9) at $k = 1$ and $l = 1$:

$$\varphi_0 = C\frac{2e^{\lambda\eta_3+\nu\eta_4-ia\lambda\cos\theta+i\nu\ln\left|\frac{1-\cos\theta}{\sin\theta}\right|}}{\sin\theta\,\Delta(r-ia\cos\theta)^{-1}}, \tag{28}$$

where

$$\eta_3 = t + r + M\ln\Delta + \frac{M^2}{\sqrt{M^2-a^2}}\ln\left|\frac{r-r_+}{r-r_-}\right|, \tag{29}$$

$$\eta_4 = \phi + \frac{a}{2\sqrt{M^2 - a^2}} \ln \left| \frac{r - r_+}{r - r_-} \right|. \tag{30}$$

A solution of Eq. (28), which is finite in time and 2π-periodic in the azimuthal angle, reads

$$\varphi_0 = C \frac{2e^{i\omega\eta_3 + im\eta_4 + a\omega\cos\theta}}{\sin\theta\,\Delta(r - ia\cos\theta)^{-1}} \left(\frac{1 - \cos\theta}{\sin\theta} \right)^{-m}, \tag{31}$$

This solution, like solution (23), also has singularities at $\theta = 0$, $\theta = \pi$ and $r = r_+$, $r = r_-$. Beyond the rotation axis and the horizons, it is also physically meaningful.

The Maxwell tensor for the ingoing NOWF is calculated by the formula

$$F_{ab} = \sqrt{2} \begin{pmatrix} 0 & \frac{a}{\Delta}U & \frac{1}{\sin\theta}V & U \\ -\frac{a}{\Delta}U & 0 & \frac{\Sigma}{\sin\theta\,\Delta}V & \frac{r^2+a^2}{\Delta}U \\ -\frac{1}{\sin\theta}V & -\frac{\Sigma}{\sin\theta\,\Delta}V & 0 & a\sin\theta V \\ -U & -\frac{r^2+a^2}{\Delta}U & -a\sin\theta V & 0 \end{pmatrix}, \tag{32}$$

where

$$U = c_1 \sin(\omega\eta_3 + m\eta_4) + c_2 \cos(\omega\eta_3 + m\eta_4) \times$$
$$\times\, e^{a\omega\cos\theta} \left(\frac{1 - \cos\theta}{\sin\theta} \right)^{-m},$$
$$V = c_1 \cos(\omega\eta_3 + m\eta_4) - c_2 \sin(\omega\eta_3 + m\eta_4) \times$$
$$\times e^{a\omega\cos\theta} \left(\frac{1 - \cos\theta}{\sin\theta} \right)^{-m}.$$

The NOWF condition (4) in the coordinate representation has the form

$$\begin{cases} (r^2 + a^2)F_{tr} - aF_{r\phi} = 0, \\ a\sin^2\theta F_{t\theta} - F_{\theta\phi} = 0, \\ \Sigma F_{t\theta} - \Delta F_{r\theta} = 0, \\ F_{tr} - \frac{a}{\Delta}F_{t\phi} = 0. \end{cases} \tag{33}$$

Finally, the energy-momentum tensor $T_{ab} = (1/2\pi) \times \times |\varphi_0|^2 n_a n_b$ of the ingoing NOWF for solution (31) looks like

$$T_{ab} = \frac{|\varphi_0|^2\Delta^2}{8\pi\Sigma^2} \begin{pmatrix} 1 & \frac{\Sigma}{\Delta} & 0 & -a\sin^2\theta \\ \frac{\Sigma}{\Delta} & \frac{\Sigma^2}{\Delta^2} & 0 & -a\sin^2\theta\frac{\Sigma}{\Delta} \\ 0 & 0 & 0 & 0 \\ -a\sin^2\theta & -a\frac{\sin^2\theta\Sigma}{\Delta} & 0 & a^2\sin^4\theta \end{pmatrix}, \tag{34}$$

$$|\varphi_0|^2 = \frac{4|C|^2\Sigma e^{2a\omega\cos\theta}}{\sin^2\theta\,\Delta^2} \left(\frac{1 - \cos\theta}{\sin\theta} \right)^{-2m}. \tag{35}$$

6. Conclusions

The equations describing one-way gravitational, electromagnetic, and neutrino fields in the Petrov-type D spacetime are reduced to a single form, which is valid for an arbitrary value of the field spin. When considering the systems of equations for the outgoing and ingoing NOWFs in the Kerr metric in the Boyer-Lindquist coordinates, they can be generalized to the same system of the first-order PDEs for an unknown function $\psi(t, r, \theta, \phi)$, similarly to that done by Teukolsky in the case of second-order equations for functions regular at $\theta = 0$ and $\theta = \pi$. The change of the function in our case [see Eqs. (9)] differs from analogous changes in the Teukolsky and Chandrasekhar approaches. This circumstance, however, does not prohibit a comparison of our solutions describing NOWFs with the solutions obtained by Teukolsky and Chandrasekhar.

We have obtained a generalized system and found its general solution, by sequentially integrating the first-order PDEs. This approach is quite different from the approaches of other authors. In particular, it allowed us to obtain an analytical solution, which is general for a certain class of fields: null one-way fields. The found solution with separated variables depends on a linear combination of the integrals of the system.

In the case of NOW Maxwell fields, the solutions describe circularly polarized waves, with the outgoing wave propagating from the Kerr black hole to the spatial infinity and the ingoing one propagating backward. The solutions describing the NOWFs are meaningful everywhere, except for the rotation axis and the horizons, where they have coordinate singularities. They were rejected by Teukolsky on the basis of their irregularity. The application of the obtained solutions to the analysis of the field behavior in the Kerr spacetime will be considered elsewhere.

By comparing the outgoing null solution of Maxwell's equations at $r \to \infty$ with the radial Teukolsky solution that is asymptotically outgoing at infinity [see Eq. (5.4) in work [1]], one can see that the functions ϕ_2 have the same $e^{i\omega r_*}/r$-asymptotics. Furthermore, the limitation of the consideration to only the outgoing null field does not result in the loss of information about the only field component that is meaningful for a remote observer, the "far field". *Vice versa*, the analytical solutions satisfying such requirements open possibilities to study the qualitative behavior of fields. This task cannot be done with the use of the Teukolsky solutions obtained in the form of series in spheroidal harmonics, because there are no recurrence relations for the coefficients in those series.

The authors express their gratitude to the anonymous referees for their useful remarks and advices.

1. S. Teukolsky. Perturbations of a rotating black hole. I. Fundamental equations for gravitational, electromagnetic, and neutrino-field perturbations. *Astrophys. J.* **185**, 635 (1973).
2. W.H. Press, S.A. Teukolsky. Perturbations of a rotating black hole. II. Dynamical stability of the Kerr metric. *Astrophys. J.* **185**, 649 (1973).
3. S.A. Teukolsky, W.H. Press. Perturbations of a rotating black hole. III. Interaction of the hole with gravitational and electromagnetic radiation. *Astrophys. J.* **193**, 443 (1973).
4. A.A. Starobinskii, S.M. Churilov. Amplification of electromagnetic and gravitational waves scattered by a rotating "black hole". *Zh. Èksp. Teor. Fiz.* **65**, 3 (1974) (in Russian).
5. S. Chandrasekhar. *The Mathematical Theory of Black Holes* (Oxford Univ. Press, 1983).
6. P.P. Fiziev. Classes of exact solutions to the Teukolsky master equation. *Class. Quant. Grav.* **27**, 135001 (2010).
7. S. Teukolsky. The Kerr metric. *Class. Quant. Grav.* **32**, 124006 (2015).
8. Y.V. Taistra. New approach to Maxwell equations decoupling in curved spacetime. In *WDS'13 Proceedings of Contributed Papers: Part III– Physics,* edited by J. Safrankova, J. Pavlu (Matfyzpress, 2013), p. 39–44.
9. V.O. Pelykh, Yu.V. Taistra. Class of general solutions for Maxwell's equations in a Kerr spacetime. *Mat. Metod. Fiz.-Mekh. Polya* **59**, 48 (2016) (in Ukrainian).
10. V.O. Pelykh, Y.V. Taistra. Solution with separable variables for null one-way Maxwell field in Kerr space-time. *Acta Phys. Pol. B Proc. Suppl.* **10**, 387 (2017).
11. W.T.M. Irvine, D. Bouwmeester. Linked and knotted beams of light. *Nature Phys.* **4**, 716 (2008).
12. H. Kedia, I. Bialynicki-Birula, D. Peralta-Salas, W. Irvine. Tying knots in light fields. *Phys. Rev. Lett.* **111**, 150404 (2013).
13. G.P. Torres del Castillo *3-D Spinors, Spin-Weighted Functions and Their Applications* (Birkhäuser, 2003).
14. S. Chandrasekhar. On algebraically special perturbations of black holes. *Proc. R. Soc. London, Ser. A* **392**, 1 (1984).
15. R.M. Wald. On perturbations of a Kerr black hole. *J. Math. Phys.* **14**, 1453 (1973).
16. J. Jezierski, T. Smolka. A geometric description of Maxwell field in a Kerr spacetime. *Class. Quant. Grav.* **33**, 125035 (2016).
17. M.V. Berry, M.R. Dennis. Knotted and linked phase singularities in monochromatic waves. *Proc. R. Soc. Lond. A* **457**, 2251 (2001).
18. R. Penrose, W. Rindler. *Spinors and Space-Time. Two-Spinor Calculus and Relativistic Fields, Vol. 1* (Cambridge Univ. Press, 1984).
19. W. Kinnersley. Type D vacuum metrics. *J. Math. Phys.* **10**, 1195 (1969).
20. *GRTensorIII for Maple* [https://github.com/grtensor/grtensor].

Received 20.08.17.
Translated from Ukrainian by O.I. Voitenko

В.О. Пелих, Ю.В. Тайстра

ОДНОНАПРЯМЛЕНІ ІЗОТРОПНІ
ПОЛЯ У ПРОСТОРІ КЕРРА

Р е з ю м е

Метою роботи є побудова у аналітичному вигляді розв'язків рівнянь безмасового поля довільного спіну у метриці Керра у вигляді ізотропних однонапрямлених – вихідних та вхідних за Чандрасекаром полів, тобто полів, які поширюються від або до чорної діри. На основі методу Ньюмена–Пенроуза у його спінорній формі розглянуто однонапрямлені ізотропні поля у просторі типу D за Петровим та знайдено у аналітичному вигляді загальний розв'язок та розв'язок із відокремленими змінними узагальнених рівнянь таких полів у метриці Керра. У частковому випадку електромагнітного поля обчислено тензор Максвелла та тензор енергії-імпульсу для вихідного та вхідного однонапрямленого поля.