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P. KOSOBUTSKY
National University "Lviv Polytechnic"
(12, S. Bandera Str., Lviv 79013, Ukraine; e-mail: petkosob@gmail.com)

ANALYTICAL RELATIONS FOR THE MATHEMATICAL EXPECTATION AND VARIANCE OF A STANDARD DISTRIBUTED RANDOM VARIABLE SUBJECTED TO THE $\sqrt{X}$ TRANSFORMATION

The mathematical expectation and the variance have been calculated for random physical variables with the standard distribution function that are transformed by functionally related direct quadratic, X^2, and inverse quadratic, $\sqrt{X}$, dependences.

Keywords: normal distribution, mathematical expectation, variance, random variables, direct quadratic transformation, inverse quadratic transformation, errors.

1. Introduction

Linear and nonlinear transformations of physical quantities form a basis for simulating the principles of device operation. In particular, the direct quadratic transformation

$$Y = \alpha X^2 \tag{1}$$

and the inverse one

$$Y = \beta\sqrt{X} \tag{2}$$

are widely used, while developing models to describe, e.g., the elastically deformed state of a static body by the potential energy method, a moving body by the kinetic energy method, electric heaters by the Joule heat generation method, and so forth. If the corresponding input parameters undergo random fluctuations, then, in most cases, a statistical model of physical regularity can be constructed by studying the nonlinearly transformed quantity. Similar problems are challenging, e.g., in optics and quantum mechanics, while simulating the wave processes with the help of a trigonometric transformation of phase relations.

However, despite that the statistic-probabilistic methods that are used to process the results of measurements or calculations have been developed quite well, the search for analytical relations that would make it possible to simplify the algorithm of analytical estimation of the probabilistic and statistical parameters of the researched models always remains to be an importnat task. For normally distributed systems, the basic quantities are the mathematical expectation $E_{X,Y}$ and the mean square variance (MSV) $\sigma_{X,Y} = \sqrt{D_{X,Y}}$, although, for the analysis of the probability distribution law $F(x)$ to be more complete, regularities in the coefficients of asymmetry, skewness, and others for the curves of the differential function $f(x) = \frac{dF(x)}{dx}$ should be modeled.

When processing the results of a physical experiment, the measured values of a physical quantity, the random variable (RV) sample, are mainly subjected to arithmetic, trigonometric, logarithmic, and so on transformations. However, direct non-linear RV transformations of the type $Y = g(X) = X^2$ or $\cos X$, as well as the corresponding inverse transformations $Z = g^{-1}(X) = \sqrt{X}$ or $\arccos X$, respectively, and other ones, unlike linear transformations of the type $aX + b$, change the distribution density $f(x)$, which can make the algorithm of error calculation much more complicated and can dictate the introduction of other distribution parameters. For a normally $N(m_X, \sigma_X)$ distributed RV, one of such problems was formulated in works [1, 2]. For the direct, $Y = g(X) = (X^2, \cos X^2)$, and the inverse to them, $Z = g^{-1}(X) = \left(\sqrt{X}, \arccos X\right)$, transformations of the sample elements, the cited author proposed the so-called "error propagation" rules. They consist in a formal reduction of the indices in the solutions of corresponding square equations. However, a proper probabilistic and statistic substantiation of the method was not given. In this work, analytical formulas for the calculation of the basic probability distribution parameters $E_{X,Y}$ and $\sigma_{X,Y}$ for the transformation $Z = g^{-1}(X) = \sqrt{X}$ of a standard $N(0, \sigma_X)$ distributed RV X, which is inverse to the quadratic transformation $Y = g(X) = X^2$, have been substantiated on the basis of fundamental points of probability theory and mathematical statistics [3, 4].

2. Theoretical Analysis of Statistical Averaging and Discussion of the Results Obtained

Despite that the statistical model of quadratic transformation has been repeatedly discussed in the literature (see, e.g., works [3, 4]), the analysis in most cases was ended, when the distribution function for the probability density of transformed RV had been determined. Therefore, in order to apply the transformation inverse to the quadratic one, let us find the function $f_W(w)$ describing the probability density distribution for the random variable W.

Let the RV X be subjected to the bi-lateral ($-\infty < < x < +\infty$) quadratic transformation (1). Our task consists in substantiating the variance $D_Y = D_{\sqrt{X}}$ and the mean $\bar{Y} = \overline{\sqrt{X}}$ for the inverse transformation (2). For this purpose, let us consequently apply a two-stage transformation of the following type to the RV X:

$$X \xrightarrow{Y=\sqrt{X}} \text{RV}\,\sqrt{X} \xrightarrow{W=\alpha Y^4} \text{RV}\,W. \tag{3}$$

Here, the transformation function looks like

$$y = \beta\sqrt{x} \tag{4}$$

at the first stage and

$$w = \alpha y^4 \tag{5}$$

at the second one. The transformation according to algorithm (3) is ended by the quadratic transformation (1) of the RV X. Therefore, let us analyze the regularities of the quadratic transformation.

In the unlimited interval of values $x \in (-\infty, +\infty)$, the function

$$w = g(x) = \alpha x^2 \tag{6}$$

of the quadratic transformation (1) has two roots,

$$x_1 = -\sqrt{\frac{w}{\alpha}} \text{ at } x < 0, \text{ and } x_2 = +\sqrt{\frac{w}{\alpha}} \text{ at } x \geq 0. \tag{7}$$

Therefore, it is convenient to divide the interval $(-\infty, +\infty)$ into two subintervals, $(-\infty, 0)$ and $(0, +\infty)$, in which function (6) is monotonic. There are no negative w-values, and the set of points $X_1 \leq \leq x \leq X_2$ is a set of $g(w)$ at $w \geq 0$. In the regions of monotonicity, the function $w = \alpha x^2$ has the inverse function

$$g_1(w) = -\frac{1}{\sqrt{\alpha}}\sqrt{w} \tag{8}$$

and the first derivative

$$\frac{d}{dw}g_1(w) = -\frac{1}{2\sqrt{\alpha}}\frac{1}{\sqrt{w}} \tag{9}$$

in the interval $(-\infty, 0)$, as well as the inverse function

$$g_2(w) = +\frac{1}{\sqrt{\alpha}}\sqrt{w} \tag{10}$$

and the first derivative

$$\frac{d}{dw}g_2(w) = +\frac{1}{2\sqrt{\alpha}}\frac{1}{\sqrt{w}} \tag{11}$$

in the interval $(0, +\infty)$. Then, according to the transformation formula [3]

$$f_W(w) = f_X\left[g^{-1}(w)\right]\left|\frac{d}{dw}g^{-1}(w)\right| = \frac{f_X\left[g^{-1}(w)\right]}{\left|\frac{dw}{dx}\Big|_{x=g^{-1}(w)}\right|}, \tag{12}$$

the probability density function $f_W(w)$ transformed by expression (6) has the form

$$f_W(w) = \frac{1}{2\sqrt{\alpha}}\frac{1}{\sqrt{w}}f_X\left(+\sqrt{\frac{w}{\alpha}}\right) + + \frac{1}{2\sqrt{\alpha}}\frac{1}{\sqrt{w}}f_X\left(-\sqrt{\frac{w}{\alpha}}\right). \tag{13}$$

In the case of the RV X with the standard distribution function

$$f_X(x) = \sqrt{\frac{p}{\pi}}\exp\left(-px^2\right) \tag{14}$$

(here, $p = \frac{1}{2\sigma_X^2}$) transformed by law (6), the function $f_W(w)$ looks like

$$f_W(w) = \frac{1}{2\sqrt{\alpha}}\sqrt{\frac{p}{\pi}}\frac{1}{\sqrt{w}}e^{-p\frac{w}{\alpha}} + \frac{1}{2\sqrt{\alpha}}\frac{1}{\sqrt{w}}\sqrt{\frac{p}{\pi}}e^{-p\frac{w}{\alpha}} = = \sqrt{\frac{p}{\pi\alpha}}\frac{1}{\sqrt{w}}\exp\left(-\frac{p}{\alpha}w\right). \tag{15}$$

As follows from Eq. (15), the quadratic transformation changes the distribution function for the RV X.

Despite that the function $f_W(w) \to \infty$ at $w \to 0$, it remains normalized:

$$C_w\sqrt{\frac{p}{\pi\alpha}}\int\limits_0^{\infty} w^{-1/2}\exp\left(-\frac{p}{\alpha}w\right)dw = = C_w\sqrt{\frac{p}{\pi\alpha}}\sqrt{\frac{\pi\alpha}{p}} = 1 \Rightarrow C_w = 1, \tag{16}$$

so that $C_w = 1$, where the tabular integral (860.05) from book [7] was used. Provided that the normalization condition (16) is satisfied, the equation for the variance D_W reads

$$D_W = \int\limits_0^{\infty}\left(w - \bar{W}\right)^2 f(w)dw = = \int\limits_0^{\infty}\left(w^2 + \bar{W}^2 - 2w\bar{W}\right)f(w)dw = = \int\limits_0^{\infty} w^2 f(w)dw + \bar{W}^2\int\limits_0^{\infty} f(w)dw - 2\bar{W}\int\limits_0^{\infty} wf(w)dw = = \bar{W^2} + \bar{W}^2 - 2\bar{W}^2 = \bar{W^2} - \bar{W}^2, \tag{17}$$

which is typical of statistically independent RVs [1]. The integration limits in Eq. (17) are put in agreement with the set of non-negative values of the RV $w \geq 0$ transformed by law (1).

Now, let us construct a system of equations for the variances of the initial RV and the RV transformed by the quadratic algorithm $X \xrightarrow{W=\alpha X^2}$ RV αX^2:

$$D_X = \bar{X^2} - \bar{X}^2, \tag{18a}$$

$$D_W = \bar{W^2} - \bar{W}^2, \tag{18b}$$

or

$$D_X = \bar{X^2} - \bar{X}^2, \tag{18c}$$

$$D_W = \bar{W^2} - \alpha^2\left(\bar{X^2}\right)^2, \tag{18d}$$

$$\bar{W} = C_W\sqrt{\frac{p}{\pi\alpha}}\int\limits_0^{+\infty} w^{1/2}\exp\left(-\frac{p}{\alpha}w\right)dw = = \frac{1}{2}\frac{\alpha}{p}\sqrt{\frac{p}{\pi\alpha}}\sqrt{\frac{\pi\alpha}{p}} = \alpha\sigma_X^2, \tag{19}$$

where the tabular integral (860.04) from work [7] was used. For a harmonic oscillator, $W = \frac{1}{2}\alpha x^2$, so that $\bar{W} = \frac{1}{2}\alpha\sigma_X^2$. The mean of the square $\bar{W^2}$ equals

$$\bar{W^2} = C_W\sqrt{\frac{p}{\pi\alpha}}\int\limits_0^{+\infty} w^{3/2}\exp\left(-\frac{p}{\alpha}w\right)dw = = \frac{1\cdot 3}{2^2}\frac{(p/\alpha)^{1/2}}{\sqrt{\pi}}\frac{\sqrt{\pi}}{(p/\alpha)^{5/2}} = \frac{3}{4}\left(\frac{\alpha}{p}\right)^2 = 3\alpha^2\sigma_X^4, \tag{20}$$

where the tabular integral (860.06) from work [7] was used. For a harmonic oscillator, $\bar{W^2} = \frac{3}{4}\alpha^2\sigma_X^2$, and the MSV for the energy equals $\sqrt{\bar{W^2}} = \frac{\sqrt{3}}{2}\alpha\sigma_X^2$.

The equation for the variance in the case of transformation (6) looks like

$$D_W = 3\alpha^2\sigma_X^4 - \left(\alpha\sigma_X^2\right)^2 = 2\alpha^2\sigma_X^4 > 0 \tag{21}$$

and satisfies the requirement that the variance is nonnegative. Therefore, the MSV σ_W equals

$$\sigma_W = \sqrt{2}\alpha\sigma_X^2. \tag{22}$$

Let us compare the results obtained and the results known from the literature. For the normal

[1] For independent RVs, the covariance equals zero [3–5]. The correlation coefficient for two RVs can equal zero, even if they are not independent. On the contrary, if the correlation coefficient differs from zero, the two RVs cannot be independent [6].

$N(m_X, \sigma_X^2)$ distributed RV X, the quadratic transformation $Y = X^2$ gives the following value for the variance: $D_{X^2} = 2D_X^2 + 4E_X^2 D_X$ [1]. If the initial RV X has a standard distribution $N(0, \sigma_X^2)$, then $D_{X^2} = 2\sigma_X^4$, which agrees with Eq. (21).

Now, let us simulate the quadratic transformation RV $W \to \alpha X^2$ as two consequent stages (3). The fractional transformation $Y = \sqrt{X}$ is used at the first stage, and the transformation $W = \alpha Y^4$ at the second one. The equation $y = \beta\sqrt{x}$ has no solutions in the interval $y < 0$, and the cumulative distribution function $F(y) = 0$. In the interval $y \geq 0$, we have $\sqrt{x} \leq \frac{y}{\beta}$ at $-\left(\frac{y}{\beta}\right)^2 \leq x \leq +\left(\frac{y}{\beta}\right)^2$, and the equation $y = \beta\sqrt{x}$ has one root $x = +\frac{y^2}{\beta^2}$ at $x \geq 0$ [5], where the y-value never becomes negative. The function $y = \beta\sqrt{x}$ has the inverse function

$$g_{-1}(x) = \frac{y^2}{\beta^2}, \tag{23}$$

and the first derivative

$$\frac{dg_{-1}(x)}{dy} = \frac{2y}{\beta^2}. \tag{24}$$

Therefore, the distribution function $f_Y(y)$ for the transformed RV equals

$$f_Y(y) = \frac{2y}{\beta^2} f_X\left(\frac{y^2}{\beta^2}\right). \tag{25}$$

Since the initial RV X is distributed according to the law $N(0, \sigma_X)$, function (25) looks like

$$f_Y(y) = \frac{2}{\beta^2}\sqrt{\frac{p}{\pi}}\, y \exp\left(-\frac{p}{\beta^4}y^4\right). \tag{26}$$

The plot of function (26) starts from the point $(0, 0)$ and reaches a maximum at the point with the most probable coordinate $y_{\max} = \sqrt[4]{\sigma_X^2/2}$ or, in the coordinate system of initial RV X, $x_{\max} = (y_{\max})^2 = \sqrt[2]{\frac{\sigma_X^2}{2}} = \frac{\sigma_X}{\sqrt{2}}$.

Let us substantiate the normalization condition and determine the normalization constant C_Y for function (26):

$$C_Y \frac{2}{\beta^2}\sqrt{\frac{p}{\pi}} \int_0^\infty y \exp\left(-\frac{p}{\beta^4}y^4\right) dy = = C_Y \frac{2}{\beta^2}\sqrt{\frac{p}{\pi}}\frac{1}{4}\left(\frac{p}{\beta^4}\right)^{-2/4}\Gamma\left(\frac{2}{4}\right) = = C_Y \frac{1}{2} = 1 \Rightarrow C_Y = 2. \tag{27}$$

so that $C_Y = 2$, where the tabular integral (3.478 (1)) from book [8] was used. Then the equation for the variance of type (17) for statistically independent RVs transformed according to law (4) looks like

$$D_{\sqrt{X}} = \int_0^\infty \left(\sqrt{x} - \overline{\sqrt{X}}\right)^2 f(y)dy = = \int_0^\infty \left(\left(\sqrt{x}\right)^2 + \overline{\sqrt{X}}^2 - 2\sqrt{x}\overline{\sqrt{X}}\right) f(y)dy = = \int_0^\infty \left(\sqrt{x}\right)^2 f(y)dy + \overline{\sqrt{X}}^2 \int_0^\infty f(y)dy - 2\overline{\sqrt{X}} \times \times \int_0^\infty \sqrt{x} f(y)dy = \overline{\left(\sqrt{X}\right)^2} + \overline{\sqrt{X}}^2 \int_0^\infty f(y)dy - 2\overline{\sqrt{X}}^2 = = \overline{\left(\sqrt{X}\right)^2} - \overline{\sqrt{X}}^2, \tag{28}$$

where, according to Eq. (27), the means $\overline{\sqrt{x}}$ and $\overline{(\sqrt{x})^2}$ are calculated by the formulas

$$\overline{\sqrt{X}} = \frac{4}{\beta^2}\sqrt{\frac{p}{\pi}} \int_0^{+\infty} y^2 \exp\left(-\frac{p}{\beta^4}y^4\right) dy = = \frac{4}{\beta^2}\sqrt{\frac{p}{\pi}}\frac{1}{4}\frac{\beta^3}{\sqrt{p}\sqrt[4]{p}}\Gamma\left(\frac{3}{4}\right) = \beta\Gamma\left(\frac{3}{4}\right)\frac{\sqrt[4]{2}}{\sqrt{\pi}}\sqrt{\sigma_X}, \tag{29}$$

and

$$\overline{\left(\sqrt{X}\right)^2} = \frac{4}{\beta^2}\sqrt{\frac{p}{\pi}} \int_0^{+\infty} y^3 \exp\left(-\frac{p}{\beta^4}y^4\right) dy = = \frac{4}{\beta^2}\sqrt{\frac{p}{\pi}}\frac{1}{4}\frac{\beta^4}{p}\Gamma\left(\frac{4}{4}\right) = \sqrt{\frac{2}{\pi}}\beta^2\sigma_X \neq 0, \tag{30}$$

respectively.

Let us verify that the variance

$$D_{\sqrt{X}} = \frac{\beta^2\sigma_X}{\sqrt{2\pi}} - \beta^2\sigma_X\left(\Gamma\left(\frac{3}{4}\right)\frac{\sqrt[4]{2}}{\sqrt{\pi}}\right)^2 = = \beta^2\sigma_X\left(\sqrt{\frac{2}{\pi}} - \left(\Gamma\left(\frac{3}{4}\right)\sqrt[4]{\frac{2}{\pi^2}}\right)^2\right) > 0. \tag{31}$$

is non-negative. For the standard $N(0, \sigma_X^2)$ distributed initial RV X subjected to the transformation $Y = \beta\sqrt{X}$, it is analytical relations (29)–(31) that the author of work [1] tried to obtain. However, the corresponding expressions derived in this work and in work [1] cannot be compared, because the latter cannot be applied to a standard $N(0,\sigma_X^2)$ distributed initial RV X. For a normal $N(m_X,\sigma_X^2)$ distributed initial RV X, the mathematical expectation $m_X \neq 0$. Therefore, the corresponding statistical averaging becomes more complicated due to the appearance of the product of three functions with random variables in the integrands, which requires an additional theoretical study.

Let us finish the second stage of the two-stage RV X transformation (3) and determine regularities for the probability density distribution function $f_U(u)$ of the RV $Y = \beta\sqrt{X}$ transformed according to the law $U = \gamma Y^4$. For simplicity, we put $\beta = 1$. Then the function $u = g(y) = \gamma y^4$ has the inverse functions

$$y = \pm\left(\frac{u}{\gamma}\right)^{1/4}, \tag{32}$$

and the derivatives

$$\left|\frac{d}{du}\left(\frac{u}{\gamma}\right)^{1/4}\right| = \left|\frac{1}{4}\gamma^{-1/4}u^{-3/4}\right|. \tag{33}$$

A random variable X was first transformed according to the law $Y = \sqrt{X}$, so that its domain of definition is $y \geq 0$. Therefore, the function w equals $w = \alpha y^4 = \gamma y^4$, and the probability density function $f_U(u)$ looks like

$$f_U(u) = 2\sqrt{\frac{p}{\pi}}\left(\frac{u}{\gamma}\right)^{1/4}\frac{1}{4}\gamma^{-1/4}u^{-3/4}\exp\left(-\frac{p}{\gamma}u\right) = \frac{1}{\sqrt{\pi}}\frac{1}{2\sigma_X\sqrt{w}}\exp\left(-\frac{w}{2\alpha\sigma_X^2}\right) = f_W(w). \tag{34}$$

This result coincides with Eq. (15), which confirms the correctness of the transformations and calculations made above.

Hence, if a standard $N(0,\sigma_X)$ distributed random variable X is subjected to a nonlinear transformation of the radical type, $Y = \sqrt{X}$, the mathematical expectation for the transformed quantity is equal to

$$\overline{\sqrt{X}} = \frac{1}{\sqrt[4]{2}}\,\Gamma\left(\frac{3}{4}\right)\sqrt{\frac{2}{\pi}\sigma_X} \cong 0.691 \times \sqrt{\frac{2}{\pi}}\sqrt{\sigma_X} \cong 0.822\sqrt{\sigma_X},$$

and the mean square variance to

$$\sigma_{\sqrt{X}} = \sqrt{\frac{2}{\pi}\sigma_X}\sqrt{1-\frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{4}\right)^2} \cong 0.391\sqrt{\frac{2}{\pi}\sigma_X} \cong 0.312\sqrt{\sigma_X}.$$

3. Conclusion

The analytical relations obtained for the estimation of the mathematical expectation, the variance, and the mean square error of a standard $N(0,\sigma_X)$ distributed random variable X subjected to the transformation $Y = \sqrt{X}$, which is inverse to the quadratic one $Z = X^2$, have been substantiated for the first time. The mathematical expectation equals $\overline{\sqrt{X}} \cong 0.822\sqrt{\sigma_X}$, and the mean square variance $\sigma_{\sqrt{X}} \cong 0.312\sqrt{\sigma_X}$.

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П. Кособуцький

АНАЛІТИЧНІ СПІВВІДНОШЕННЯ ОБЧИСЛЕННЯ МАТЕМАТИЧНОГО СПОДІВАННЯ І СЕРЕДНЬОЇ КВАДРАТИЧНОЇ ПОХИБКИ СТАНДАРТНО $N(0,\sigma_X)$ РОЗПОДІЛЕНОЇ ВИПАДКОВОЇ ВЕЛИЧИНИ, ПІДДАНОЇ ПЕРЕТВОРЕННЮ $\sqrt{X}$

Р е з ю м е

Обчислено математичне сподівання і дисперсія фізичних величин із випадковими значеннями, підпорядкованих стандартному $N(0,\sigma_X)$ розподілу та перетворені функціонально пов'язаними залежностями прямим квадратичним X^2 та оберненим вигляду $\sqrt{X}$.