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GENERALIZED UNCERTAINTY PRINCIPLE IN QUANTUM COSMOLOGY [1]

The effects of gravity which manifest themselves when performing the simultaneous measurement of two non-commuting observables in the quantum theory are discussed. Matter and gravity are considered as quantum fields. The Schrödinger-type time equation is given for the case of a finite number of degrees of freedom: one for the matter field and one for geometry. For a spatially closed system filled with dust and radiation being in definite quantum states, the solutions to the quantum equations are found, and the existence of the minimum measurable length and the minimum momentum is shown. It appears that the simultaneous measurement of fluctuations of the intrinsic and extrinsic curvatures of the spacelike hypersurface in spacetime cannot be performed with an accuracy exceeding the Planck constant. Unruh's and Bronstein's uncertainty relations are discussed.

Keywords: quantum gravity, quantum geometrodynamics, cosmology, uncertainty principle.

1. Introduction

The Heisenberg uncertainty principle states that two observables which do not commute cannot be measured simultaneously with arbitrary accuracy. This principle in its standard quantum mechanical form does not take the effects of gravity into account. At the same time, the study of the properties of quantum systems on small scales requires dealing with high energies. It is expected that, on the scales less than the Planck one, the classical concepts of space and time lose their meanings, and a fundamental revision of their interpretation is needed. How the inclusion of gravity can contribute to the uncertainty relation have been debated since the middle 1930s [1, 2]. The interest in the problem was revived since the middle 1980s, after the existence of a minimum observable length was shown in string theory. The effects of a spacetime curvature on statistical fluctuations of two observables corresponding to canonically conjugate variables may be clarified in quantum theory which treats gravity on the same grounds as quantized matter fields.

In this note, we present the results of our investigations of the fluctuations of the observables that characterize a quantum gravitational system in itself, like the intrinsic and extrinsic curvatures of the spacelike hypersurfaces in spacetime, and the solution to the problem of minimum length in the framework of an exactly solvable cosmological model.

A model with a finite number of degrees of freedom may provide a reasonable framework for addressing the problems of quantum gravity. The homogeneous minisuperspace models have been proven to be successful in classical cosmology. They have predictive power and are consistent with observations. This gives rise to the hope for that the homogeneous models could be useful in quantum cosmology as well. For such models, the quantum theory of gravity with a well-defined time variable was proposed and studied in Refs. [3–7].

2. Time Equation and Its Interpretation

Consider the homogeneous isotropic quantum gravitational system. It is described by the Schrödinger-type time equation in Planck units

$$-i\partial_T|\Psi(T)\rangle = \mathrm{H}|\Psi(T)\rangle, \qquad (1)$$

where the operator

$$\mathrm{H} = \frac{1}{2}\left(-\partial_a^2 + \kappa a^2 - 2aH_\phi - a^4\frac{\Lambda}{3}\right) \qquad (2)$$

[1] This work is based on the results presented at the XI Bolyai–Gauss–Lobachevskii (BGL-2019) Conference: Non–Euclidean, Noncommutative Geometry and Quantum Physics.

can be considered as the effective Hamiltonian which does not depend on the conformal time T explicitly; a is the cosmic scale factor which determines the geometric properties of the system in the case of the maximally symmetric geometry with the Robertson–Walker metric, $H_\phi = H_\phi(a)$ is a self-adjoint Hamiltonian of the matter sector of the system taken in the form of a uniform scalar field ϕ, Λ is the cosmological constant, and $\kappa = +1, 0, -1$ is the curvature constant for spatially closed, flat, and open geometries, respectively.

The commutation relation between a and its conjugate momentum $\pi = -i\hbar\,\partial_a$ takes the form

$$[a, -i\partial_a] = i. \tag{3}$$

The general solution of Eq. (1) can be written as

$$|\Psi(T)\rangle = \sum_{n,k} e^{\frac{i}{2}E_{n(k)}(T-T_0)} C_{nk}(T_0|u_k\rangle|f_{n(k)}\rangle, \tag{4}$$

where T_0 is an arbitrary constant taken as a time reference point. The state vectors $|u_k\rangle$ and $|f_{n(k)}\rangle$ satisfy the equations

$$\langle u_k|H_\phi|u_{k'}\rangle = M_k(a)\,\delta_{kk'}, \tag{5}$$

$$\left(-\partial_a^2 + \kappa a^2 - 2aM_k(a) - a^4\frac{\Lambda}{3}\right)|f_{n(k)}\rangle =$$
$$= E_{n(k)}|f_{n(k)}\rangle, \tag{6}$$

$M_k(a)$ is the proper mass-energy of a new effective matter in the discrete and/or continuous kth state. It is supposed that the vectors $|u_k\rangle$ and $|f_{n(k)}\rangle$ form the complete sets of orthonormalized functions. The eigenvalue $E_{n(k)}$ determines the energy density of relativistic matter, $\rho_\gamma = a^{-4}E_{n(k)}$, and n enumerates discrete and/or continuous states of the system with matter in the fixed kth state. The coefficient C_{nk} gives the probability $|C_{nk}(T_0)|^2$ to find the system in the nth state of relativistic matter and the kth state of effective matter at the instant of time T_0.

3. Uncertainty Relation

The uncertainty relation between the scale factor and its conjugate momentum consistent with the commutation relation (3) has a form

$$\Delta a \Delta\pi \geq \frac{\hbar}{2}, \tag{7}$$

where $\Delta a = \sqrt{\langle a^2\rangle - \langle a\rangle^2}$ is the root-mean-square deviations of a and a similar expression for $\Delta\pi$, the brackets denote the mean values.

For the cosmological system with zero cosmological constant filled with dust and relativistic matter, the eigenvalue $E_{n(k)} = 2n + 1 - M_k^2$, and the state vector is given by

$$|f_{n(k)}\rangle \equiv f_n(\xi_k) = N_{nk}\, e^{-\frac{1}{2}\xi_k^2} H_n(\xi_k), \tag{8}$$

where $\xi_k = a - M_k$, H_n is the Hermite polynomial, N_{nk} is the normalizing constant, and the wave function is normalized on the interval $[-M_k, \infty)$.

With an accuracy of order $e^{-M_k^2}$ [3], we have

$$\langle a^2\rangle = n + \frac{1}{2} + M_k^2, \quad \langle a\rangle = M_k, \tag{9}$$

where the averaging is performed over states (8). Then, in ordinary physical units,

$$\Delta a = l_\mathrm{P}\sqrt{n + \frac{1}{2}}, \tag{10}$$

where $l_\mathrm{P} = \sqrt{G\hbar/(c^3)}$ is the Planck length. For the momentum, we obtain

$$\langle \pi^2\rangle = n + \frac{1}{2}, \quad \langle\pi\rangle = 0 \tag{11}$$

and

$$\Delta\pi = m_\mathrm{P}\,c\sqrt{n + \frac{1}{2}}, \tag{12}$$

where $m_\mathrm{P}c = \hbar/l_\mathrm{P}$ is the Planck momentum. As a consequence, we get the uncertainty product of the same form as for a harmonic oscillator,

$$\Delta a\,\Delta\pi = \left(n + \frac{1}{2}\right)\hbar \geq \frac{\hbar}{2}. \tag{13}$$

From Eqs. (10) and (12), one can see that the fluctuations Δa and $\Delta\pi$ take minimum values in a ground (vacuum) state with $n = 0$,

$$\Delta a_{\min} = \frac{l_\mathrm{P}}{\sqrt{2}}, \quad \Delta\pi_{\min} = \frac{m_\mathrm{P}\,c}{\sqrt{2}}. \tag{14}$$

The size of fluctuations increases as the square root $\sqrt{n}$. It is of interest to estimate the size of fluctuations in a subsystem having the mass-energy of the observable part of our universe $l \sim 10^{28}$ cm. The cosmological parameters $E_{n(k)} \sim 10^{118} \ll M_k^2 \sim 10^{122}$ (i.e.

$\rho_\gamma \sim 10^{-10}\,\mathrm{GeV\,cm^{-3}}$ and $\rho_m \sim 10^{-5}\,\mathrm{GeV\,cm^{-3}}$) correspond to $n \sim 10^{122}$ and fluctuations $\Delta a \sim l \sim \sim 10^{28}$ cm. In such a description, the observable part of the universe appears as a gigantic fluctuation [7] which brings us back to Boltzmann's speculations about the origin of the observable universe (cf. Ref. [8]).

Expressions (14) solve the problem of existence of a minimum observable length and a minimum momentum in the context of the exactly solvable cosmological model.

The uncertainty relation (7) establishes, in fact, a connection between fluctuations of the quantities which determine the intrinsic and extrinsic curvatures of the spacelike hypersurface in spacetime. By associating the quantum operators to the scalar curvature ${}^{(3)}R$ and the extrinsic curvature tensor $K_{ij} = = -\frac{1}{2}\partial^{(3)}g_{ij}/\partial\tau$, where ${}^{(3)}g_{ij}$ is the 3-metric, and τ is the proper time, Eq. (7) can be written explicitly in terms of curvature fluctuations,

$$\frac{\Delta^{(3)}R}{|^{(3)}R|}\ \ \Delta K\ {}^{(3)}V \gtrsim 4\pi\hbar, \qquad (15)$$

where $K = K^i_i$, and ${}^{(3)}V \sim \frac{4}{3}\pi a^3$ is the 3-volume of the measurement (observed part of the system). The uncertainty relation (15) demonstrates that the product of the relative fluctuation of the scalar curvature and the fluctuation of the extrinsic curvature in the observed volume must be greater than the Planck constant.

The uncertainty relation (7) can be reduced to the Unruh's uncertainty relation for fluctuations of the metric and the Einstein tensor. In contrast to the formal assumption that Einstein's equations are valid in the quantum regime as well [9], it was shown in Ref. [5] that the equations of quantum cosmology can be reduced to the Einstein–Friedmann equations which contain the terms of quantum corrections to the total energy density and pressure. Then the rate of change of the momentum in time is given by the equation $\dot{\pi} = -\frac{1}{2}a^2 T^\alpha_\alpha + \kappa$, where T^α_α is the trace of the stress tensor. Considering the fluctuations of the quantities in spatial directions, in a comoving reference frame, one can express the T^x_x component of the stress tensor as $T^x_x = -\mathrm{p}$, where p is the pressure defined as the force acting on the surface element having an area of $\sim a^2$ in the direction of x. In that case, the fluctuation of the momentum can be estimated as $\Delta\pi \sim \Delta T^x_x a^2 \delta\tau$, where $\delta\tau$ is a time interval, and $\Delta T^x_x \sim T^x_x$. The metric component g_{xx} can be taken in the form $g_{xx} = a^2\gamma_{xx}$, where γ_{xx} is the comoving spatial metric component whose fluctuation can be neglected, $\Delta\gamma_{xx} = 0$. Then the fluctuations Δg_{xx} and Δa are connected between themselves: $\Delta g_{xx}/g_{xx} = 2\Delta a/a$. As a result, in the rest frame, relation (7) takes the form

$$\Delta g_{xx}\Delta T^x_x \gtrsim \hbar\frac{g_{xx}}{\delta\tau^{(3)}V}, \qquad (16)$$

where ${}^{(3)}V \sim a^3$ is the 3-volume. Introducing the Einstein tensor $G^x_x = 8\pi T^x_x$ (in units $G = c = 1$) and defining the 4-volume ${}^{(4)}V \sim \delta\tau^{(3)}V$, we rewrite the preceding relation in Unruh's form

$$\Delta g_{xx}\Delta G^{xx} \gtrsim \hbar\frac{8\pi}{{}^{(4)}V}. \qquad (17)$$

The established connection between Eqs. (7) and (17) may be interpreted as clarifying the physical meaning of Eq. (17).

4. Concluding Remarks

Let us briefly discuss the uncertainty relation obtained by Bronstein in his pioneering papers [1, 2]. If one considers the motion of a test body with mass $m = \rho V$, where ρ is body's density and V is its volume, the uncertainty of the momentum of the test body Δp_x is composed of two summands, namely of the usual quantum mechanical term inversely proportional to an uncertainty in the coordinate, $\hbar/\Delta x$, and a term connected with the gravitational field created by the measuring device itself due to the recoil during the measurement procedure. The second summand can be brought to the form

$$\frac{\Delta p_x}{\hbar} > \left(\frac{G}{\hbar c}m^2\right)^{1/3}\frac{1}{L}, \qquad (18)$$

where $L \sim V^{1/3}$. Using the relation between the uncertainty of the Christoffel symbol Γ^1_{00} and the uncertainty of the momentum Δp_x, $\Delta\Gamma^1_{00} \approx \Delta p_x/mT$, where T is the time period of the momentum measurement, one can obtain Bronstein's original inequality

$$\Delta\Gamma^1_{00} > \left(\frac{\hbar}{V}\right)^{2/3}\left(\frac{G}{c\rho}\right)^{1/3}\frac{1}{T}. \qquad (19)$$

This was a rough approximation, but allowing one to estimate the applicability limits of general relativity [10]. An overview of the current state of the problem can be found, e.g., in Ref. [11] (and in references therein).

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УЗАГАЛЬНЕНИЙ ПРИНЦИП
НЕВИЗНАЧЕНОСТІ У КВАНТОВІЙ КОСМОЛОГІЇ

Р е з ю м е

Вивчаються ефекти гравітації, що проявляються при одночасному вимірюванні двох некомутуючих спостережуваних у квантовій теорії. Матерія та гравітація розглядаються як квантові поля. Часове рівняння типу Шредінгера наведено для випадку скінченної кількості ступенів вільності: одного для матеріального поля та одного для геометрії. Для просторово замкненої системи, заповненої пилом і випромінюванням, що знаходяться у визначених квантових станах, знайдено розв'язки квантових рівнянь та показано існування мінімальної вимірюваної довжини та мінімального імпульсу. Виявляється, що одночасне вимірювання флуктуацій внутрішньої та зовнішньої кривини просторово-подібної гіперповерхні у просторі-часі не може бути виконане з точністю, що перевищує сталу Планка. Обговорюються співвідношення невизначеності Анру та Бронштейна.