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A LOOK AT MULTIPLICITY DISTRIBUTIONS VIA MODIFIED COMBINANTS

The experimentally measured multiplicity distributions exhibit, after a closer inspection, the peculiarly enhanced void probability and the oscillatory behavior of modified combinants. We show that both these features can be used as additional sources of information, not yet fully explored, on the mechanism of multiparticle production. We provide their theoretical understanding within the class of compound distributions.

Keywords: multiplicity distributions, combinants, void probabilities, compound distributions.

1. Introduction

The experimentally measured (non-single diffractive (NSD) charged) multiplicity distributions, $P(N)$ (which are one of the most thoroughly investigated and discussed sources of information on the mechanism of the production process [1]), exhibit, after a closer inspection, the peculiarly enhanced void probability, $P(0) > P(1)$ [2, 3], and the oscillatory behavior of the so-called modified combinants, C_j , introduced by us in [4, 5] (and thoroughly discussed in [6, 7]; they are closely connected with the combinants C_j^* introduced in [8] and discussed occasionally for some time [9–14]). Both features were only rarely used as a source of information. We demonstrate that the modified combinants can be extracted experimentally from the measured $P(N)$ by means of a recurrence relation involving all $P(N < j)$, and that new information is hidden in their specific distinct oscillatory behavior, which, in most cases, is not observed in the C_j obtained from the $P(N)$ commonly used to fit experimental results [4–7]. We discuss the possible sources of such behavior and the connection of C_j with the enhancement of void probabilities, and their impact on our understanding of the multiparticle production mechanism, with emphasis on understanding both phenomena within the class of compound distributions.

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2. Recurrence Relation and Modified Combinants

The dynamics of the multiparticle production process is hidden in the way, in which the consecutive measured multiplicities N are connected. There are two ways of characterizing the multiplicity distributions: by means of generating functions, $G(z) = \sum_{N=0}^{\infty} P(N)z^N$, or by some form of a recurrence relation between $P(N)$. In the first case, one uses the Poisson distribution as a reference and characterizes deviations from it by means of combinants C_N^* defined as [8]

$$C_j^* = \frac{1}{j!} \left. \frac{d^j \ln G(z)}{dz^j} \right|_{z=0}, \quad (1)$$

or by the expansion

$$\ln G(z) = \ln P(0) + \sum_{j=1}^{\infty} C_j^* z^j. \quad (2)$$

For the Poisson distribution, $C_1^* = \langle N \rangle$ and $C_{j>1}^* = 0$. The combinants were used in the analysis of experimental data in [9–14]. In [10, 13], it was demonstrated that they are particularly useful in identifying the nature of the emitting source. It turns out that, in the case of S sources emitting particles without any restrictions concerning their number, the multiplicity $P^S(N)$ is a completely symmetric function of degree N of the probabilities of emission, p_i , the generating

function of which reduces for $p_i \rightarrow 0$ to the generating function of the Poisson Distribution (PD). For all probabilities remaining the same, $p_i = p$, it reduces to the generating function of the Negative Binomial Distribution (NBD). In this case, the combinants are given by a power series

$$C_j^* = \frac{1}{j} \sum_{i=1}^S p_i^j \tag{3}$$

and are always positive. However, when each of the sources can emit only a given number of particles (let us assume, for definiteness, that at most only one particle), then $P^S(N)$ is an elementary symmetric function of degree N in the arguments, and the corresponding combinants are given by

$$C_j^* = (-1)^{j+1} \frac{1}{j} \sum_{i=1}^S \left(\frac{p_i}{1-p_i} \right)^j, \tag{4}$$

and alternate in sign for different j 's. For all probabilities remaining the same, $p_i = p$, a generating function in this scenario reduces to the generating function of the Binomial Distribution (BD) and the combinants oscillate rapidly with period equal to 2.

Note that, in both cases, we were working with probabilities p_i , which were not extracted from experiment, but their values were taken such that the measured multiplicity distributions are reproduced. They are then usually represented by one of the known theoretical formulae for multiplicity distributions, $P(N)$, which can be defined either by the generating functions mentioned above or by some recurrence relations connecting different $P(N)$. In the simplest (and most popular) case, one assumes that the multiplicity N is directly influenced only by its neighboring multiplicities, $(N \pm 1)$, i.e., we have

$$(N+1)P(N+1) = g(N)P(N), \quad g(N) = \alpha + \beta N. \tag{5}$$

This recurrence relation yields BD (when $\alpha = Kp/(1-p)$ and $\beta = -\alpha/K$), PD (when $\alpha = \lambda$ and $\beta = 0$), and NBD (when $\alpha = kp$ and $\beta = \alpha/k$, where p denotes the particle emission probability). Usually, the first choice of $P(N)$ in fitting the data is a single NBD [15] or two- [16, 17], three- [18], or multicomponent NBDs [19] (or some other forms of $P(N)$ [1, 15, 20]). However, such a procedure only improves the agreement at large N , whereas the ratio

$R = \text{data/fit}$ still deviates dramatically from unity at small N for all fits [4, 5]. This means that the measured $P(N)$ contains information which is not yet captured by the rather restrictive recurrence relation (5). Therefore, in [4], we proposed to use a more general form of the recurrence relation (used, e.g., in counting statistics when dealing with multiplication effects in point processes [21]):

$$(N+1)P(N+1) = \langle N \rangle \sum_{j=0}^N C_j P(N-j). \tag{6}$$

This relation connects multiplicities N by means of some coefficients C_j , which contain the memory of particle $N+1$ about all the $N-j$ previously produced particles. The most important feature of this recurrence relation is that C_j can be directly calculated from the experimentally measured $P(N)$ by reversing Eq. (6) [4-7]:

$$\langle N \rangle C_j = (j+1) \left[\frac{P(j+1)}{P(0)} \right] - \langle N \rangle \sum_{i=0}^{j-1} C_i \left[\frac{P(j-i)}{P(0)} \right]. \tag{7}$$

The modified combinants C_j defined by the recurrence relation (7) are closely related to the combinants C_j^* defined by Eq. (1), namely,

$$C_j = \frac{j+1}{\langle N \rangle} C_{j+1}^*. \tag{8}$$

Using Leibnitz's formula for the j^{th} derivative of the quotient of two functions $x = G'(z)/G(z)$,

$$x^{(j)} = \frac{1}{G} \left(G^{(j)} - j! \sum_{k=1}^j \frac{G^{(j+1-k)}}{(j+1-k)!} \frac{x^{(k-1)}}{(k-1)!} \right), \tag{9}$$

where $G'(z)/G(z) = [\ln G(z)]'$ and $G(z)^{(N)}/N!|_{z=0} = P(N)$, we immediately obtain the recurrence relation (7).

The modified combinants, C_j , share with the combinants C_j^* the apparent ability of identifying the nature of the emitting source mentioned above (with, respectively, Eq. (3) corresponding to the NBD case with no oscillations, and Eq. (4) corresponding to the rapidly oscillating case of BD). This also means that C_j can be calculated from the generating function $G(z)$ of $P(N)$,

$$\langle N \rangle C_j = \frac{1}{j!} \left. \frac{d^{j+1} \ln G(z)}{dz^{j+1}} \right|_{z=0}. \tag{10}$$

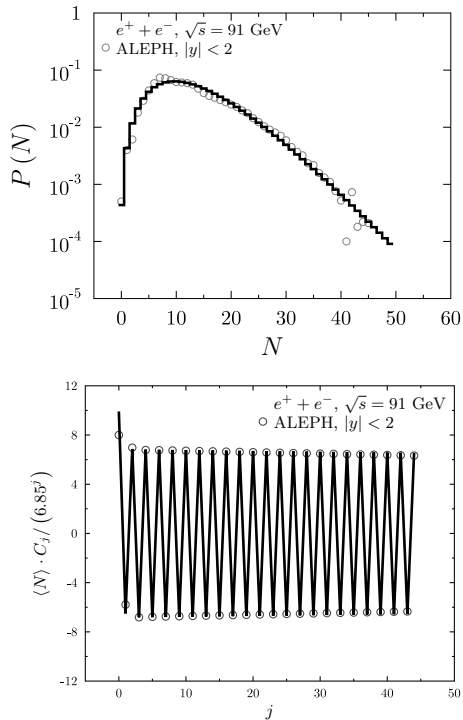


Fig. 1. Upper panel: Data on $P(N)$ measured in e^+e^- collisions by the ALEPH experiment at 91 GeV [23] are fitted by the distribution obtained from the generating function given by the product $G(z) = G_{\text{BD}}(z)G_{\text{NBD}}(z)$ with the parameters: $k' = 1$ and $p' = 0.8725$ for BD and $k = 4.2$ and $p = 0.75$ for NBD. Lower panel: the modified combinants C_j deduced from these data on $P(N)$. They can be fitted by C_j obtained from the same generating function with the same parameters, as used for fitting $P(N)$

Thus, whereas the recurrence relation, Eq. (7), allows us to obtain the C_j from the experimental data on $P(N)$, Eq. (10) allows for their calculation from the distribution defined by the generating function $G(z)$.

Note that C_j provide a similar measure of fluctuations as the set of cumulant factorial moments, K_q , which are very sensitive to the details of the multiplicity distribution and are frequently used in phenomenological analyses of data (cf., [1, 22]),

$$K_q = F_q - \sum_{i=1}^{q-1} \binom{q-1}{i-1} K_{q-i} F_i, \quad (11)$$

where $F_q = \langle N(N-1)(N-2) \dots (N-q+1) \rangle$ are the factorial moments, and K_q can be expressed as

an infinite series in C_j ,

$$K_q = \sum_{j=q}^{\infty} \frac{(j-1)!}{(j-q)!} \langle N \rangle C_{j-1}. \quad (12)$$

However, while the cumulants are best suited to study densely populated regions of the phase space, combinants are better suited for the study of sparsely populated regions, because, according to Eq. (7), the calculation of C_j requires only a finite number of probabilities $P(N < j)$ (which may be advantageous in applications).

The modified combinants share with the cumulants the property of additivity. For a random variable composed of independent random variables, with its generating function given by the product of their generating functions, $G(x) = \prod_j G_j(x)$, the corresponding modified combinants are given by the sum of the independent components. To illustrate this property, let us consider the e^+e^- data and use the generating function $G(z)$ formally treated as a generating function of the multiplicity distribution $P(N)$, in which N consists of both the particles from BD (N_{BD}) and from NBD (N_{NBD}):

$$N = N_{\text{BD}} + N_{\text{NBD}}. \quad (13)$$

In this case, the multiplicity distribution can be written as

$$P(N) = \sum_{i=0}^{\min\{N, k'\}} P_{\text{BD}}(i) P_{\text{NBD}}(N-i), \quad (14)$$

and the respective modified combinants as

$$\langle N \rangle C_j = \langle N_{\text{BD}} \rangle C_j^{(\text{BD})} + \langle N_{\text{NBD}} \rangle C_j^{(\text{NBD})}. \quad (15)$$

Figure 1 shows the results of attempts to fit both the experimentally measured [23] multiplicity distributions and the corresponding modified combinants C_j calculated from these data (cf. [24] for details). The fits shown in Fig. 1 correspond to the parameters: $k' = 1$ and $p' = 0.8725$ for BD and $k = 4.2$ and $p = 0.75$ for NBD.

Concerning the void probabilities at all energies of interest, one observes that $P(0) > P(1)$, a feature which cannot be reproduced by any composition of NBD used to fit the data [7]. To visualize the importance of this result, we note firstly that $P(0)$ is

strongly connected with the modified combinants C_j , in fact:

$$P(0) = \exp \left(- \sum_{j=0}^{\infty} \frac{\langle N \rangle}{j+1} C_j \right). \quad (16)$$

From Eq. (7), one can deduce that the $P(0) > P(1)$ property is possible only when $\langle N \rangle C_0 < 1$. For most multiplicity distributions, $P(2) > P(1)$, which results in an additional condition, $C_1 > C_0(2 - \langle N \rangle C_0)$; taken together, this means that $C_1 > C_0$. However, because of the normalization condition $\sum_{j=0}^{\infty} C_j = 1$, such an initial increase of C_j cannot continue for all ranks j , and we should observe some kind of nonmonotonic behavior of C_j with rank j in this case. This means that all multiplicity distributions, for which the modified combinants C_j decrease monotonically with rank j , do not exhibit the enhanced void probability.

3. Compound Distributions

To continue, we use the idea of *compound distributions* (CD), which are applicable, when (as in our case) the production process consists of a number M of some objects (clusters/fireballs/etc.) produced according to a distribution $f(M)$ (defined by a generating function $F(z)$), which subsequently decay independently into a number of secondaries, $n_{i=1, \dots, M}$, following some other (always the same for all M) distribution, $g(n)$ (defined by a generating function $G(z)$). The resultant multiplicity distribution,

$$h \left(N = \sum_{i=0}^M n_i \right) = f(M) \otimes g(n), \quad (17)$$

is a compound distribution of f and g with the generating function

$$H(z) = F[G(z)]. \quad (18)$$

Equation (18) means that, in the case where $f(M)$ is a Poisson distribution with the generating function

$$F(z) = \exp[\lambda(z - 1)], \quad (19)$$

the combinants for any other distribution $g(n)$ with a generating function $G(z)$, which are obtained from the compound distribution $h(N) = P_{PD} \otimes g(n)$ and calculated with the use of Eq. (10), do not oscillate and are equal to

$$C_j = \frac{\lambda(j+1)}{\langle N \rangle} g(j+1). \quad (20)$$

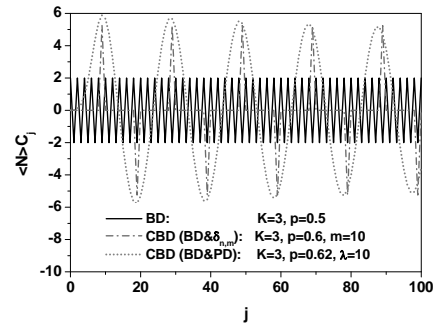


Fig. 2. C_j for BD, BD compounded with $\delta_{n,m}$ with $m = 10$ and compounded with the Poisson distribution with $\lambda = 10$

This fact explains why C_j from NBDs do not oscillate. This is because NBD is a compound distribution of the Poisson and logarithmic distributions. This means that $g(n) = -p^n / [n \ln(1 - p)]$, and $h(N)$ is NBD with $k = -\lambda / \ln(1 - p)$. In this case, C_j coincide with those derived before and given by Eq. (3). Actually, this reasoning applies to all more complicated compound distributions, with any distribution itself being a compound Poisson distribution. This property limits the set of distributions $P(N)$ leading to oscillating C_j , to BD, and to all compound distributions based on it. In this case, the period of oscillations is determined by the number of particles emitted from the source. For the compound distributions based on BD with $P(n) = \delta_{n,m}$, we have

$$C_j = (-1)^{j/m+1} \frac{K}{\langle N \rangle} \left(\frac{p}{1-p} \right)^{j/m+1}, \quad (21)$$

(for $j = mk$ and $C_j = 0$ for $j \neq mk$, where $k = 1, 2, 3, \dots$). For broader distributions $P(n)$, we get a smoother C_j dependence on rank j . For example, for $P(n)$ given by the Poisson distribution (with expected value λ), we obtain a Compound Binomial Distribution (CBD) with the generating function

$$H(z) = \{p \exp[\lambda(z - 1)] + 1 - p\}^K, \quad (22)$$

and the modified combinants are given by

$$C_j = \frac{(-1)^{j+1} K e^\lambda \lambda^{j+1} \frac{1-p}{p}}{\langle N \rangle \left(e^\lambda \frac{1-p}{p} + 1 \right)^{j+1}} A_j \left(e^\lambda \frac{p-1}{p} \right), \quad (23)$$

where $A_j(x)$ are the Eulerian polynomials. As an illustration, we show in Fig. 2 that, by compounding

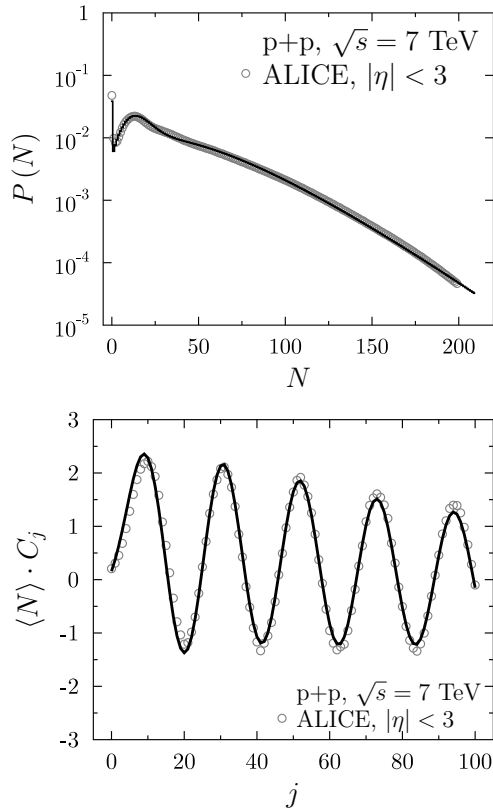


Fig. 3. Multiplicity distributions $P(N)$ measured in pp collisions by ALICE [25] (upper panel) and the corresponding modified combinants C_j (lower panel). Data are fitted using a two compound distribution (BD+NBD) given by Eqs. (25) and (24) with the parameters: $K_1 = K_2 = 3$, $p_1 = 0.9$, $p_2 = 0.645$, $k_1 = 2.8$, $k_2 = 1.34$, $m_1 = 5.75$, $m_2 = 23.5$, $w_1 = 0.24$ and $w_2 = 0.76$

BD with a Poisson distribution, one gains control over the period of oscillations (now equal to 2λ) and their amplitude. However, it turns out that such a combination does not allow us to fit data.

4. Multicomponent

The situation improves substantially, when one uses a multi-CBD based on Eq. (22). But the agreement is not yet satisfactory. It turns out that the situation improves dramatically, if one replaces the Poisson distribution by NBD and, additionally, uses a two-component version of such CBD with

$$P(N) = \sum_{i=1,2} w_i h(N; p_i, K_i, k_i, m_i) \quad (24)$$

with the generating function of each component equal to

$$H(z) = \left[p \left(\frac{1-p'}{1-p'z} \right)^k + 1-p \right]^K. \quad (25)$$

In such a case, as can be seen in Fig. 3, one gains a satisfactory control over the periods of oscillations, their amplitudes, and their behavior as a function of the rank j . Moreover, one can nicely fit $P(N)$ and C_j . Of special importance is the fact that the enhancement $P(0) > P(1)$ is also reproduced in this approach.

The above result also explains the apparent success in fitting the experimentally observed oscillations of C_j by using a weighted sum of the three NBD used in [26]. Such a distribution uses freely selected weights and parameters (p, k) of NBDs and, therefore, resembles the compound distribution of BD with NBD. However, we note that the sum of M variables (with $M = 0, 1, 2, \dots$), each from NBD characterized by parameters (p, k) , is described by NBD characterized by (p, Mk) . Therefore, as discussed before, it cannot reproduce the void probability $P(0)$. This can be reproduced only in the case where $M = 0, 1, \dots, K$ is distributed according to BD, and we have a K -component NBD (where the consecutive NBDs have precisely defined parameters k),

$$P(N) = \sum_{M=0}^K P_{\text{BD}}(M) P_{\text{NBD}}(N; p, Mk). \quad (26)$$

In this case, one also has the $M = 0$ component, which is lacking in the previous multi-NBD case used in [26]. This is the reason for that, whereas the compound (BD&NBD) distribution reproduces the void probability, $P(0)$, the single NBD (or any combination of NBDs) do not. This means that the observation of the peculiar behavior of the void probability discussed above signals the necessity of using some compound distribution based on BD to fit data for $P(N)$ (and the C_j obtained from it).

5. Summary and Conclusions

Since the time of Ref. [8], one encounters essentially no detailed experimental studies of the combinants and only rather sporadic attempts at their phenomenological use to describe the multiparticle production processes. We demonstrate that the modified combinants C_j are a valuable tool for the in-

vestigations of multiplicity distributions, and C_j deduced from the measured multiplicity distributions, $P(N)$, could provide additional information on the dynamics of the particle production. This, in turn, could allow us to reduce the number of possible interpretations presented so far and, perhaps, answer some of the many still open fundamental questions (that this is possible, despite experimental errors, has been shown in [7, 26]). Finally, let us note that a large number of papers suggest some kind of universality in the mechanisms of hadron production in e^+e^- annihilations and in pp and $p\bar{p}$ collisions. This arises from observations of the average multiplicities and relative dispersions in both types of processes (cf., e.g., [27, 28]). However, as we have shown here, the modified combinant analysis reveals differences between these processes. Namely, while, in e^+e^- annihilations, we observe oscillations of C_j with period 2, the period of oscillations in pp collisions is ~ 10 times longer, and the amplitude of oscillations in both types of processes differs dramatically. At the moment, this problem remains open and awaits a further investigation.

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ПОГЛЯД НА МНОЖИННІ РОЗПОДІЛИ ЧЕРЕЗ МОДИФІКОВАНІ КОМБІНАНТИ

Резюме

Експериментально виміряні розподіли по множинності після їх ретельного аналізу демонструють незвично підвищену ймовірність порожнечі і осциляторну поведінку модифікованих комбінантів. Ми показуємо, що обидві ці риси можна використати як додаткові джерела інформації, ще не використані в повній мірі в механізмах багаточастинкового народження. Ми надаємо їх теоретичну інтерпретацію в термінах компаундних розподілів.