https://doi.org/10.15407/ujpe63.7.584

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THE MASSLESS LIMIT OF BARGMANN–WIGNER EQUATIONS FOR A MASSIVE GRAVITON

Information about the discovery of gravity waves attract attention to the graviton's mass problem. The massive graviton is a spin-2 particle with a non-zero mass. In this work, relativistic wave equations for a massive graviton have been studied in the limiting case of zero particle mass. The equations for the non-zero-mass graviton are based on the Bargmann–Wigner equations in the five-dimensional space-time with the $(++++-)$ signature. In the massless limit of massive graviton, all states with possible helicity values – 0 (LL-graviton), ±1 (TL-graviton), and ±2 (TT-graviton) – are preserved.

Keywords: Bargmann–Wigner equation, massive graviton, wave equations.

1. Introduction

Observations of gravity waves in 2015–2017 attracted attention to the old issue about graviton's mass, i.e. to the problem of a massive graviton [1, 2]. The massive graviton is a hypothetical particle with a spin of 2, whose mass m is evaluated to be less than 10^{-22}–10^{-32} eV. The aim of this study is to consider the massive graviton as a massless particle in the five-dimensional space-time and to demonstrate that all five states of its helicity polarization survive in the massless limit in the Minkowski space-time. This result contradicts the conventional viewpoint [3].

2. The Landau–Peierls Wave Equations for Light Quanta and the Bronshtein Equations for Gravitational Quanta

In 1930, L. Landau and R. Peierls were the first who considered the issue about the wave function of a light quantum. At the beginning of their work [4], L. Landau and R. Peierls reasonably assumed that the light quantum had to be described by Maxwell's equations in vacuum (hereafter, we adopt that $c = \hbar = 1$)

$$\dot{\mathfrak{E}} = \quad \mathbf{rot}\mathfrak{H}, \quad \mathbf{div}\mathfrak{E} = 0, \tag{1}$$

$$\dot{\mathfrak{H}} = -\mathbf{rot}\mathfrak{E}, \quad \mathbf{div}\mathfrak{H} = 0. \tag{2}$$

According to Landau and Peierls, the vectors $\mathfrak{E}$ and $\mathfrak{H}$ in Eqs. (1) and (2) are complex-valued quantities.

Landau and Peierls imposed additional restrictions on $\mathfrak{E}$ and $\mathfrak{H}$, which excluded the solutions of Eqs. (1) and (2) with negative energies. Denoting the photon "wave function" as $\mathfrak{F} \equiv \mathfrak{E}$ (or $\mathfrak{F} \equiv \mathfrak{H}$), the cited authors wrote the following "wave equations" for the "wave function" $\mathfrak{F}$:

$$\dot{\mathfrak{F}} = -\sqrt{\Delta}\mathfrak{F}, \tag{3}$$
$$\mathbf{div}\mathfrak{F} = 0, \tag{4}$$

where $\sqrt{\Delta}$ is the integro-differential operator

$$\sqrt{\Delta}\mathfrak{E}(\mathbf{x}) = \frac{1}{2\pi^2 i}\Delta \int \frac{\mathfrak{E}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} d^3y. \tag{5}$$

Equations (3) and (4) has a plane-wave solution

$$\mathfrak{F} = e^{i(\mathbf{kx}-\omega t)}\mathbf{f}, \tag{6}$$

in which

$$\omega = |\mathbf{k}|, \tag{7}$$
$$(\mathbf{kf}) = 0. \tag{8}$$

Equation (7) differs from the consequences of Maxwell's equations (1) and (2). Namely,

$$\omega = \pm|\mathbf{k}|. \tag{9}$$

In due course, it became clear that it is Maxwell's equations (1) and (2) with the complex $\mathfrak{E}$ and $\mathfrak{H}$ that are the wave equations for light quanta (photons).

In order to elucidate the physical quantum-mechanical content of Maxwell's equations as the wave equations for photons, it is expedient to introduce six complex-valued quantities, the Riemann–Silberstein vectors $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}$ [5–7], instead of six complex-valued quantities $\boldsymbol{\mathfrak{E}}$ and $\boldsymbol{\mathfrak{H}}$:

$$\boldsymbol{\psi}_\pm = \mathfrak{E} \pm i\mathfrak{H}. \tag{10}$$

Then Maxwell's equations (1) and (2) read

$$i\frac{\partial}{\partial t}\boldsymbol{\psi}_\pm = \pm\mathbf{rot}\boldsymbol{\psi}_\pm, \tag{11}$$

$$\mathbf{div}\boldsymbol{\psi}_\pm = 0. \tag{12}$$

The change from the vectors $\mathfrak{E}$ and $\mathfrak{H}$ to the vectors $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}$ has the following advantages.

1. Maxwell's equations are transformed into two independent pairs of equations for two functions, $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$.

2. When subjected to the transformations of the proper Lorentz group, the functions ψ_+ and ψ_- are transformed quite simply and independently of each other. For instance, when changing to a reference frame that moves relatively to another reference frame with a velocity $\mathbf{v}$, the new functions look like

$$\boldsymbol{\psi}'_\pm = \frac{\boldsymbol{\psi}_\pm \pm i[\mathbf{v}, \boldsymbol{\psi}_\pm]}{\sqrt{1-\mathbf{v}^2}}. \tag{13}$$

3. Equation (11) has the form of Schrödinger's equation

$$i\frac{\partial}{\partial t}\psi = H\psi \tag{14}$$

with the Hamiltonian $H = \pm\,\mathrm{rot}$.

4. The physical meaning of the Hamiltonian $H = \pm\,\mathrm{rot}$ is as follows: $H = \pm(\mathbf{sp})$, where $\mathbf{s}$ is the photon spin operator, $(s_i)_{kl} = -i\varepsilon_{ikl}$; $\mathbf{p}$ is the photon momentum operator, $p_i = -i\frac{\partial}{\partial x_i}$; and ε_{ikl} is the completely antisymmetric Levi–Civita unit symbol, $\varepsilon_{123} = 1$.

Thus, if photons have certain energies and momenta, Eqs. (11) describe photons with the right (R or $+$) and left (L or $-$) helicities.

As was elucidated in work [8], the Weyl equations [9] for a massless neutrino ($s = \frac{1}{2}$), Maxwell's wave equations for a photon ($s = 1$), and the Bronshtein wave equations [10] for a graviton ($s = 2$) have the same group-theoretic nature, and the corresponding Hamiltonians look like $H = \pm\frac{1}{s}(\mathbf{sp})$ (for more details, see works [11–15]).

In the work by M.P. Bronshtein [10], the equations for weak gravitational waves were presented in a form that made obvious their relation to Maxwell ones. In particular, Maxwell's equations (1), (2) can be written as follows:

$$\begin{aligned}\frac{\partial}{\partial t}E_i &= \varepsilon_{ikl}\frac{\partial}{\partial x_k}H_l, \quad \frac{\partial}{\partial x_i}E_i = 0,\\ \frac{\partial}{\partial t}H_i &= -\varepsilon_{ikl}\frac{\partial}{\partial x_k}E_l, \quad \frac{\partial}{\partial x_i}H_i = 0.\end{aligned} \tag{15}$$

Einstein's equations for weak gravitational waves in the Bronshtein form look like

$$\begin{aligned}\frac{\partial}{\partial t}E_{ij} &= \varepsilon_{ikl}\frac{\partial}{\partial x_k}H_{lj}, \quad \frac{\partial}{\partial x_i}E_{ij} = 0,\\ \frac{\partial}{\partial t}H_{ij} &= -\varepsilon_{ikl}\frac{\partial}{\partial x_k}E_{lj}, \quad \frac{\partial}{\partial x_i}H_{ij} = 0,\end{aligned} \tag{16}$$

where E_{ij} and H_{ij} are symmetric traceless tensors [1]

$$\begin{aligned}E_{ij} &= R_{4j4i} = \frac{1}{4}\varepsilon_{ikl}\varepsilon_{jmn}R_{klmn},\\ H_{ij} &= \frac{i}{2}\varepsilon_{imn}R_{4jmn} = \frac{i}{2}\varepsilon_{imn}R_{mn4j},\end{aligned} \tag{17}$$

and $R_{\mu\nu\rho\sigma}$ is the Einstein curvature tensor.

In his work [10], M.P. Bronshtein did not introduce the notions "graviton" and "wave function of a gravitational quantum". He considered the complex tensors E_{ij} and H_{ij} as solutions of Eqs. (16). Those solutions are nothing else but the wave functions of gravitons, and Eqs. (16) are the corresponding wave equations.

3. Wave Equations for Photons and Gravitons in the Spinor Form

In 1929, on P. Ehrenfest's request, B.L. van der Waerden developed the spinor analysis [16]. In 1931, O. Laporte and G.E. Uhlenbeck for the first time

[1] To verify the symmetry of the right-hand side of Eqs. (16) with respect to the subscripts i and j, let us consider the difference $\varepsilon_{ikl}\frac{\partial}{\partial x_k}H_{lj} - \varepsilon_{jkl}\frac{\partial}{\partial x_k}H_{li}$. Using property (17) and the relation $\varepsilon_{ikl}\varepsilon_{mnl} = \delta_{im}\delta_{kn} - \delta_{in}\delta_{km}$, one can demonstrate that this difference equals zero. Therefore, the tensor is symmetric with respect to the permutation $i \leftrightarrow j$. The symmetry of the other tensor is proved analogously.

considered Maxwell's equations in the spinor form [17]. They recalled about the Riemann–Silberstein vector and used it in their work (both Riemann and Silberstein used only one vector, $\mathfrak{H} - i\mathfrak{E}$). Using the Riemann–Silberstein vectors (10), we can write Eqs. (11) and (12) in the form

$$\left(\frac{\partial}{\partial t} I \pm \sigma \frac{\partial}{\partial \mathbf{x}}\right) \psi_{(\pm)} = 0, \tag{18}$$

where $\psi_{(\pm)}$ is the 2×2-matrix $(\psi_{(\pm)})_\alpha^\beta = (\boldsymbol{\psi}_\pm \boldsymbol{\sigma})_\alpha^\beta$, and $(\boldsymbol{\sigma})_\alpha^\beta$ are the standard Pauli matrices. Equations (11) and (12) can be obtained from Eqs. (18) and the following formula for the product of Pauli matrices:

$$\sigma_i \sigma_k = \delta_{ik} I + i\varepsilon_{ikl}\sigma_l. \tag{19}$$

Instead of 10 complex variables E_{ij} and H_{ij}, let us introduce 10 complex variables (analogs of the Riemann–Silberstein vectors)

$$\psi_{(\pm)_{ij}} = E_{ij} \pm iH_{ij}, \tag{20}$$

and define the spin tensor

$$\psi_{(\pm)_{\alpha\beta\gamma\delta}} = \psi_{(\pm)_{ij}} (\sigma_i)_{\alpha\beta} (\sigma_j)_{\gamma\delta}, \tag{21}$$

where $(\sigma_i)_{\alpha\beta} = (\sigma_i)_\alpha^\gamma \varepsilon_{\gamma\beta}$ are symmetric matrices, i.e. $(\sigma_i)_{\alpha\beta} = (\sigma_i)_{\beta\alpha}$, and $\varepsilon_{\gamma\beta}$ is the completely antisymmetric Levi–Civita unit symbol, $\varepsilon_{12} = 1$. It can be shown that the spin tensor (21) is completely symmetric.

In the spinor form, the Weyl equations look like

$$\left(\frac{\partial}{\partial t} I \pm \sigma \frac{\partial}{\partial \mathbf{x}}\right)_\alpha^\eta \psi_{(\pm)_\eta} = 0, \tag{22}$$

Maxwell's equations like

$$\left(\frac{\partial}{\partial t} I \pm \sigma \frac{\partial}{\partial \mathbf{x}}\right)_\alpha^\eta \psi_{(\pm)_{\eta\beta}} = 0, \tag{23}$$

and the Bronshtein equations like

$$\left(\frac{\partial}{\partial t} I \pm \sigma \frac{\partial}{\partial \mathbf{x}}\right)_\alpha^\eta \psi_{(\pm)_{\eta\beta\gamma\delta}} = 0. \tag{24}$$

The spinor forms of those equations convincingly demonstrate that the latter are related to one another.

Formulas (22) to (24) are relativistic wave equations in the non-relativistic notation. Their relativistic invariance is not evident, and this is their shortcoming. In the notation introduced by van der Waerden (ordinary and dotted spinor indices), they acquire the following forms:

- Eqs. (22):

$$\begin{aligned} &(\sigma_\mu)^{\dot\alpha\alpha} \frac{\partial}{\partial x_\mu} \psi_{(-)\alpha}(x) = 0, \\ &(\sigma_\mu)_{\alpha\dot\alpha} \frac{\partial}{\partial x_\mu} \psi_{(+)}^{\dot\alpha}(x) = 0, \end{aligned} \tag{25}$$

- Eqs. (23):

$$\begin{aligned} &(\sigma_\mu)^{\dot\alpha\alpha} \frac{\partial}{\partial x_\mu} \psi_{(-)\alpha\beta}(x) = 0, \\ &(\sigma_\mu)_{\alpha\dot\alpha} \frac{\partial}{\partial x_\mu} \psi_{(+)}^{\dot\alpha\dot\beta}(x) = 0, \end{aligned} \tag{26}$$

- and Eqs. (24):

$$\begin{aligned} &(\sigma_\mu)^{\dot\alpha\alpha} \frac{\partial}{\partial x_\mu} \psi_{(-)\alpha\beta\gamma\delta}(x) = 0, \\ &(\sigma_\mu)_{\alpha\dot\alpha} \frac{\partial}{\partial x_\mu} \psi_{(+)}^{\dot\alpha\dot\beta\dot\gamma\dot\delta}(x) = 0, \end{aligned} \tag{27}$$

where the matrices $(\sigma_\mu)^{\dot\alpha\alpha}$ and $(\sigma_\mu)_{\alpha\dot\alpha}$ are defined as follows:

$$(\sigma_\mu)^{\dot\alpha\alpha} = (\sigma, iI), \quad (\sigma_\mu)_{\alpha\dot\alpha} = (\sigma. - iI). \tag{28}$$

Equations (25) to (27) can be written in a more compact form, if we change from two-component spinors to four-component Dirac bispinors. In this case, when considering, e.g., a photon, it is expedient to return back from the Riemann–Silberstein vectors to the complex-valued Landau–Peierls vectors $\mathfrak{E} \equiv \mathbf{E}$ and $\mathfrak{H} \equiv \mathbf{H}$, as well as to the four-dimensional antisymmetric complex electromagnetic field tensor $F_{\mu\nu}(x) = -F_{\nu\mu}(x) = (\mathbf{E}(x), \mathbf{H}(x))$, and to consider the latter as the wave function of a photon. In the spinor notation, the both pairs of Maxwell's equations

$$F_{\mu\nu,\nu} = 0, \quad F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0 \tag{29}$$

read

$$(\gamma_\mu)_\alpha^\beta F_{\rho\sigma,\mu} (\gamma_\rho\gamma_\sigma)_\beta^\delta = 0, \ \text{ i.e. } \ \gamma_\mu F_{\rho\sigma,\mu}\gamma_\rho\gamma_\sigma = 0, \tag{30}$$

where γ_μ are the Dirac matrices. At the same time, for a massless notoph [18] $F_\mu(x) = (\mathbf{F}(x), iF_0(x))$, the wave equations

$$F_{\mu,\mu} = 0, \quad F_{\mu,\nu} - F_{\nu,\mu} = 0, \tag{31}$$

acquire the form [2]

$$(\gamma_\mu)^\beta_\alpha F_{\rho,\mu}(\gamma_\rho)^\delta_\beta = 0, \text{ i.e. } \gamma_\mu F_{\rho,\mu}\gamma_\rho = 0. \tag{32}$$

4. Three Variants of Proca Equations. Photon and Notoph

The relativistic wave equations for a massive spin-1 particle (the Proca equations) can be written in three (at $m \neq 0$) forms [22]:

$$F_{\mu\nu,\nu} = m^2 F_\mu, \tag{33}$$

$$F_{\mu,\nu} - F_{\nu,\mu} = F_{\mu\nu}, \tag{34}$$

$$W = |\mathbf{E}|^2 + |\mathbf{H}|^2 + m^2(|\mathbf{F}|^2 + |F_0|^2), \tag{35}$$

where W is the energy density. Proca considered his equations to be more suitable than the Dirac equations for the description of an electron.

The Proca equations and the positive energy density can also be written in the form [23]

$$F_{\mu\nu,\nu} = F_\mu, \tag{36}$$

$$F_{\mu,\nu} - F_{\nu,\mu} = m^2 F_{\mu\nu}. \tag{37}$$

$$W = m^2(|\mathbf{E}|^2 + |\mathbf{H}|^2) + |\mathbf{F}|^2 + |F_0|^2. \tag{38}$$

In a certain sense, Eqs. (33)–(35) are symmetric to Eqs. (36)–(38): in the massless limit, we lose the notoph in the former case, and the photon in the latter. If Eqs. (33)–(35) describe a "massive photon", then Eqs. (36)–(38) describe a "massive notoph". In work [23], the both indicated variants of Proca equations were analyzed in detail, by using the Bargmann–Wigner equations [24].

The third variant of the Proca equations was used in the work by Bass and Schrödinger [25]. In the massless limit of those equations, both a photon and a notoph "survive":

$$F_{\mu\nu,\nu} = mF_\mu, \tag{39}$$

$$F_{\mu,\nu} - F_{\nu,\mu} = mF_{\mu\nu}. \tag{40}$$

$$W = |\mathbf{E}|^2 + |\mathbf{H}|^2 + |\mathbf{F}|^2 + |F_0|^2. \tag{41}$$

In Eqs. (39)–(41), $F_{\mu\nu}$ and F_μ have the same dimension, because they both are equitable components of the same multicomponent wave function. The third variant of the Proca equations can be simply generalized to the case of five-dimensional space-time, which will be used in this work. It should be noted that the five-dimensional description of photons and gravitons was used for the first time by J.K. Lubański in 1942 in the theory of relativistic equations for particles with an arbitrary spin [26]. The fifth additional space-like coordinate is quite equitable with three other space-like coordinates. This five-dimensionality is much simpler, and it is not related to the five-dimensional theories by T. Kaluza [27] and O. Klein [28].

5. Bargmann–Wigner Equations for a Massive Graviton

Let us firstly consider an ordinary massive non-relativistic particle of spin 2. Hence, we assume that the wave function of a massive graviton is a completely symmetric spin-tensor of rank 4,

$$\varphi_{abcd} = \varphi_{abcd}(\mathbf{x}, t). \tag{42}$$

As was proved by E. Majorana in 1928 [29], an arbitrary wave function (42) can always be represented in the form

$$\varphi_{abcd} = A\{\varphi^{(1)}_a \varphi^{(2)}_b \varphi^{(3)}_c \varphi^{(4)}_d\}, \tag{43}$$

where A is a constant, the notation $\{...\}$ means a complete symmetrization over all indices, and $\varphi^{(i)}_a$ are certain two-component spinors [3]. When changing to the relativistic theory, if the spin equals $\frac{1}{2}$, the non-relativistic spinor $\varphi_a (a = 1, 2)$ has to be substituted by the 4-component Dirac bispinor ψ_α $(\alpha = 1, 2, 3, 4)$, which satisfies the Dirac equations

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)^\beta_\alpha \psi_\beta(x) = 0, \tag{44}$$

where $x_\mu = (\mathbf{x}, it)$ is the space-time coordinate of the particle, and m the particle mass. It is quite reasonable to replace all four non-relativistic spinors in

[2] For more about the spinor analysis and the application of spinors to the theory of relativistic wave equations and to the general theory of relativity, see works [19–21].

[3] The Majorana theorem was rediscovered by R. Penrose in 1960 [30] and used to analyze the algebraic properties of the curvature tensor in the general theory of relativity, when classifying the types of gravitational fields according to Petrov [31].

Eq. (43) by the Dirac bispinors and to assume that the wave function of a massive graviton looks like

$$\psi_{\alpha\beta\gamma\delta}(x) = A\{\psi^{(1)}_\alpha \psi^{(2)}_\beta \psi^{(3)}_\gamma \psi^{(4)}_\delta\}. \quad (45)$$

The wave function (45) evidently satisfies the Bargmann–Wigner equations [24]

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)^\lambda_\alpha \psi_{\lambda\beta\gamma\delta}(x) = 0. \quad (46)$$

Since the spin tensor $\psi_{\alpha\beta\gamma\delta}$ is completely symmetric, there is no difference, on which of the indices the matrix $\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)$ acts.

6. Five-Dimensional Bargmann–Wigner Equations for a Massive Graviton

The Dirac equations (44) can be expressed in the five-dimensional form. First, Eqs. (44) are multiplied by $i\gamma_5$,

$$\left(i\gamma_5\gamma_\mu \frac{\partial}{\partial x_\mu} + im\gamma_5\right)\psi = 0. \quad (47)$$

Then a fifth coordinate X_5, a new wave function

$$\Psi(x, X_5) = e^{imX_5}\psi(x), \quad (48)$$

and new matrices Γ_A $(A = 1, 2, 3, 4, 5)$, namely,

$$\Gamma_\mu = i\gamma_5\gamma_\mu, \Gamma_5 = \gamma_5, \quad (49)$$

are introduced.

Five coordinates $(X_\mu = x_\mu, X_5)$ will be denoted by the capital letter X. Then the Dirac equation for the wave function (48) can be written as

$$\Gamma_A \frac{\partial}{\partial X_A}\Psi(X) = 0. \quad (50)$$

By performing an analogous procedure with Eq. (46), the latter can be rewritten in the form

$$\frac{\partial}{\partial X_A}(\Gamma_A)^\lambda_\alpha \Psi_{\lambda\beta\gamma\delta}(X) = 0. \quad (51)$$

7. Five-Dimensional Bargmann–Wigner Equations for a Massive Graviton in the Tensor Form

Let us first consider the case of massive photon. Analogously to Eq. (45), we assume that the wave function of a massive photon is a completely symmetric tensor of rank 2

$$\psi_{\alpha\beta}(x) = A\{\psi^{(1)}_\alpha \psi^{(2)}_\beta\}, \quad (52)$$

where A is a constant. It is evident that this wave function satisfies the Bargmann–Wigner equation

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)^\lambda_\alpha \psi_{\lambda\beta}(x) = 0. \quad (53)$$

In the five-dimensional space $X_A = (X_\mu = x_\mu, X_5)$, the wave function

$$\Psi_{\alpha\beta}(X) = e^{imX_5}\psi_{\alpha\beta}(x) \quad (54)$$

satisfies the following equation, which is similar to Eq. (51):

$$\frac{\partial}{\partial X_A}(\Gamma_A)^\lambda_\alpha \Psi_{\lambda\beta}(X) = 0. \quad (55)$$

Let us introduce the infinitesimal operators $(S_{AB})^\beta_\alpha$ of the generalized Lorentz group $SO(4, 1)$. They act on the Dirac bispinors in the five-dimensional space with five coordinates X_A and the invariant form X_AX_A:

$$S_{AB} = \frac{\Gamma_A\Gamma_B - \Gamma_B\Gamma_A}{4i}. \quad (56)$$

Let us consider new matrices

$$(S_{AB})^{\alpha\beta} = (C^{-1})^{\alpha\gamma}(S_{AB})^\beta_\gamma, \quad (57)$$

where C is the antisymmetric matrix of the charge conjugation, $C_{\alpha\beta} = -C_{\beta\alpha}$. Ten matrices $(S_{AB})^{\alpha\beta}$ are characterized by the following important symmetry properties:

$$(S_{AB})^{\alpha\beta} = -(S_{BA})^{\alpha\beta} = (S_{AB})^{\beta\alpha}. \quad (58)$$

Let us introduce the tensor wave function of a massive photon,

$$\Phi_{AB}(X) = -\Phi_{BA}(X) = (S_{AB})^{\alpha\beta}\Psi_{\alpha\beta}(X). \quad (59)$$

It is easy to be convinced that the wave function (59) satisfies the equations

$$\Phi_{AB,B} = 0, \quad (60)$$

$$\Phi_{AB,C} + \Phi_{BC,A} + \Phi_{CA,B} = 0, \quad (61)$$

where the comma means a partial derivative, i.e. $\Phi_{,A} \equiv \frac{\partial}{\partial X_A}\Phi$.

Let us introduce a wave function in the Minkowski space $F_{AB}(x)$,

$$\Phi_{AB}(X) = e^{imX_5}F_{AB}(x), \quad (62)$$

and the notation

$$F_{\mu 5}(x) = iF_\mu(x). \tag{63}$$

For the functions $F_{\mu\nu}$ and F_μ, Eqs. (60) and (61) are transformed into the Proca equations (39) and (40), respectively, for a massive particle with spin 1,

$$F_{\mu\nu,\nu} = mF_\mu, \tag{64}$$

$$F_{\mu,\nu} - F_{\nu,\mu} = mF_{\mu\nu}. \tag{65}$$

In notations (62)–(64), Eq. (61) takes the form

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0, \tag{66}$$

and Eq. (60) becomes

$$F_{\mu,\mu} = 0. \tag{67}$$

The massless limit of the Proca equations (64) and (65) was first analyzed by Bass and Schrödinger in 1955 [25]. If $m = 0$, Eqs. (64)–(67) give rise to the following expressions:

$$F_{\mu\nu,\nu} = 0, \quad F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0, \tag{68}$$

$$F_{\mu,\mu} = 0, \quad F_{\mu,\nu} - F_{\nu,\mu} = 0. \tag{69}$$

Equations (68) are ordinary Maxwell's equations for the complex fields $F_{\mu\nu}(x)$, i.e. the wave equations for photons (T-photons, according to work [25]). Equations (69) correspond to particles that were called L-photons in work [25] and, later, "notophs" in work [18]. From Eq. (69), it follows that

$$F_\mu = \frac{\partial\phi}{\partial x_\mu}, \quad \Box\phi = 0, \tag{70}$$

i.e. the L-photon (or the notoph) is an ordinary massless scalar particle. Thus, one can see that, in the case of massive photon, which has three polarization degrees, all three states survive at $m \to 0$, in contrast to the fault conclusion made by Wigner [3] [4].

Analogously to Eq. (59), the tensor wave function of a massive graviton looks like

$$G_{ABCD}(X) = S^{\alpha\beta}_{AB} S^{\gamma\delta}_{CD} \Psi_{\alpha\beta\gamma\delta}(X). \tag{71}$$

[4] Namely, according to Wigner, only maximum spin projections on the particle motion direction survive in the massless limit $m \to 0$. In the general case of spin S, these are the projections S and $-S$.

It is clear that

$$G_{ABCD}(X) = -G_{BACD}(X) = -G_{ABDC}(X) = \\ = G_{CDAB}(X). \tag{72}$$

From the generalized Pauli–Fierz relations [32], it follows that

$$G_{ABCB}(X) = 0, \tag{73}$$

$$\varepsilon_{ABCDE} G_{ABCD}(X) = 0. \tag{74}$$

At the same time, it follows from the Bargmann–Wigner equations (51) that the new wave function satisfies the equations

$$G_{ABCD,D}(X) = 0, \tag{75}$$

$$G_{ABCD,E} + G_{ABDE,C} + G_{ABEC,D} = 0. \tag{76}$$

Those equations are not independent: the former can be obtained from the latter by applying the contraction operation over two indices. Let us verify that the symmetry properties of the tensor G_{ABCD} restrict the number of its components to the same value as the symmetry properties of the wave function in the Bargmann–Wigner form $\Psi_{\alpha\beta\gamma\delta}$ do, which is a completely symmetric 4-rank spin-tensor in the 4-dimensional spinor space. The number of components of a completely symmetric n-th rank tensor in an m-dimensional space can be found by the formula

$$N = \frac{(n+m-1)!}{n!\,(m-1)!}, \tag{77}$$

which gives $N = 35$ in our case. At the same time, relations (72) restrict the number of the tensor G_{ABCD} components to $10 \times (10+1)/2 = 55$. But 15 equations (73) and 5 equations (74) reduce their total number to $N = 55 - 15 - 5 = 35$. Thus, the number of components in the wave functions $\Psi_{\alpha\beta\gamma\delta}$ and G_{ABCD} coincide.

One can see that the symmetric properties of the graviton wave function (71) as a tensor are identical to the property of the linearized Weyl tensor [33] in the 5-dimensional space.

8. Generalized Proca Equations for a Massive Graviton

Using Eq. (73), the tensor function of a massive graviton can be presented in the four-dimensional form:

$$G_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}, \tag{78}$$

$$G_{5\nu\rho\sigma} = iH_{\rho\sigma\nu}, \tag{79}$$

$$G_{5\mu5\nu} = H_{\mu\nu} = -R_{\mu\nu}, \tag{80}$$

$$R_{\mu\mu} = R = 0. \tag{81}$$

Substituting these properties into Eqs. (73)–(76), we obtain the following expressions:

$$R_{\mu\nu\rho\sigma,\lambda} + R_{\mu\nu\sigma\lambda,\rho} + R_{\mu\nu\lambda\rho,\sigma} = 0, \tag{82}$$

$$H_{\nu\sigma,\lambda} - H_{\nu\lambda,\sigma} = mH_{\sigma\lambda\nu}. \tag{83}$$

From Eqs. (75), it follows that

$$R_{\mu\nu\rho\sigma,\sigma} = -mH_{\mu\nu\rho}. \tag{84}$$

Equation

$$H_{\mu\nu\rho,\sigma} - H_{\mu\nu\sigma,\rho} = -mR_{\mu\nu\rho\sigma} \tag{85}$$

gives rise to the equality

$$H_{\rho\sigma\nu,\sigma} = -mH_{\nu\rho}. \tag{86}$$

From Eq. (76), it also follows that

$$H_{\rho\sigma\nu,\lambda} + H_{\sigma\lambda\nu,\rho} + H_{\lambda\rho\nu,\sigma} = 0. \tag{87}$$

In the four-dimensional notation, the following important relation can be obtained from Eq. (75):

$$H_{\mu\nu,\nu} = 0. \tag{88}$$

Let us consider Eqs. (78)–(88) for a massive graviton in the massless limit. At $m = 0$, Eqs. (83) become

$$H_{\nu\sigma,\lambda} - H_{\nu\lambda,\sigma} = 0. \tag{89}$$

Hence, the tensor $H_{\nu\sigma}$ turns out a second-order derivative of a scalar function:

$$H_{\nu\sigma} = \frac{\partial^2\phi}{\partial x_\nu \partial x_\sigma}. \tag{90}$$

Following the work by Bass and Schrödinger, let us call the particles that correspond to those equations as LL-gravitons. Let us write the previous expressions in the zero-mass case. Equations (82), (87), and (88) remain unchanged. Equations (84), (85), and (86) are transformed into the equations

$$R_{\mu\nu\rho\sigma,\sigma} = 0, \tag{91}$$

$$H_{\mu\nu\rho,\sigma} - H_{\mu\nu\sigma,\rho} = 0, \tag{92}$$

$$H_{\rho\sigma\nu,\sigma} = 0, \tag{93}$$

Three symmetry relations remain:

$$H_{\rho\sigma\sigma} = 0, \tag{94}$$

$$H_{\mu\nu\rho} + H_{\nu\rho\mu} + H_{\rho\mu\nu} = 0, \tag{95}$$

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0. \tag{96}$$

By considering Eqs. (92), (94), and (95) in more details, the following expressions can be obtained:

$$H_{\mu\nu\rho} = \frac{\partial f_{\mu\nu}}{\partial x_\rho}, \tag{97}$$

$$\frac{\partial f_{\mu\nu}}{\partial x_\nu} = 0, \tag{98}$$

$$\frac{\partial f_{\mu\nu}}{\partial x_\rho} + \frac{\partial f_{\nu\rho}}{\partial x_\mu} + \frac{\partial f_{\rho\mu}}{\partial x_\nu} = 0. \tag{99}$$

Again following work [25], the particle that satisfies those equations will be called as TL-graviton.

Equations (81), (91), and (86) bring us to the Bronshtein equations [10], which describe particles with a helicity of ± 2, i.e. TT-gravitons. It should be noted that the helicity of a superposition of the left and right gravitons can be an arbitrary real number within an interval of $[-2, +2]$, including zero. The zero case is an analog of the linear photon polarization. The gravitational waves that were registered in 2015–2017 were composed of gravitons of this type. This remark is also applicable to TL-gravitons. Note also that, in the case of Einstein's equations, the components of the tensor $R_{\mu\nu\rho\sigma}$ are real-valued numbers. But, in the case of Bronshtein equations, they are complex numbers, because the tensor $R_{\mu\nu\rho\sigma}$ is the wave function of a graviton.

9. Conclusions

By reducing the Bargmann–Wigner equations for a massless graviton in the 5-dimensional space, we have obtained a system of wave equations, which describes the dynamics of a massive graviton in the 4-dimensional space. In the zero-mass limit, those equations transform into the Klein–Gordon equations for the wave function of a massless scalar particle, Maxwell's equations for particles with helicities $+1$ and -1 in vacuum , and the Bronshtein equation for gravitons with helicities $+2$ and -2. The obtained equations generalize the Proca equations in the Bass and Schrödinger formulation onto the spin-2 case and prove the possibility of preserving all 5 helicity polarization states of a massive particle with spin 2 in the massless limit. By analogy with the work by Bass and Schrödinger, particles with those states are called LL-gravitons (helicity 0), TL-gravitons (helicity ± 1), and TT-gravitons (helicity ± 2). It was TT-gravitons that were described by M. Bronshtein for the first time [10].

1. A.S. Goldhaber, M.M. Nieto. Photon and graviton mass limits. *Rev. Mod. Phys.* **82**, 939 (2010).
2. C. de Rham, J.T. Deskins, A.J. Tolley, S.-Y. Zhou. Graviton mass bounds. *Rev. Mod. Phys.* **89**, 025004 (2017).
3. E.P. Wigner. Relativistic invariance and quantum phenomena. *Rev. Mod. Phys.* **29**, 255 (1957).
4. L. Landau, R. Peierls. Quantum electrodynamics in the configuration space. *Zs. Phys.* **62**, 188 (1930).
5. I. Bialynicki-Birula. Photon wave function. *Progress in Optics* **36**, edited by E. Wolf (Elsevier, 1996) [arXiv: quant-ph/0508202].
6. I. Bialynicki-Birula. Photon as a quantum particle. *Acta Phys. Pol. B* **37**, 935 (2006).
7. I. Bialynicki-Birula, Z. Bialynicka-Birula. The role of the Riemann–Silberstein vector in classical and quantum theories of electromagnetism. *J. Phys. A* **46**, 053001 (2013).
8. Yu.P. Stepanovskyi. The little Lorentz group and wave equations of free massless fields with arbitrary spin. *Ukr. J. Phys.* **9**, 1165 (1964).
9. H. Weyl. Electron and gravitation. *Z. Phys.* **56**, 330 (1929).
10. M.P. Bronstein. Quantization of gravitational waves. *J. Exper. Theor. Phys.* **6**, 195 (1936).
11. Yu.P. Stepanovsky. On wave equations of massless fields. *Teor. Matem. Fiz.* **47**, 343 (1981).
12. Yu.P. Stepanovsky. From Maxwell equation to Berry's phase and sonoluminescence: Problems of theory of electromagnetic and other massless fields. *Electromag. Phenom.* **1**, 180 (1998).
13. Yu.P. Stepanovsky. On massless field and infinite component relativistic wave equations. *Nucl. Phys. B. Proc. Suppl.* **102** (1), 407 (2001).
14. Yu.P. Stepanovsky. Ettore Majorana and Matvei Bronstein (1906–1938): Men and scientists. In: *Advances in the Interplay between Quantum and Gravity Physics* (Kluwer Academic Publishers, 2002), p. 435 [ISBN: 978-94-010-0347-6].
15. Yu.P. Stepanovsky. In: *Problems in Contemporary Physics* (KIPT, 2008) (in Russian) [ISBN: 978-966-2136-15-9].
16. B.L. Van der Waerden. Spinor analysis. *Nachr. Ges. Wiss. Göttingen, Math.-Phys.* 100 (1929).
17. O. Laporte, G.E. Uhlenbeck. Application of spinor analysis to the Maxwell and Dirac equations. *Phys. Rev.* **37**, 1380 (1931).
18. V.I. Ogievetskii, I.V. Polubarinov. Notoph and its possible interactions. *Yader. Fiz.* **4**, 216 (1966).
19. Yu.B. Rumer. *Spinor Analysis* (Librocom, 2010) (in Russian) [ISBN: 978-5-397-01381-9].
20. Yu. B. Rumer, A.I. Fet. *Group Theory and Quantum Fields* (Nauka, 1977) (in Russian).
21. R. Penrose, W. Rindler. *Spinors and Space-Time* (Cambrige Univ. Press, 1984) [ISBN: 0521337070].
22. A. Proca. Wave theory of positive and negative electrons. *J. Phys. Radium* **7**, 347 (1936).
23. V.V. Dvoeglazov. Photon-notoph equations. *Phys. Scripta* **64**, 201 (2001).
24. V. Bargmann, E.P. Wigner. Group theoretical discussion of relativistic wave equations. *Proc. Nat. Acad. Sci. USA* **34**, 211 (1948).
25. L. Bass, E. Schrödinger. Must the photon mass be zero? *Proc. R. Soc. London A* **232**, 1 (1955).
26. J.K. Lubański. Sur la theorie des particules elementaires de spin quelconque. I. *Physica* **9**, 310 (1942).
27. T. Kaluza. Zum unitatsproblem der physik. *Sitzungsber. Preuss. Akad. Wiss. Berlin. (Math. Phys.)* 966 (1921).
28. O. Klein. Quantentheorie und fünfdimensionale relativitätstheorie. *Zeit. Phys.* **37**, 895 (1926).
29. E. Majorana. Atomi orientati in campo magnetico variabile. *Nuovo Cimento* **9**, 43 (1932).
30. R. Penrose. A spinor approach to general relativity. *Ann. Phys.* **10**, 171 (1960).
31. A.Z. Petrov. Clasification of spaces defining gravity fields. *Uchen. Zapisk. Kazan. Gosud. Univer.* **114**, 55 (1954).
32. Yu.P. Stepanovsky. Complete set of Fierz's relations in six-dimensional form. *Ukr. J. Phys.* **11**, 1191 (1966).
33. H. Weyl. Reine infinitesimalgeometrie. *Mat. Zeit.* **2**, 384 (1918).

Received 26.01.18.
Translated from Ukrainian by O.I. Voitenko

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БЕЗМАСОВА ГРАНИЦЯ РІВНЯНЬ БАРГМАНА–ВІГНЕРА ДЛЯ МАСИВНОГО ГРАВІТОНА

Р е з ю м е

Дані про відкриття гравітаційних хвиль привернули увагу до питання існування маси у гравітонів, тобто до питання масивного гравітона. Масивний гравітон – це частинка зі спіном 2 та ненульовою масою. Метою роботи є дослідження границі релятивістських хвильових рівнянь масивного гравітона для випадку нульової маси частинки. Рівняння для гравітона ненульової маси базуються на рівняннях Баргмана–Вігнера у п'ятивимірному просторі-часі із сигнатурою $(++++-)$. Безмасова границя масивного гравітона зберігає усі можливі стани поляризації. Ці стани відповідають LL-гравітону (спіральність 0), TL-гравітону (спіральність ± 1) та TT-гравітону (спіральність ± 2).