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QUANTUM-MECHANICAL MODEL OF AN ELECTRON WITH SELF-CONSISTENT ELECTROSTATIC FIELD

A possibility to construct a theory for an electron on the basis of the Dirac equation, where the electromagnetic field potentials are defined as those created by the electron itself, has been analyzed. It is shown that the energy conservation law is obeyed for the combined electromagnetic + bispinor field of an isolated electron. A stationary quasilinear system of equations for the electrostatic + bispinor field is formulated in terms of the quaternion algebra. The quasilinear problem for the electrostatic model of an electron is analyzed. The absence of singularities in the bispinor field components and the density of the electric charge distributed within electron's central region whose radius is about the Compton length is demonstrated.

Keywords: Dirac equation, Klein–Gordon equation, charge conservation law, electromagnetic field, bispinor, quaternion.

1. Introduction

The electron is the main object to study in quantum electrodynamics. The latter owes its main success to the relativistic Dirac equation [1]. Dirac's theory turned out sufficient to describe all electron-spin effects in given electromagnetic fields. It provided a basis for the formation of a lepton group in the theory of elementary particles. But, in the case of heavier particles, baryons, the Dirac equation failed. Dirac himself explained this fact by the absence of an internal structure in an electron and considered his theory to be adapted exclusively to describe interparticle interactions, where the electron participates. His viewpoint was adopted in modern quantum electrodynamics [1, 2]. Perhaps, this taboo may be the origin of the lack of publications in the modern literature concerning theories, where the electron would be deprived of a point-like, i.e. structureless, character.

All attempts to relate electron's parameters to peculiarities in the spatial distribution of the electron charge within a confined region remained in the past. Also unrecognized remained the nonlinear modification of Maxwell's equations by Born, which confines the electron field magnitude and the energy of this field, but retains the point character of an electron in the field theory and relativistic mechanics [3]. When commenting on this and other examples of substantial efforts aimed at developing an electron model with non-zero dimensions of the electron charge localization region that would not create problems for the field theory, R. Feynman emotionally appraised this situation as dramatic in the whole [4]. It is this viewpoint on the described problem of the electron theory that allows the studies of the field non-singular structures localized within the classical boundaries of electron's central region to be considered as challenging.

The main idea of the presented work consists in the identification of the continuity equation, which follows from the Dirac equation, and the charge conservation law in the equations describing the electromagnetic field generated by the electron. In other words, the real-valued quantity $\bar{\psi}\psi$ normalized to unity, where ψ is the bispinor field in Dirac's theory, is interpreted as the charge density ρ normalized to the electron charge, i.e. $\rho = e\bar{\psi}\psi$. This assumption allows the bispinor and electromagnetic fields of the electron to be combined into a closed system, which is characterized by its own law of energy conservation. Such an approach makes it possible to formulate a quasilinear Dirac equation with a self-consistent field, where the electromagnetic potentials are not considered as external field potentials, as is usually postulated, but as created by the spatially distributed charge $e\bar{\psi}\psi$ of the electron and its current.

Those terms were used to formulate an electrostatic model for a free electron isolated from external influences and possessing a centrosymmetric bispinor+electric field. The analysis of the structure of its components was performed, which proved the finiteness of the quantity ρ and its localization in the central region about the Compton length in size, $r_e\hbar/(mc) = 3.86 \times 10^{-11}$ cm. At larger distances, the field generated by the electron corresponds to the classical Coulomb law. The self-consistent electromagnetic+bispinor field of the electron is considered in a one-particle phase space, where the actual electromagnetic field is described in the Schrödinger coordinate representation, and it is quasiclassical from the viewpoint of the quantum electrodynamics of fermionic and photonic fields.

The consistency of the approach is illustrated in Section 2, where the energy conservation law is proved for the combined system of electron's electromagnetic and bispinor fields. In Section 3, the corresponding stationary problem is formulated in the form of a system of second-order equations for spinors, which arises as a result of the splitting of the bispinor problem using the linear transformation method. This method makes it possible to represent spinors as quaternions with real parameters and use the same algebra of hypercomplex numbers that is used in the theory of Dirac matrices. Section 4 is devoted to the ultimate formulation of the centrosymmetric problem that is invariant with respect to the coordinate inversion. The electrostatic field plays a decisive role in this formulation, whereas the magnetic field effects are neglected. The system has the form of second-order differential equations for the scalar and radial components of quaternions, as well as the Poisson equation for the electrostatic potential. Those equations are somewhat similar to the equations known from the theory of a hydrogen atom. The asymptotic analysis shows the regular character of electron's fields at the center of the coordinate system, where the charge density saturates, and the electric field disappears. In such a way, the radius of the electron charge localization region was estimated to be of the order of the Compton length r_e.

2. Energy Conservation Law for the Electromagnetic and Quantum Fields of an Electron

The proof of the energy conservation law for the electromagnetic and quantum fields of an isolated electron is a useful introduction to the self-consistent model of an electron. The conservation law follows from the Dirac equation for the bispinors ψ,

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi, \tag{1a}$$

where

$$\begin{aligned} &H = e\varphi + \boldsymbol{\alpha}\cdot\mathbf{P} - \varepsilon_0\beta,\\ &\varepsilon_0 = mc^2,\\ &\mathbf{P} = -ic\hbar\nabla - e\mathbf{A}, \end{aligned} \tag{1b}$$

and the following second-order equation, which is a direct consequence of this equation and generalizes the Klein–Gordon scalar equation [1, 5]:

$$K\psi = i\hbar e\boldsymbol{\alpha}\cdot(\mathbf{E} + i\,\iota\,\mathbf{H})\,\psi, \tag{2a}$$

where

$$K = \left(i\hbar\frac{\partial}{\partial t} - e\varphi\right)^2 - \mathbf{P}^2 - \varepsilon_0^2 \tag{2b}$$

is the Klein–Gordon scalar operator. In Eqs. (1) and (2), a somewhat modified notation for the four-row matrices in the Dirac theory is used. As a rule, they are written in the form of two-row matrices, the elements of which are two-row matrices **0**, **I,** and the Pauli matrices $\boldsymbol{\sigma}$'s , and they determine the multiplication algebra for the linear forms $\boldsymbol{\sigma}\cdot\mathbf{a} = \sigma_x a_x + \sigma_y a_y + \sigma_z a_z$ with arbitrary vectors **a** and **b**:

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad \beta = \begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\quad \iota = \begin{pmatrix} 0 & I\\ I & 0\end{pmatrix},$$

$$(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{b},$$

$(\boldsymbol{\alpha}\cdot\mathbf{a})(\boldsymbol{\alpha}\cdot\mathbf{b}) = \mathbf{a}\cdot\,\mathbf{b} + \grave{\imath}\boldsymbol{\alpha}\cdot\,\mathbf{a}\times\mathbf{b}$.

The electromagnetic field potentials in the Dirac equation are determined by external electric charges and current in the framework of the electrodynamic theory:

$$\Box\varphi = -4\pi\rho, \quad \Box\mathbf{A} = -4\pi\mathbf{j}/c,$$
$$\mathbf{E} = -\nabla\varphi - \frac{\partial}{c\partial t}\mathbf{A}, \quad \mathbf{H} = \operatorname{rot}\mathbf{A}, \tag{3}$$

where $\Box$ is the d'Alembert operator. The ψ-bispinor values can affect the quantities φ, $\mathbf{A}$, $\mathbf{E}$, and $\mathbf{H}$ only in the framework of an additional description of the external field sources ρ and $\mathbf{j}$. The system of equations (3) is supplemented by the conservation law for the charge distributed in space,

$$\frac{\partial}{\partial t}\rho + \operatorname{div}\mathbf{j} = 0. \tag{4}$$

The Dirac theory also generates the continuity equation (4), if the notations ρ and $\mathbf{j}$ are given a quantum-mechanical meaning,

$$\rho = e\bar{\psi}\psi, \quad \mathbf{j} = ce\bar{\psi}\boldsymbol{\alpha}\psi, \tag{5}$$

where the bar over the bispinors means their Dirac conjugation. But the quantum-mechanical theory deals with a single particle, whereas the quantity $\bar{\psi}\psi$ in Eq. (4) is interpreted as the probability density for an electron to exist in the physical space, and its integrated value is equal to unity, $\int d\mathbf{r}\bar{\psi}\psi = 1$. The dimensionality of the quantity $e\bar{\psi}\psi$ is the charge density, and Eq. (4) is a quantum-mechanical extension of its conservation law.

Another conservation law follows from Eq. (2),

$$\frac{\partial}{\partial t}\left[\frac{i\hbar}{2}\left(\bar{\psi}\frac{\partial\psi}{\partial t} - \frac{\partial\bar{\psi}}{\partial t}\psi\right) + e\varphi\bar{\psi}\psi\right] -$$
$$- c\nabla\cdot\left[\frac{i\hbar}{2}c(\bar{\psi}\nabla\psi - \nabla\bar{\psi}\psi) + e\mathbf{A}\bar{\psi}\psi\right] =$$
$$= ce\mathbf{E}\cdot\,\bar{\psi}\boldsymbol{\alpha}\psi. \tag{6}$$

A similar equation is discussed to illustrate that the scalar Klein–Gordon equation cannot serve as a basis for the relativistic quantum theory, because the quantity whose the time derivative is taken in Eq. (6) cannot be considered as the probability density [6]. However, Eq. (6) can be given a different sense. It can be naturally interpreted as an energy balance equation, which becomes obvious, if the dimensionalities of its components are examined. An important result of the bispinor analysis is the presence of the right-hand side in Eq. (6), which is determined by the contribution of the electric field. If the time-differentiated quantity is interpreted as the energy density of the electron's quantum field, then the right-hand side corresponds to the source of the energy transferred to the electron current by the electric field per unit volume per unit time, $ce\,\mathbf{E}\cdot\,\bar{\psi}\boldsymbol{\alpha}\psi = \mathbf{E}\cdot\,\mathbf{j}$.

The same source is present in the conservation law for the electromagnetic energy, which follows from Maxwell's equations, where it is included with the opposite sign,

$$\frac{\partial}{\partial t}\left[\frac{1}{8\pi}(H^2 + E^2)\right] + \operatorname{div}\frac{c}{4\pi}\mathbf{E}\times\mathbf{H} = -\mathbf{E}\cdot\,\mathbf{j}. \tag{7}$$

For an isolated electron, the energy source of its own electromagnetic field is the electron charge, so it is natural to assume that the source of this field in Eq. (7) is the electric current determined in the framework of the quantum-mechanical theory, $\mathbf{j} = ce\bar{\psi}\boldsymbol{\alpha}\psi$. The simple summation of Eqs. (6) and (7) brings about the conservation law for the total energy of the quantum+electromagnetic field of the electron,

$$\frac{\partial}{\partial t}\left\{\frac{1}{8\pi}(H^2 + E^2) - \frac{\hbar}{2i}\left(\bar{\psi}\frac{\partial\psi}{\partial t} - \bar{\psi}\frac{\partial\psi}{\partial t}\right) - e\varphi\psi\bar{\psi}\right\} +$$
$$+ \operatorname{div}\left\{\frac{c}{4\pi}\mathbf{E}\times\mathbf{H} + \frac{\hbar c^2}{2i}(\bar{\psi}\nabla\psi - \nabla\bar{\psi}\psi) - ce\mathbf{A}\psi\bar{\psi}\right\} = 0. \tag{8}$$

Being written in this form, the obtained result corresponds to the Dirac equation, if the electromagnetic potentials are expressed directly through the bilinear forms of bispinors, $\Box\varphi = -4\pi e\bar{\psi}\psi$ and $\Box\mathbf{A} = -4\pi e\bar{\psi}\boldsymbol{\alpha}\psi$, provided the normalization condition $\int d\mathbf{r}\bar{\psi}\psi = 1$. Hence, the Dirac equation transforms into a quasilinear differential equation with the self-consistent field of a single electron.

It is natural to consider the problem aimed at describing the electron field in the stationary quantum-mechanical approximation, i.e. taking the wave function in the form $\psi(t,\mathbf{r}) \sim \exp\{-i\varepsilon t/\hbar\}\psi(\mathbf{r})$, in an immovable coordinate system conditionally 'pinched' to the electron's center. In this case, the energy density in Eq. (8) is given by the expression $\frac{1}{8\pi}(H^2 + E^2) - \rho\varphi + \varepsilon\bar{\psi}\psi$, which illustrates the heuristic aspect of the proposed approach to the possibility of interpreting the positive-valued quantity $\bar{\psi}\psi$ in the quantum-relativistic model of the electron.

3. Quasilinear System of Equations with the Self-Consistent Field

The second-order equation (2) doubles (from 4 to 8) the number of complex parameters to be determined. Therefore, the solution of this equation must be somehow conditioned. This problem is solved by changing to the spinors $\psi_\pm$, which are determined by the spinor components ψ_1 and ψ_2 of the bispinor ψ: $\psi_\pm = \psi_1 \pm \psi_2$. Furthermore, one can see that the consideration can be reduced to the analysis of the second-order spinor equation for either of them. The procedure that introduces new spinor components and splits Eq. (2) can be most easily implemented with the help of the unitary real transformation

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{9}$$

so that $U^2 = 1$ and

$$U\psi = U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \tag{10}$$

The operator U diagonalizes symmetric matrices. In particular,

$$U\boldsymbol{\alpha}U = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}. \tag{11}$$

Multiplying Eq. (2) on the left side by U, we obtain two second-order spinor equations

$$[K - ice\hbar\boldsymbol{\sigma} \cdot (\pm\mathbf{E} + i\mathbf{H})]\psi_\pm = 0. \tag{12}$$

In terms of $\psi_\pm$, formulas (5) look like

$$\rho = e\bar{\psi}\psi = e\bar{\psi}U^2\psi = \frac{e}{2}(\bar{\psi}_+\psi_+ + \bar{\psi}_-\psi_-), \tag{13a}$$

$$\mathbf{j} = ce\bar{\psi}\boldsymbol{\alpha}\psi = ce\bar{\psi}U^2\boldsymbol{\alpha}U^2\psi =$$
$$= \frac{ce}{2}(\bar{\psi}_+\boldsymbol{\sigma}\psi_+ - \bar{\psi}_-\boldsymbol{\sigma}\psi_-). \tag{13b}$$

The operators on the left-hand side of Eqs. (12) only differ in the sign in front of the electric field and transform into each other at the coordinate inversion operation, because the operator K contains the squared momentum $\mathbf{P}$, the latter being a polar vector in the initial equations. Hence, the difference between the operators in Eqs. (12) can be associated with the inversion of the coordinate system. The inversion of operators can be provided by multiplying Eqs. (12) by β and making use of the commutators $\beta\boldsymbol{\alpha} = -\boldsymbol{\alpha}\beta$ and $\beta\iota\boldsymbol{\alpha} = \iota\boldsymbol{\alpha}\beta$ to obtain the equation for $\beta\psi$. The factor β is known [2, 6] to ensure the invariance of the Dirac equation with respect to the coordinate inversion. At the same time, the application of the splitting procedure (10) and (11) to the product $\gamma_0\psi$ makes the permutation $\psi_+ \leftrightarrow \psi_-$. Therefore, in order to construct an inversion-invariant model, it is enough to solve, in any way, only one of the equations and afterward to perform the inversion transformation of the result to obtain the solution for the other equation.

The analysis made above dealt with the formulas written in a too general form. In particular, in the stationary approximation, $\mathbf{E} = -\nabla\varphi$ and $\operatorname{div}\mathbf{j} = 0$, which allows the solution $\mathbf{j} \neq 0$. Although problems (12) are equivalent in this sense, the inequality $\psi_+ \neq \psi_-$ always holds true. This fact means a necessity to include the effects of the own magnetic field of the electron into its self-consistent model. Such a possibility cannot be excluded in principle. Nevertheless, let us consider below a simplified electrostatic model in which the magnetic field does not play a decisive role. The main purpose of this work was to demonstrate a possibility to describe the model of the self-consistent electric+quantum field of the electron within its central region.

4. Analysis of the Electrostatic Model

The simplest representation of the electrostatic model is achieved by using quaternions instead of spinors in Eqs. (12) provided the absence of the magnetic field. It can be done by making the substitution

$$\psi_\pm \prec W_\pm\sigma_0, \tag{14a}$$

where

$$W_\pm = (w_{\pm 0} + i\boldsymbol{\sigma} \cdot \mathbf{w}_\pm) \tag{14b}$$

and

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{14c}$$

This operation complements the spinors to two-row matrices by using zero spinors, and the equality symbol is substituted by the symbol $\prec$. The quaternions

$W_\pm$, which are hypercomplex numbers, are determined by four real parameters $(w_{\pm 0}, \mathbf{w}_\pm)$. The two-row matrix σ_0 provides a one-to-one correspondence of the quaternion parameters to two complex numbers, the spinor components [7]:

$$\psi_\pm = \begin{pmatrix} a_\pm + ib_\pm \\ c_\pm + id_\pm \end{pmatrix} \prec (w_{0\pm} + i\boldsymbol{\sigma} \cdot \mathbf{w}_\pm)\sigma_0 = \\ = \begin{pmatrix} w_{0\pm} + iw_{z\pm} & 0 \\ -w_{y\pm} + iw_{x\pm} & 0 \end{pmatrix}. \qquad (15)$$

The conjugate spinor is defined by a similar formula,

$$\bar{\psi}_\pm \prec \sigma_0 \bar{W}_\pm,$$

where

$$\bar{W}_\pm = (w_{\pm 0} - i\boldsymbol{\sigma} \cdot \mathbf{w}_\pm).$$

In other words, the Dirac conjugation is ensured by the classical conjugation of the quaternion.

The correctness of the application of quaternions instead of spinors, as well as biquaternions instead of bispinors, can be demonstrated by the example of the calculation of the Dirac numbers, bilinear forms of the ρ- and $\mathbf{j}$-types. For instance, for the current j_z [see Eq. (13b)], we have to use multiplication rules (3). In so doing, the simplification of linear forms (they arise as a result of the quaternion multiplication) under the action of projective factors σ_0 should be taken into account: $\sigma_0\sigma_x\sigma_0 = \sigma_0\sigma_y\sigma_0 = 0$. Hence, $\sigma_0\boldsymbol{\sigma}\sigma_0 = \sigma_0\sigma_z\sigma_0\mathbf{e}_z$ and

$$j_z = \frac{1}{2}ce\sigma_0(\bar{W}_+\sigma_z W_+ - \bar{W}_-\sigma_z W_-)\sigma_0 =$$

$$= \frac{1}{2}ce\sigma_0[(w_{+0} - i\boldsymbol{\sigma} \cdot \mathbf{w}_+)\sigma_z(w_{+0} + i\boldsymbol{\sigma} \cdot \mathbf{w}_+) -$$

$$-(w_{-0} - i\boldsymbol{\sigma} \cdot \mathbf{w}_-)\sigma_z(w_{-0} + i\boldsymbol{\sigma} \cdot \mathbf{w}_-)]\sigma_0 =$$

$$= \frac{1}{2}ce\sigma_0\sigma_z\sigma_0\{[(w_{+0}^2 + \mathbf{w}_+^2) +$$

$$+ 2\mathbf{e}_z \cdot \mathbf{w}_+ \times (w_{+y}\mathbf{e}_x - w_{+x}\mathbf{e}_y)] - [(w_{-0}^2 + \mathbf{w}_-^2) +$$

$$+ 2\mathbf{e}_z \cdot \mathbf{w}_- \times (w_{-y}\mathbf{e}_x - w_{-x}\mathbf{e}_y)]\}.$$

Finally, since $\sigma_0\sigma_z\sigma_0 = \sigma_0$, it is easy to obtain the following ultimate expression for j_z, which is identical to the spinor definition of this quantity [Eq. (13)], if correspondence (15) of the real parameters $(w_{\pm 0}, \mathbf{w}_\pm)$ to the parameters (a, b, c, d) is taken into account:

$$j_z = \frac{ce}{2}\sigma_0\{[w_{+0}^2 + w_{+z}^2 - w_{+x}^2 - w_{+y}^2] -$$
$$-[w_{-0}^2 + w_{-z}^2 - w_{-x}^2 - w_{-y}^2]\} =$$
$$= \sigma_0\left\{\frac{ce}{2}(\bar{\psi}_+\sigma_z\psi_+ - \bar{\psi}_-\sigma_z\psi_-)\right\}.$$

The Dirac numbers occupy the upper left corner of the matrix, whereas the other matrix elements equal zero. The main advantage of applying the quaternions is the ability to arrange eight spinor parameters in the form of two vector sets and two scalars, similarly to the vector formulation of Maxwell's equations (it is well known [7] that the latter can also be conveniently formulated in terms of quaternions).

However, the application of quaternions to analyze problem (2) is restricted, because the operators on the left-hand side of Eq. (2) or (12) cannot be defined in the field of quaternions with real parameters even in the framework of the stationary problem, if the effects of magnetic potential are made allowance for. Identical modifications in the Dirac equation are possible to remove obstacles to the quaternion formalism [7]. However, the application of quaternions has no obstacles in the framework of the simplified electrostatic approximation and allows the algebra of hypercomplex numbers (3) to be used when formulating a system of equations for the real-valued scalar and vector components of all fields. Equations (12) are convenient to be written in the dimensionless form by expressing the quantities $\mathbf{r}$, φ, ψ, and ε in the $\hbar/(mc)$-, $e\,[\hbar/(mc)]^{-1}$-, $[\hbar/(mc)]^{-3/2}$-, and mc^2-units, respectively:

$$[K \pm i\delta_e\boldsymbol{\sigma} \cdot \nabla\varphi](w_{\pm 0} + i\boldsymbol{\sigma} \cdot \mathbf{w}_\pm) = 0,$$

where

$$K = \Delta + (\varepsilon - \delta_e\varphi)^2 - 1,$$

or, for real-valued quantities,

$$Kw_{\pm 0} \mp \delta_e\nabla\varphi \cdot \mathbf{w}_\pm = 0, \qquad (16a)$$
$$K\mathbf{w}_\pm \pm \delta_e\nabla\varphi w_{\pm 0} \mp \delta_e\nabla\varphi \times \mathbf{w}_\pm = 0, \qquad (16b)$$

where $\delta_e = e^2/(c\hbar) = 1/137$. The system of equations also includes the Poisson equation

$$\Delta\varphi = 2\pi(w_{+0}^2 + w_{-0}^2 + \mathbf{w}_+^2 + \mathbf{w}_-^2) \qquad (16c)$$

and the normalization condition

$$\frac{1}{2}\int d\mathbf{r}(w_{+0}^2 + w_{-0}^2 + \mathbf{w}_+^2 + \mathbf{w}_-^2) = 1, \quad (16d)$$

the both being expressed in terms of new variables. In the formulated quasilinear problem, the parameter is determined from the standard requirement that a complete solution of system (16) must exist.

The system of equations (16) will be satisfied, if we assume that the vector field $\mathbf{w}_\pm$ and the gradient $\nabla\varphi$ are collinear. In the centrosymmetric approximation, the choice of the set $\varphi(\mathbf{r}) = \varphi(r)$, $\mathbf{w}_\pm(\mathbf{r}) = \mathbf{e}_r w_{\pm r}(r)$, and $\mathbf{e}_r = \mathbf{r}/r = \nabla r$ written in the spherical coordinates makes it possible to formulate a system of equations that is invariant with respect to the coordinate inversion by applying the simple substitution ($w_{0+} = w_{0-}, \mathbf{w}_+ = -\mathbf{w}$). Taking into account that $\Delta\mathbf{e}_r w_{\pm r}(r) = \mathbf{e}_r(\Delta_r - 2r^{-2})w_{\pm r}$, this choice brings about a system of scalar equations for only two quantities: $w_r = w_{+r} = -w_{-r}$ and $w_0 = w_{+0} = w_{-0}$. Then, in accordance with definitions (14) and (16), the formula for the charge density reads $\rho = w_0^2 + w_r^2$. Finally, it is convenient to use the substitutions $w_r = \chi/r$, $w_0 = \eta/r$, $\varphi = q/r$, and $\tilde\rho = \rho/r^2$, and to get rid of the radial Laplacians,

$$\Delta_r w_r = r^{-2}(r^2 w_r')' = \chi''/r,$$
$$\Delta_r w_0 = \eta''/r,$$
$$\Delta_r\varphi = q''/r,$$

so that Eqs. (16) acquire the final form

$$\eta'' + [(\varepsilon - \delta_e\varphi)^2 - 1]\eta - \delta_e\varphi'\chi = 0, \quad (17)$$

$$\chi'' + \left[(\varepsilon - \delta_e\varphi)^2 - 1 - \frac{2}{r^2}\right]\chi + \delta_e\varphi'\eta = 0, \quad (18)$$

$$q'' = 4\pi\frac{\tilde\rho}{r}, \quad (19)$$

$$4\pi\int_0^\infty dr\tilde\rho = 1, \quad (20)$$

where $\tilde\rho = \chi^2 + \eta^2$.

The limiting value of the potential φ at $r \to \infty$ can be chosen to equal zero, which corresponds to the classical definition: an arbitrary choice of the potential is guaranteed by the gradient invariance of the Dirac equation [2]. The integration of the Poisson equation (19) within an arbitrary interval $(0, R)$ as $R \to \infty$ with regard for the normalization condition (20) for the charge density $\tilde\rho$ results in the classical expression for the asymptotics of the electric field and its potential,

$$\int_0^R r dr q'' = Rq'(R) - q(R) = R^2\varphi'(R) \underset{R\to\infty}{\longrightarrow} -1,$$

i.e.

$$\varphi(r\to\infty) = \frac{1}{r}, \quad q(r\to\infty) = 1$$

provided that the function $q(r)$ is differentiable at $r = 0$ and $q(r = 0) = 0$. The Poisson equation written in terms of the quantity q demonstrates indirectly what the potential and charge density distributions should be in the regular electron model. The function $q(r) = r\varphi(r)$ is convex and grows monotonically from zero to unity in the interval $(0, \infty)$. Accordingly, the derivative q' decreases from the maximum positive value $q'(0) = \varphi(0) = \varphi_0$ to zero. The reduced charge density $\tilde\rho$ equals zero not only at $r \to \infty$ but also at electron's center. The regular amplitudes η and χ are also characterized by the behavior $(\eta, \chi) \to 0$ at $r \to 0$. Therefore, $\tilde\rho$ is a small parameter of an order of at least r^2 at $r \to 0$ and symmetric at the formal substitution $r \to -r$. At the same time, according to the Poisson equation, there must be $q/r \approx \varphi_0 + ar^2$, where $a < 0$. Those general conclusions drawn from the requirements of regularity for the amplitudes have to be confirmed by the analysis of Eqs. (17) and (18).

The coefficients depending on φ, as well as the coefficient $2/r^2$, in front of the operators are small in comparison with unity and vanish at $r \to \infty$. The contribution of the others can be compensated only by the differential operator. The behavior of the amplitudes η and χ is governed by identical equations, e.g., $\eta'' - (1 - \varepsilon^2)\eta = 0$ for η. An acceptable solution is the exponential damping law in space

$$(\eta, \chi) \sim e^{-\kappa_\varepsilon r}, \quad (21)$$

where $\kappa_\varepsilon = \sqrt{1 - \varepsilon^2}$. This formula is a modification of the known Yukawa result $\psi \sim e^{-r}/r$ obtained for the scalar solution of the Klein–Gordon equation and expressed on the Compton-length scale [4,6]. The performed analysis restricts the conclusion about the exponential localization of the amplitudes of the electron's quantum field by the critical requirement $\varepsilon^2 < 1$.

Equations (17) and (18) are similar to the equations for the radial functions in the hydrogen atom theory. Therefore, a similar analysis can be applied at the center ($r \to 0$) in our case. The behavior of the amplitudes of the radial functions is determined by the condition that the singular coefficient $-2/r^2$ should be annihilated with the help of the differential operator, i.e. $\chi'' - 2\chi/r^2 = 0$, whence the positive solution $s = 2$ is obtained for the asymptotics $\chi = r^s$. The operator in the equation for the scalar amplitude η does not contain singular components at $r = 0$. However, taking the small contributions of the electric field and the radial function χ into account, the asymptotic equation reads

$$\eta'' + [(\varepsilon - \delta_e\varphi_0)^2 - 1]\eta = 0.$$

Its solution depends on the sign of the difference $(\varepsilon - - \delta_e\varphi_0)^2 - 1$, i.e. on the parameter ratio $\varepsilon/\left(\delta_e\varphi_0\right)$. The solution looks like

$$\eta \sim \sin\left(r\sqrt{(\varepsilon - \delta_e\varphi_0)^2 - 1}\right)$$

if $(\varepsilon - \delta_e\varphi_0)^2 > 1$. Otherwise, i.e. if $1 > (\varepsilon - \delta_e\varphi_0)^2$, the asymptotics satisfying the requirement $\eta(0) = 0$ has the form

$$\eta \sim \sinh\left(r\sqrt{1 - (\varepsilon - \delta_e\varphi_0)^2}\right).$$

Therefore, the above-presented attributes of the regular character of the analytical model for the field distribution in the electrostatic model of electron become confirmed. The main requirement concerning the second order of smallness, $\tilde{\rho} \sim r^2$, is satisfied by means of the corresponding behavior of the scalar amplitude, $\eta \sim r$.

5. Conclusions

The considered centrosymmetric model of the electron corresponds to the physical scenario in which Coulomb's law is realized beyond the central region of the electron, whereas the electron charge is smoothly distributed within this region. The estimated value of the size of this region r_e, which was obtained in the Compton-length units, differs from the classical value of the electron radius $r_c = e^2/\left(mc^2\right)$. The size of the electron charge localization region turns out in between the electron radius r_c and the Bohr orbit radius $r_\mathrm{B} = \hbar^2/\left(me^2\right)$. The corresponding proportion is

$$r_c : r_e : r_\mathrm{B} = \delta_e : 1 : \delta_e^{-1},$$

where $\delta_e = 1/137$.

The ultimate answer concerning the adequacy of the proposed model can be obtained by numerically solving the system of equations (17)–(20). In particular, it would be very interesting to calculate the quantity ε, which determines the bispinor contribution to the total electron energy. Note that, according to formula (8), the total energy density of a stationary combined field is determined by the sum

$$\frac{1}{8\pi}(H^2 + E^2) - \rho\varphi + \varepsilon\bar{\psi}\psi.$$

Hence, the value of the parameter ε is crucially important for the proposed model. The determination of this quantity cannot be considered as an example describing the application of the standard quantum-mechanical spectral theory. The Hamiltonian in the Dirac equation with the self-consistent field is not a linear operator and, therefore, generates a confined spectrum.

It is important to estimate the integrated energy value

$$\varepsilon + \int d\mathbf{r}\left(\frac{E^2}{8\pi} - \rho\varphi\right)$$

for the electrostatic model and compare it with unity, which corresponds to the electron rest energy. The problem of calculating the spectrum for a certain quantity in the stationary quasilinear Dirac equation has common roots with the problem of finding the existence conditions for solitons, which are considered in the plasma theory and nonlinear optics. The issue of the confined spectrum of this quantity can be raised, if we take into account that the existence of a muon is theoretically supposed as an excited state of an electron.

The approach used in this work can be verified in the course of a broader study of electron's field structure making allowance for internal currents and magnetostatic effects, because the proposed theory is not exhaustive without them. The determination of the magnetic moment of an electron requires a more complicated analysis of the Dirac equation modified in terms of biquaternions.

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КВАНТОВА МОДЕЛЬ ЕЛЕКТРОНА З САМОУЗГОДЖЕНИМ ЕЛЕКТРОСТАТИЧНИМ ПОЛЕМ

Р е з ю м е

Розглянуто можливість побудови теорії електрона на основі рівняння Дірака, у якому потенціали електромагнітного поля визначаються як такі, що створюються самим електроном. Показано, що для сукупного електромагнітного і біспінорного поля ізольованого електрона виконується закон збереження енергії. Сформульовано стаціонарну квазілінійну систему рівнянь для сукупного електростатичного і біспінорного поля в термінах алгебри кватерніонів. Виконано аналіз квазілінійної задачі для електростатичної моделі електрона; показано відсутність сингулярності компонент біспінорного поля та густини електричного заряду, розподіленого в межах центральної області радіусом порядку комптонівської довжини.