doi: 10.15407/ujpe62.03.0263

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PACS 03.65.-w, 03.65.Nk, 21.45.-v

SOLUTION OF THE LIPPMANN–SCHWINGER EQUATION FOR A PARTIAL WAVE TRANSITION MATRIX WITH REPULSIVE COULOMB INTERACTION

A special case where the Lippmann–Schwinger integral equation for the partial wave two-body Coulomb transition matrix for likely charged particles with a negative energy has an analytical solution has been considered. Analytical expressions for the partial s-, p-, and d-wave Coulomb transition matrices for repulsively interacting particles at the ground-state energy have been derived, by using the Fock method of stereographic projection of the momentum space onto a four-dimensional unit sphere.

Keywords: partial wave transition matrix, Coulomb interaction, Lippmann–Schwinger equation, Fock method, analytical solution.

1. Introduction

The Coulomb transition matrix (t-matrix) characterizes all properties of a two-body system with the Coulomb interaction. In the momentum space, it is a scalar function of three variables: the initial and final momenta, and the energy. The presence of bound states for a system with opposite charges results in the appearance of energy poles in the t-matrix, whose residuals are connected with wave functions of the system in those states. In the case of repulsive Coulomb interaction between two charges of the same sign, the corresponding t-matrix has no energy poles. At positive energies, the analytical properties of the Coulomb t-matrix–in the case of short-range interaction potentials, they manifest themselves as a singular branch point with a cut along the positive energy axis–and the corresponding unitarity conditions on the energy surface and beyond it are more complicated (see review [1]).

Knowledge of the two-body Coulomb transition matrix is especially important when studying the properties of atomic and nuclear systems consisting of three or more charges with the use of the Faddeev [2,3] and Faddeev–Yakubovsky [4] integral equation methods. For such systems, three-body Faddeev equations are known to become non-Fredholm already below the decay threshold. The extraction of the main Coulomb singularity and the regularization of three-body equations in this case were proposed by Veselova [5] with the help of the known Gorshkov procedure for two-body systems [6]. The problem of regularization of the integral equations for four-body systems containing charged particles was considered in work [7]. Earlier information about the properties of the two-body off-shell Coulomb transition matrix can be found in review [1].

There are a number of representations for the two-body Coulomb transition matrix [8–16]. Of special interest is the study of the Coulomb transition matrix, by taking advantage of the Coulomb system symmetry in the Fock four-dimensional Euclidean space [17]. Earlier, the Fock method was applied in Bratsev–Trifonov's [10] and Schwinger's [12] works in order to derive the Coulomb Green's function in the one-parameter integral form. Expressions for the three-dimensional Coulomb transition matrix with explicitly singled out transferred momentum and energy singularities were obtained in works [14] (for negative energies, $E < 0$) and [15] (for zero and positive energies, $E \geq 0$).

For the first time, a possibility to derive an analytical expression for partial wave two-body Coulomb transition matrices at the ground bound state energy was examined for oppositely charged particles (with the attractive interaction) in the previous work [18]. In this work, on the basis of the Fock method of stereographic projection of the three-dimensional momentum space onto a four-dimensional unit sphere [17], the form of the partial wave Coulomb transition matrices for a system of two likely charged bodies (with the repulsive Coulomb interaction) is analyzed. The consideration begins in Section 2, where the expression obtained earlier in work [14] for the three-dimensional Coulomb transition matrix at the negative energy is used. In Section 3, a general expression for the off-shell partial wave Coulomb t-matrix at the negative energy is derived. Section 4 is devoted to the study of the partial wave Coulomb t-matrix at the ground bound state energy, and it is shown that a simple analytical expression for the partial wave t-matrix can be obtained in this case. Explicit analytical expressions for the s-, p-, and d-wave components of the Coulomb t-matrix are presented. Final remarks and conclusions are made in Section 5.

2. Three-Dimensional Coulomb Transition Matrix at the Negative Energy with Explicitly Singled Out Singularities

The three-dimensional Coulomb transition matrix $\langle \mathbf{k}|t(E)|\mathbf{k}'\rangle$ satisfies the inhomogeneous Lippmann–Schwinger integral equation

$$\langle \mathbf{k}|t(E)|\mathbf{k}'\rangle = \langle \mathbf{k}|v|\mathbf{k}'\rangle + + \int \frac{d\mathbf{k}''}{(2\pi)^3}\langle \mathbf{k}|v|\mathbf{k}''\rangle \frac{1}{E-\frac{k''^2}{2\mu}}\langle \mathbf{k}''|t(E)|\mathbf{k}'\rangle. \tag{1}$$

Here, the free term $\langle \mathbf{k}|v|\mathbf{k}'\rangle$ is determined by the Coulomb interaction potential $v(r) = q_1q_2/r$, where q_i is the charge of the i-th particle (i = 1, 2), and r is the distance between particles 1 and 2. In the momentum space, this term looks like

$$\langle \mathbf{k}|v|\mathbf{k}'\rangle = \frac{4\pi q_1 q_2}{|\mathbf{k}-\mathbf{k}'|^2}, \tag{2}$$

where $\mathbf{k}$ and $\mathbf{k}'$ are relative momenta corresponding to the radius-vectors $\mathbf{r}$ and $\mathbf{r}'$, respectively, in the coordinate space. The kernel of the integral equation (1) is a product of the operator of Coulomb interaction potential (2) and the free Green operator

$$\langle \mathbf{k}|g_0(E)|\mathbf{k}'\rangle = \frac{(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')}{E-\frac{k^2}{2\mu}}, \tag{3}$$

where the quantity E is the total energy of the relative motion of particles 1 and 2, and $\mu = m_1m_2/(m_1+m_2)$ is their reduced mass.

In this work, the consideration is confined to the problem of Coulomb scattering of two off-energy-shell likely charged particles in the case of negative energy

$$E = -\frac{\hbar^2\kappa^2}{2\mu}. \tag{4}$$

The consideration is based on the solution of the integral equation (1) for the three-dimensional off-shell Coulomb transition matrix with the explicitly singled out transferred momentum and energy singularities, which were obtained by us earlier [14]:

$$\langle \mathbf{k}|t(E)|\mathbf{k}'\rangle = \frac{8\pi q_1q_2\kappa^2}{(k^2+\kappa^2)(k'^2+\kappa^2)\sin\omega} \times \times\Big[\cot\frac{\omega}{2} - \pi\gamma\cos\gamma\omega - \gamma\sin 2\gamma\omega\ln\Big(\sin\frac{\omega}{2}\Big) + . + 2\pi\gamma\; c(\gamma)\;\cot\gamma\pi\sin\gamma\omega + \gamma\cos\gamma\omega\int\limits_0^{\omega} d\varphi\;\sin\gamma\varphi\times \times\cot\frac{\varphi}{2} + 2\gamma^2\sin\gamma\omega\int\limits_{\omega}^{\pi} d\varphi\;\sin\gamma\varphi\;\ln\Big(\sin\frac{\varphi}{2}\Big)\Big], \tag{5}$$

where

$$\gamma = \frac{\mu q_1 q_2}{\hbar^2\kappa} \tag{6}$$

is the dimensionless Coulomb parameter, and $\hbar$ the reduced Planck's constant. The variable ω in Eq. (5) stands for the angle between two 4-dimensional unit vectors $e \equiv (\mathbf{e}, e_0)$ and $e' \equiv (\mathbf{e}\prime, e'_0)$ in the four-dimensional Euclidean space introduced by Fock [17]:

$$\mathbf{e} = \frac{2\kappa\mathbf{k}}{\kappa^2+k^2}, \quad e_0 = \frac{\kappa^2-k^2}{\kappa^2+k^2}, \\ \mathbf{e}' = \frac{2\kappa\mathbf{k}'}{\kappa^2+k'^2}, \quad e'_0 = \frac{\kappa^2-k'^2}{\kappa^2+k'^2}, \tag{7}$$

ISSN 2071-0186. Ukr. J. Phys. 2017. Vol. 62, No. 3

$$\cos\omega = e\,e' = \mathbf{e}\,\mathbf{e}' + e_0\,e_0'. \tag{8}$$

The three-dimensional vectors $\mathbf{k}$ and $\mathbf{k}'$ lie in a hyperplane, which is a stereographic projection of a sphere with the unit radius. The variable ω is determined by the relation

$$\sin^2\frac{\omega}{2} = \frac{\kappa^2|\mathbf{k}-\mathbf{k}'|^2}{(k^2+\kappa^2)(k'^2+\kappa^2)}, \quad 0 \le \omega \le \pi. \tag{9}$$

The function $c(\gamma)$ in Eq. (5) looks like

$$c(\gamma) = \frac{1}{2}\left(1 - \frac{1}{\pi}\int\limits_0^\pi d\varphi\,\sin\gamma\varphi\,\cot\frac{\varphi}{2}\right) \tag{10}$$

or, in terms of the gamma, $\Gamma(x)$, and digamma, $\psi(x) \equiv d\ln\Gamma(x)/dx$, functions [18],

$$c(\gamma) = \theta(-\gamma) + \frac{\sin\gamma\pi}{2\pi}\left[\psi\left(\frac{|\gamma|+1}{2}\right) - \right.$$
$$\left. -\,\psi\left(\frac{|\gamma|}{2}\right) - \frac{1}{|\gamma|}\right], \tag{11}$$

where $\theta(x)$ is the Heaviside step function,

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

The first three terms in the square brackets in Eq. (5) contain transferred momentum singularities: $|\mathbf{k}-\mathbf{k}'|^{-2}$, $|\mathbf{k}-\mathbf{k}'|^{-1}$ and $\ln\{\kappa|\mathbf{k}-\mathbf{k}'|/[(k^2 + \kappa^2)^{1/2}(k'^2+\kappa^2)^{1/2}]\}$, respectively. The other three terms in Eq. (5) are smooth functions of $|\mathbf{k}-\mathbf{k}'|$.

The fourth term in expression (5) contains energy singularities. They arise only in the case of attractive Coulomb potential (with opposite electric charges, $q_1q_2 < 0$), when the Coulomb parameter γ accepts negative integer values corresponding to the spectrum of bound states of a two-particle system with the energies

$$E_n = -\frac{\mu(q_1q_2)^2}{2\hbar^2 n^2}, \quad n = 1, 2, 3, \ldots. \tag{12}$$

According to Eqs. (4) and (6), the corresponding values of the parameter κ and the Coulomb parameter γ are equal to

$$\kappa_n = \frac{\sqrt{-2\mu E_n}}{\hbar} = \frac{\mu|q_1q_2|}{\hbar^2 n}$$

and

$$\gamma_n = \frac{\mu q_1 q_2}{\hbar^2\kappa_n} = \frac{q_1q_2}{|q_1q_2|}n, \tag{13}$$

respectively. At those points, $\gamma = \gamma_n = -n$, so that the function $\cot\gamma\pi$ has pole singularities, and the function $c(\gamma)$ differs from zero, $c(-n) = 1$.

In the case of repulsive Coulomb potential ($\gamma > 0$), the expression for $c(\gamma)$ equals zero at positive integer γ-values, $c(n) = 0$, and the fourth term in Eq. (5) is finite and equal to

$$\rho(\gamma) \equiv \left.\frac{2\pi\gamma c(\gamma)}{\tan\gamma\pi}\right|_{\gamma\to n} = \rho_n, \tag{14}$$

where

$$\rho_n = 2nc'(n), \quad c'(n) = -\frac{1}{2\pi}\int\limits_0^\pi d\varphi\,\cos n\varphi\,\cot\frac{\varphi}{2} \tag{15}$$

or, using the function $\beta(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right]$,

$$\rho_n = (-1)^n\left[2n\beta(n) - 1\right]. \tag{16}$$

The ultimate expression for ρ_n looks like

$$\rho_n = (-1)^n - 2n\ln 2 - 2n\sum_{m=1}^{n}\frac{(-1)^m}{m}. \tag{17}$$

3. Partial Wave Component of the Coulomb Transition Matrix with Negative Energy

Using the partial-wave method and expanding the matrix elements of the Coulomb potential and the transition matrix with a negative energy in series in Legendre polynomials $P_l(x)$,

$$\begin{aligned} \langle\mathbf{k}|v|\mathbf{k}'\rangle &= \sum_{l=0}^{\infty}(2l+1)v_l(k,k')P_l(\hat{\mathbf{k}}\,\hat{\mathbf{k}}'), \\ \langle\mathbf{k}|t(E)|\mathbf{k}'\rangle &= \sum_{l=0}^{\infty}(2l+1)t_l(k,k';E)P_l(\hat{\mathbf{k}}\,\hat{\mathbf{k}}'), \end{aligned} \tag{18}$$

where $\hat{\mathbf{k}}$ is a unit vector along the vector $\mathbf{k}$, and $\hat{\mathbf{k}}\,\hat{\mathbf{k}}' = \cos\theta$, the one-dimensional integral equation for the partial wave component of the transition matrix can be written in the form

$$t_l(k,k';E) = v_l(k,k') +$$
$$+\int\limits_0^\infty \frac{dk''k''^2}{2\pi^2}v_l(k,k'')\frac{1}{E - \frac{k''^2}{2\mu}}t_l(k'',k';E). \tag{19}$$

The inhomogeneous term and the kernel of this equation contain a partial wave component of the Coulomb interaction potential,

$$v_l(k,k') = \frac{1}{2}\int\limits_0^\pi d\theta\ \sin\theta\ P_l(\cos\theta)\langle \mathbf{k}|v|\mathbf{k}'\rangle. \qquad (20)$$

According to definition (18), the partial wave component of the Coulomb transition matrix $t_l^{\rm C}(k,k';E)$ equals

$$t_l(k,k';E) = \frac{1}{2}\int\limits_0^\pi d\theta\ \sin\theta\ P_l(\cos\theta)\ \langle \mathbf{k}|t(E)|\mathbf{k}'\rangle. \qquad (21)$$

Taking into account that expression (5) for the three-dimensional Coulomb transition matrix $\langle \mathbf{k}\,|t(E)|\,\mathbf{k}'\rangle$ depends on the angle ω between the unit vectors e and e' in the four-dimensional Fock space, it is convenient to change in Eq. (21) from the integration over the angle θ between the vectors $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}'$ to the integration over the angle ω. From expression (9) describing the relationship between the angles θ and ω, it follows that

$$\begin{aligned} \cos\theta &= \frac{\xi}{\eta} - \frac{1}{\eta}\sin^2\frac{\omega}{2} = \frac{2\xi - 1 + \cos\omega}{2\eta}, \\ \sin\theta\ d\theta &= \frac{1}{2\eta}\sin\omega\ d\omega, \end{aligned} \qquad (22)$$

where

$$\begin{aligned} \xi &= \frac{\kappa^2(k^2+k'^2)}{(k^2+\kappa^2)(k'^2+\kappa^2)}, \\ \eta &= \frac{2\kappa^2 kk'}{(k^2+\kappa^2)(k'^2+\kappa^2)}. \end{aligned} \qquad (23)$$

Then formula (21) can be rewritten in the form

$$t_l(k,k';E) = \frac{1}{4\eta}\times$$
$$\times\int\limits_{\omega_0}^{\omega_\pi} d\omega\ \sin\omega\ P_l\left(\frac{2\xi - 1 + \cos\omega}{2\eta}\right)\langle \mathbf{k}|t(E)|\mathbf{k}'\rangle. \qquad (24)$$

The integration limits in Eq. (24) are determined by the expressions

$$\omega_0 = 2\arcsin\sqrt{\xi-\eta}, \quad \omega_\pi = 2\arcsin\sqrt{\xi+\eta}, \qquad (25)$$

so that

$$\begin{aligned} \cos\omega_0 &= 1 - 2\xi + 2\eta, \\ \cos\omega_\pi &= 1 - 2\xi - 2\eta, \\ \sin\omega_0 &= 2\sqrt{\xi-\eta}\sqrt{1-\xi+\eta}, \\ \sin\omega_\pi &= 2\sqrt{\xi+\eta}\sqrt{1-\xi-\eta}. \end{aligned} \qquad (26)$$

Substituting expression (5) for the three-dimensional transition matrix into Eq. (24), we obtain the following formula for the partial wave Coulomb transition matrix $t_l(k,k';E)$ at $E<0$:

$$t_l(k,k';E) = \frac{\pi q_1 q_2}{kk'}\int\limits_{\omega_0}^{\omega_\pi} d\omega\ P_l\left(\frac{2\xi-1+\cos\omega}{2\eta}\right)\times$$
$$\times\Big\{\cot\frac{\omega}{2} - \pi\gamma\ \cos\gamma\omega - \gamma\ \sin 2\gamma\omega\ \ln\Big(\sin\frac{\omega}{2}\Big) +$$
$$+ 2\pi\gamma\ c(\gamma)\ \cot\gamma\pi\ \sin\gamma\omega + \gamma\ \cos\gamma\omega x_\gamma(\omega) +$$
$$+ 2\gamma^2\sin\gamma\omega y_\gamma(\omega)\Big\}, \qquad (27)$$

where

$$\begin{aligned} x_\gamma(\omega) &= \int\limits_0^\omega d\varphi\ \sin\gamma\varphi\ \cot\frac{\varphi}{2}, \\ y_\gamma(\omega) &= \int\limits_\omega^\pi d\varphi\ \sin\gamma\varphi\ \ln\Big(\sin\frac{\varphi}{2}\Big). \end{aligned} \qquad (28)$$

The partial wave Coulomb transition matrix $t_l(k,k';E)$ is a function of three independent variables: k, k', and E. The quantities ξ and η, as well as the integration limits ω_0 and ω_π in expression (27), also depend on those variables. The quantity κ is connected with the energy E by formula (4). By definition (6), the Coulomb parameter γ in expression (27) depends on κ and therefore on the energy E. Note that, in expression (27) for the t-matrix, the Coulomb interaction intensity q_1q_2 is contained both in the preintegral factor and, in the Coulomb parameter γ [see Eq. (6)], in the terms in the curly braces in the integrand.

4. Partial Wave Coulomb Transition Matrices for Likely Charged Particles at the Ground Bound State Energy

Expression (27) for the Coulomb transition matrix contains the double integration over φ and ω, which is rather difficult. It is easy to see that, for separate values of the Coulomb parameter γ (which corresponds to certain energy values E), the integration over φ and ω in Eq. (27) can be made explicitly. In such cases, simple analytical expressions for the partial wave Coulomb t-matrix can be obtained. In particular, the integration in the expressions for $x_\gamma(\omega)$ and $y_\gamma(\omega)$ in Eq. (27) becomes simpler for integer values of the Coulomb parameter $\gamma = \gamma_n$ [Eq. (13)] corresponding to the energy spectrum of bound states of two-particle systems [Eq. (12)] with the energies $E = E_n$.

Let us consider the form of the off-shell partial wave transition matrices for a repulsive Coulomb interaction potential between likely charged particles ($q_1 q_2 > > 0$) at the ground bound state energy $E = E_n$. In this case, the Coulomb parameter is determined by expression (13) with $n = 1$ and equal to

$$\gamma = \gamma_1 = 1; \tag{29}$$

the fourth term in the curly braces in Eq. (27), according to Eq. (14), is simplified to $\rho_1 \sin\omega$; and the integration in the fifth and sixth terms is carried as follows:

$$x_1(\omega) = \int_0^\omega d\varphi \sin\varphi \, \cot\frac{\varphi}{2} = \omega + \sin\omega,$$
$$y_1(\omega) = \int_\omega^\pi d\varphi \, \sin\varphi \, \ln\left(\sin\frac{\varphi}{2}\right) = \tag{30}$$
$$= -\cos^2\frac{\omega}{2} - 2\sin^2\frac{\omega}{2} \, \ln\left(\sin\frac{\omega}{2}\right).$$

As a result, formula (27) for the partial wave Coulomb transition matrices (with $l = 0, 1, 2, ...$) in the case of repulsive interaction at $\gamma = 1$ (which corresponds to the energy $E = E_1$) reads

$$t_l^r(k, k' - b_1) = \frac{\pi q_1 q_2}{kk'} \int_{\omega_{01}}^{\omega_{\pi 1}} d\omega P_l \left(\frac{2\xi_1 - 1 + \cos\omega}{2\eta_1}\right) \times$$
$$\times \left\{\cot\frac{\omega}{2} - \pi\cos\omega + \omega\cos\omega + (\rho_1 - 1)\sin\omega - \right.$$
$$\left. - 2\sin\omega \, \ln\left(\sin\frac{\omega}{2}\right)\right\}, \tag{31}$$

where, in accordance with Eq. (7),

$$\rho_1 = 1 - 2\ln 2. \tag{32}$$

The quantities ξ_1, η_1, ω_{01}, and $\omega_{\pi 1}$ in Eq. (31) are determined by the expressions for ξ, η, ω_0, and ω_π, respectively, in accordance with their definitions (23) and (25) and calculated at the point $\kappa = \kappa_1$:

$$\xi_1 = \frac{\kappa_1^2 (k^2 + k'^2)}{(k^2 + \kappa_1^2)(k'^2 + \kappa_1^2)},$$
$$\eta_1 = \frac{2\kappa_1^2 k k'}{(k^2 + \kappa_1^2)(k'^2 + \kappa_1^2)},$$
$$\omega_{01} = 2\arcsin\sqrt{\xi_1 - \eta_1},$$
$$\omega_{\pi 1} = 2\arcsin\sqrt{\xi_1 + \eta_1}. \tag{33}$$

Note that the first term in the braces in the general expression (31) for the partial wave Coulomb transition matrix corresponds to the Born approximation:

$$t_l^{\mathrm{Born}}(k, k'; -b_1) = v_l(k, k') =$$
$$= \frac{2\pi q_1 q_2}{kk'} Q_l \left(\frac{k^2 + k'^2}{2kk'}\right). \tag{34}$$

Here, the function $Q_l(x)$ is the Legendre function of the second kind [18]:

$$Q_l(x) = \frac{1}{2} P_l(x) \ln\left(\frac{x+1}{x-1}\right) - W_{l-1}(x), \tag{35}$$

where

$$W_{-1}(x) = 0, \quad W_{l-1}(x) = \sum_{k=1}^{l} \frac{1}{k} P_{l-k}(x) P_{k-1}(x).$$

Owing to the orthogonality of the Legendre polynomials,

$$\int_{\omega_0}^{\omega_\pi} d\omega \, \sin\omega \, P_l \left(\frac{2\xi_1 - 1 + \cos\omega}{2\eta_1}\right) =$$
$$= 2\eta_1 \int_0^\pi d\theta \, \sin\theta \, P_l(\cos\theta) = 4\eta_1 \delta_{l0}, \tag{36}$$

the fourth term in Eq. (31) – it contains $\sin\omega$ – makes contribution different from zero only for the partial s-wave Coulomb t-matrix.

In the simplest case with $l = 0$, by integrating over ω in expression (31), we obtain the following formula for the partial s-wave Coulomb transition matrix for two likely charged particles (for $q_1 q_2 > 0$):

$$t_0^r(k, k'; E_1) = \frac{\pi q_1 q_2}{kk'} \left\{4(\rho_1 - 1)\eta_1 - (2\xi_1 - 1) \times \right.$$
$$\times \ln\left(\frac{\xi_1 + \eta_1}{\xi_1 - \eta_1}\right) - 2\eta_1 \ln\left(\xi_1^2 - \eta_1^2\right) -$$
$$\left. - \left[(\pi - \omega_{\pi 1})\sin\omega_{\pi 1} - (\pi - \omega_{01})\sin\omega_{01}\right]\right\}. \tag{37}$$

Note that, in the case of attractive Coulomb interaction (at $q_1 q_2 < 0$), the corresponding partial s-wave

Coulomb transition matrix has a pole-like singularity at the energy $E = E_1$, because it contains $\cot\gamma\pi$.

Analogously, using formula (30) and integrating over ω in expression (31), we obtain the following formulas for the partial p- and d-wave Coulomb transition matrices in the case of repulsive interaction between the particles (at $q_1q_2 > 0$):

$$t_1^r(k,k';-b_1) = \frac{\pi q_1 q_2}{kk'}\left\{-1 - \frac{1}{\eta_1}\left[(\xi_1^2 - \xi_1 - \eta_1^2)\times\right.\right.$$
$$\times \ln\left(\frac{\xi_1+\eta_1}{\xi_1-\eta_1}\right) + \frac{1}{8}(\omega_{\pi 1} - \omega_{01})(2\pi - \omega_{\pi 1} - \omega_{01}) +$$
$$+ \frac{1}{2}(2\xi_1 - 1)[(\pi - \omega_{\pi 1})\sin\omega_{\pi 1} - (\pi - \omega_{01})\sin\omega_{01}] +$$
$$\left.\left. + \frac{1}{8}[(\pi - \omega_{\pi 1})\sin 2\omega_{\pi 1} - (\pi - \omega_{01})\sin 2\omega_{01}]\right]\right\}, \quad (38)$$

$$t_2^r(k,k';-b_1) = \frac{\pi q_1 q_2}{kk'}\left\{-\frac{1}{\eta_1}\left(\xi_1 + \frac{3}{2}\right) -\right.$$
$$- \frac{1}{\eta_1^2}\left[\left(\xi_1^3 - \frac{3}{2}\xi_1^2 - \xi_1\eta_1^2 + \frac{1}{2}\eta_1^2\right)\ln\left(\frac{\xi_1+\eta_1}{\xi_1-\eta_1}\right) +\right.$$
$$+ \frac{3}{16}(2\xi_1 - 1)(\omega_{\pi 1} - \omega_{01})(2\pi - \omega_{\pi 1} - \omega_{01}) +$$
$$+ \left(\frac{3}{2}\xi_1^2 - \frac{3}{2}\xi_1 - \frac{1}{2}\eta_1^2 + \frac{21}{32}\right)[(\pi - \omega_{\pi 1})\sin\omega_{\pi 1} -$$
$$- (\pi - \omega_{01})\sin\omega_{01}] + \frac{3}{16}(2\xi_1 - 1)[(\pi - \omega_{\pi 1})\sin 2\omega_{\pi 1} -$$
$$- (\pi - \omega_{01})\sin 2\omega_{01}] + \frac{1}{32}[(\pi - \omega_{\pi 1})\sin 3\omega_{\pi 1} -$$
$$\left.\left. - (\pi - \omega_{01})\sin 3\omega_{01}]\right]\right\}. \quad (39)$$

Making allowance for the relations

$$\cos\omega_{\pi 1} + \cos\omega_{01} = -2(2\xi_1 - 1),$$
$$\cos\omega_{\pi 1} - \cos\omega_{01} = -4\eta_1,$$

which follow from Eq. (26), and the expressions

$$(\pi - \omega_{\pi 1})\sin 2\omega_{\pi 1} - (\pi - \omega_{01})\sin 2\omega_{01} =$$
$$= 2(2\xi_1 - 1)A_- - 4\eta_1 A_+,$$
$$(\pi - \omega_{\pi 1})\sin 3\omega_{\pi 1} - (\pi - \omega_{01})\sin 3\omega_{01} =$$
$$= \left[4(2\xi_1 - 1)^2 + 16\eta_1^2 - 1\right]A_- + 16(2\xi_1 - 1)\eta_1 A_+,$$

where

$$A_\pm \equiv (\pi - \omega_{\pi 1})\sin\omega_{\pi 1} \pm (\pi - \omega_{01})\sin\omega_{01},$$

formulas (38) and (39) for the partial p- and d-wave components of the t-matrix can be written in simpler forms:

$$t_1^r(k,k';-b_1) = \frac{\pi q_1 q_2}{kk'}\left\{-1 - \frac{1}{\eta_1}\left[(\xi_1^2 - \xi_1 - \eta_1^2)\times\right.\right.$$
$$\times \ln\left(\frac{\xi_1+\eta_1}{\xi_1-\eta_1}\right) + \frac{1}{8}(\omega_{\pi 1} - \omega_{01})(2\pi - \omega_{\pi 1} - \omega_{01}) -$$
$$- \frac{1}{4}[(\pi - \omega_{\pi 1})\sin\omega_{\pi 1}\cos\omega_{01} -$$
$$\left.\left. - (\pi - \omega_{01})\cos\omega_{\pi 1}\sin\omega_{01}]\right]\right\}, \quad (40)$$

$$t_2^r(k,k';-b_1) = \frac{\pi q_1 q_2}{kk'}\left\{-\frac{1}{\eta_1}\left(\xi_1 + \frac{3}{2}\right) -\right.$$
$$- \frac{1}{\eta_1^2}\left[\left(\xi_1^3 - \frac{3}{2}\xi_1^2 - \xi_1\eta_1^2 + \frac{1}{2}\eta_1^2\right)\ln\left(\frac{\xi_1+\eta_1}{\xi_1-\eta_1}\right) +\right.$$
$$+ \frac{3}{16}(2\xi_1 - 1)(\omega_{\pi 1} - \omega_{01})(2\pi - \omega_{\pi 1} - \omega_{01}) +$$
$$+ \frac{1}{4}[(\pi - \omega_{\pi 1})\sin\omega_{\pi 1} - (\pi - \omega_{01})\sin\omega_{01}] -$$
$$- \frac{1}{8}(2\xi_1 - 1)[(\pi - \omega_{\pi 1})\sin\omega_{\pi 1}\cos\omega_{01} -$$
$$\left.\left. - (\pi - \omega_{01})\cos\omega_{\pi 1}\sin\omega_{01}]\right]\right\}. \quad (41)$$

For comparison, we present the formulas for the corresponding partial transition matrices with $l = 1$ and 2, which were obtained for the case of attractive Coulomb interaction ($q_1q_2 < 0$) [18]:

$$t_1^a(k,k'-b_1) = \frac{\pi q_1 q_2}{kk'}\left\{4\xi_1 - 3 - \frac{1}{\eta_1}\left[(\xi_1^2 - \xi_1 - \eta_1^2)\times\right.\right.$$
$$\times \ln\left(\frac{\xi_1+\eta_1}{\xi_1-\eta_1}\right) - \frac{1}{8}(\omega_{\pi 1} - \omega_{01})(2\pi - \omega_{\pi 1} - \omega_{01}) +$$
$$+ \frac{1}{4}[(\pi - \omega_{\pi 1})\sin\omega_{\pi 1}\cos\omega_{01} -$$
$$\left.\left. - (\pi - \omega_{01})\cos\omega_{\pi 1}\sin\omega_{01}]\right]\right\}, \quad (42)$$

$$t_2^a(k,k';-b_1) = \frac{\pi q_1 q_2}{kk'}\left\{\frac{1}{\eta_1}\left(4\xi_1^2 - 5\xi_1 - \frac{8}{3}\eta_1^2 + \frac{3}{2}\right) -\right.$$
$$- \frac{1}{\eta_1^2}\left[\left(\xi_1^3 - \frac{3}{2}\xi_1^2 - \xi_1\eta_1^2 + \frac{1}{2}\eta_1^2\right)\ln\left(\frac{\xi_1+\eta_1}{\xi_1-\eta_1}\right) -\right.$$
$$- \frac{3}{16}(2\xi_1 - 1)(\omega_{\pi 1} - \omega_{01})(2\pi - \omega_{\pi 1} - \omega_{01}) -$$

$$-\frac{1}{4}\left[(\pi-\omega_{\pi 1})\sin\omega_{\pi 1}-(\pi-\omega_{01})\sin\omega_{01}\right]-$$
$$-\frac{1}{8}(2\xi_1-1)\big[(\pi-\omega_{\pi 1})\sin\omega_{\pi 1}\cos\omega_{01}-$$
$$-(\pi-\omega_{01})\cos\omega_{\pi 1}\sin\omega_{01}\big]\Big]\Big\}. \quad (43)$$

In view of the sign difference for q_1q_2 in the coefficients before the braces in Eqs. (40), (41) and Eqs. (42), (43), we obtain that the formulas for the corresponding partial wave transition matrices differ in the cases of attractive and repulsive Coulomb interactions only by their first terms and the signs in front of the second terms. The other terms are the same.

5. Discussion and Conclusions

The off-shell Coulomb transition matrix is directly connected with the Coulomb Green's function and includes all information about the system of interacting particles. In the previous work [18], a possibility to derive an analytical expression for the off-shell Coulomb transition matrix for two particles with the use of the Fock method of stereographic projection of the momentum space onto a four-dimensional unit sphere was studied. In the case of attractive Coulomb interaction between opposite charges ($q_1q_2 < 0$), simple analytical expressions for the partial p-, d-, and f-wave transition matrices at the ground bound state energy $E = E_1$, i.e. $t_l^a(k, k'; E_1)$ with $l = 1$, 2, and 3, were obtained.

Note that knowledge of the partial wave Coulomb transition matrix $t_\lambda(k, k'; E_n)$, the bound state wave function, and its derivatives is necessary, in particular, when determining the electric 2^λ-pole polarizability α_λ ($\lambda = 1, 2, 3, ...$) of a two-particle Coulomb bound system in the state with the energy $E = E_n$ [20].

In this work, the Fock method is applied in order to derive partial wave two-particle transition matrices in the case of repulsive Coulomb interaction (likely charged particles, $q_1q_2 > 0$) at the energy $E = E_1$. Rather simple analytical expressions are obtained for the partial s-, p-, and d-wave transition matrices at the ground bound state energy, i.e. $t_l^r(k, k'; E_1)$ with $l = 0$, 1, and 2 [formulas (37), (40), and (41), respectively].

It is of interest that, in the case of particles with likely charges, for which bound states do not exist at all, the simplification of expressions for the partial wave Coulomb t-matrices takes place at the discrete energies that correspond to the spectrum of bound states for oppositely charged particles.

It should be pointed out that a possibility to have a simple analytical form for the partial wave Coulomb t-matrix is associated with a possibility to carry out the analytical integration over φ and ω in expressions (28) for $x_\gamma(\omega)$ and $y_\gamma(\omega)$ and in expression (27) for $t_l(k, k'; E)$. In particular, such integration can be done at the energy values that are equal to the energies of the ground and excited bound states in the discrete spectrum E_n, $n = 1, 2, 3, ...$ [formula (12)]. The procedure can be realized for the partial wave Coulomb matrices $t_l^r(k, k'; E_n)$ that describe a system with repulsive forces (with likely charged particles, $q_1q_2 > 0$) at all n- and l-values. Analytical expressions for the Coulomb transition matrices $t_l^a(k, k'; E)$ describing a system with attractive forces (with oppositely charged particles, $q_1q_2 < 0$) can be obtained only at n- and l-values that do not correspond to bound states, when the corresponding transition matrix has a pole-like singularity (at each n-value and the orbital momentum values $l \le n-1$). A pole singularity arises, for instance, in the partial wave Coulomb transition matrices $t_0^a(k, k'; E)$ at $E = E_1$, in the $t_0^a(k, k'; E)$ and $t_1^a(k, k'; E)$ matrices at $E = E_2$, and so forth.

Note that the partial wave Coulomb transition matrix (27) acquires a simple analytical form not only at the energy values corresponding to the discrete spectrum of bound states [Eq. (12)], which is equivalent, in accordance with Eq. (13), to integer values of Coulomb parameter (6). Specifically, a similar simplification can also be obtained for the Coulomb parameter value $\gamma = \frac{1}{2}$, which is equivalent to the negative energy $E = 4E_1$.

The work was supported by the National Academy of Sciences of Ukraine (theme No. 0112U000054).

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Received 20.11.16.
Translated from Ukrainian by O.I. Voitenko

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РОЗВ'ЯЗАННЯ РІВНЯННЯ ЛІПМАНА–ШВІНГЕРА ДЛЯ ПАРЦІАЛЬНОЇ МАТРИЦІ ПЕРЕХОДУ З ВІДШТОВХУВАЛЬНОЮ КУЛОНІВСЬКОЮ ВЗАЄМОДІЄЮ

Р е з ю м е

Досліджено випадок, коли можливе аналітичне розв'язання інтегрального рівняння Ліпмана–Швінгера для парціальної двочастинкової кулонівської матриці переходу для однойменно заряджених частинок при від'ємній енергії. За допомогою фоківського методу стереографічного проектування імпульсного простору на чотиривимірну одиничну сферу одержано аналітичні вирази для s-, p- і d-хвильових парціальних кулонівських матриць переходу для частинок з відштовхувальною взаємодією при енергії основного стану.