
<https://doi.org/10.15407/ujpe65.6.500>

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IDEAL BOSE GAS IN SOME DEFORMED TYPES OF THERMODYNAMICS. CORRESPONDENCE BETWEEN DEFORMATION PARAMETERS

Two approaches to the construction of thermodynamics in the framework of the q - and μ -formalisms, which correspond to certain deformations of the algebra of the creation–annihilation operators, have been considered. By comparing the obtained results, an approximate, independent of the space dimension, correspondence was revealed between the second virial coefficients for the ideal q - and μ -deformed Bose gases. The corresponding discrepancy arises only at the level of the third virial coefficient. A method for emulating the μ -deformed Bose gas up to the third virial coefficient inclusive by means of the two-parametric nonadditive Polychronakos statistics is demonstrated.

Keywords: q - and μ -deformed thermodynamics, q - and μ -deformed algebras, Jackson derivative, cluster integrals, virial coefficients, non-additive Polychronakos statistics.

1. Introduction

The history of non-standard quantum distributions goes back to the 1940s, when Giovanni Gentile and his colleagues tried to apply a statistics intermediate between the bosonic and fermionic ones [1] to describe the properties of liquid helium [2, 3]. In effect, this approach, which consists in the application of non-standard statistics to effectively model real physical systems, is the main motivation for studying various ways to modify quantum distributions [4–8]. Both the effective consideration of the interaction between particles [9, 10] and cosmological models [11, 12] can be taken as examples. As a result, it often becomes possible to describe complicated physical phenomena with the help of a simpler mathematical apparatus, which is especially important while studying many-particle systems.

One of the variants of non-standard statistics is the deformation of commutation relations between the creation and annihilation operators [13–19]. The corresponding quantum algebras are now actively used to analyze complex physical systems [20]. For example, the thermodynamics of Bose systems and the phenomenon of the Bose condensation were studied with their help in works [21, 22]. In work [23], it was shown how the structure of particles and the inter-

action between them can be efficiently taken into account by means of $\tilde{\mu}$ - and q -deformed bosons. The authors of works [24, 25] considered μ - and q -deformed bosons as dark matter models.

In this work, the thermodynamics of ideal q - and μ -deformed Bose gases [26–29] is considered. It can be constructed in the framework of the formalism of the q - and μ -calculi, respectively [29, 30]. The aim of the work consists in determining the correspondence between those two approaches by finding a relation between the deformation parameters. With the help of a modification of the statistics of another type – namely, the phenomenological choice of the expression for occupation numbers – a more accurate simulation of one of the examined types of deformed thermodynamics can be achieved as well.

The paper has the following structure. In Section 2, the properties of a q -deformed algebra are discussed, and one of the ways to construct the q -deformed thermodynamics is demonstrated. In Section 3, a μ -deformed algebra is described, as well as the concept of constructing the relevant thermodynamics. For the sake of completeness, the main ideas of works [26, 29, 31, 32] are also briefly summarized in Sections 2 and 3. Section 4 is devoted to general relations for the virial and cluster expansions. It also includes the results of numerical calculations for the virial coefficients in the case of q -

and μ -deformations. A correspondence between the parameters of such systems is established as well. In Section 6, a two-parameter model of the non-additive Polychronakos statistics is used as an approximation to calculate the virial coefficients of a μ -deformed boson system. Brief conclusions can be found at the end of the paper.

2. q -Deformed Algebra and Thermodynamics of q -Deformed Gas

Let us consider the main steps of constructing a q -deformed algebra and the corresponding thermodynamics [13, 32]. The creation, a^\dagger , and annihilation, a , operators satisfy the following commutation relation written in terms of the q -deformed commutator:

$$[a, a^\dagger]_q = aa^\dagger - qa^\dagger a = 1. \tag{2.1}$$

This is only one of the possible ways to generalize the ordinary commutator (see, e.g., works [14, 15]).

The particle number operator N is not reduced to the product $a^\dagger a$ in the q -deformed case. Its commutators with the operators a^\dagger and a look like

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a. \tag{2.2}$$

Additionally,

$$[a, a] = [a^\dagger, a^\dagger] = 0. \tag{2.3}$$

By applying the creation operator a^\dagger to the vacuum state $|0\rangle$, we obtain the space of vectors $|n\rangle$,

$$(a^\dagger)^n |0\rangle = \text{const} |n\rangle, \tag{2.4}$$

which are eigenvectors of the operator N ,

$$N|n\rangle = n|n\rangle. \tag{2.5}$$

At the same time,

$$a|0\rangle = 0. \tag{2.6}$$

The vector $|n\rangle$ can be written in the form

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]_q!}} |0\rangle, \tag{2.7}$$

where the q -factorial $[n]_q!$ is defined as

$$[n]_q! = [n]_q [n-1]_q \dots [1] \tag{2.8}$$

in terms of the so-called (basic) q -numbers

$$[n]_q = \frac{q^n - 1}{q - 1}. \tag{2.9}$$

Ultimately, we obtain

$$a^\dagger |n\rangle = \sqrt{[n+1]_q} |n+1\rangle, \tag{2.10}$$

$$a |n\rangle = \sqrt{[n]_q} |n-1\rangle. \tag{2.11}$$

In the limit $q \rightarrow 1$, the basic number $[n]_q$ equals n , and all the relations given above become the standard bosonic ones.

The pair products of the creation and annihilation operators look like

$$a^\dagger a = [N]_q, \quad aa^\dagger = [N+1]_q, \tag{2.12}$$

and the following expression can be formally obtained for the operator N :

$$N = \frac{1}{\ln q} \ln [1 + (q-1)a^\dagger a]. \tag{2.13}$$

Let us analyze one of the methods for constructing the thermodynamics of a q -deformed gas [27, 32]. In the approach of the grand canonical ensemble, the Hamiltonian is written in the form

$$H = \sum_j (\epsilon_j - \mu) N_j, \tag{2.14}$$

where ϵ_j is the energy, and N_j the operator of the particle number, the both corresponding to the j -th state, and μ is the chemical potential. The corresponding grand partition function equals

$$\Xi = \text{Sp} e^{-\beta H}, \tag{2.15}$$

where $\beta = 1/T$ is the inverse temperature. According to the standard definition of the mean value of the thermodynamic operator O ,

$$\langle O \rangle = \frac{1}{\Xi} \text{Sp} (O e^{-\beta H}), \tag{2.16}$$

the mean occupation number in the j -th state equals

$$[n_j]_q = \frac{1}{\Xi} \text{Sp} (e^{-\beta H} a_j^\dagger a_j). \tag{2.17}$$

Let n_j be an eigenvalue of the operator N_j . From the formula

$$N_j = \frac{1}{\ln q} \ln [1 + (q-1)a_j^\dagger a_j], \tag{2.18}$$

taking into account that $a_j^\dagger a_j = [n_j]_q$, and after a series of transformations, we obtain the following expression for the occupation numbers:

$$n_j = \frac{1}{\ln q} \ln \frac{z^{-1} e^{\beta \varepsilon_j} - 1}{z^{-1} e^{\beta \varepsilon_j} - q}, \quad (2.19)$$

where $z = e^{\beta \mu}$ is the activity.

A consistent construction of q -deformed thermodynamics involves the application of the Jackson derivative instead of the conventional one [31]. With this purpose in view, let us consider the logarithm of the grand partition function

$$\ln \Xi = - \sum_j \ln(1 - z e^{-\beta \varepsilon_j}) \quad (2.20)$$

and the total particle number

$$N = \sum_j n_j, \quad (2.21)$$

From whence, it is easy to see that the standard relation

$$N \neq z \frac{\partial}{\partial z} \ln \Xi \quad (2.22)$$

becomes violated. Replacing $\frac{\partial}{\partial z}$ by the so-called Jackson derivative $D_q^{(z)}$ [33],

$$D_q^{(z)} f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad D_1^{(z)} = \frac{d}{dz}, \quad (2.23)$$

we obtain

$$\begin{aligned} z D_q^{(z)} \ln \Xi &= \left(-\frac{1}{q-1} \ln \frac{1 - z q e^{-\beta \varepsilon_j}}{1 - z e^{-\beta \varepsilon_j}} \right) = \\ &= \sum_j \frac{\ln q}{q-1} \frac{1}{\ln q} \ln \frac{z^{-1} e^{\beta \varepsilon_j} - 1}{z^{-1} e^{\beta \varepsilon_j} - q} = \\ &= \frac{\ln q}{q-1} \sum_j n_j. \end{aligned} \quad (2.24)$$

After a slight modification of the Jackson derivative,

$$\frac{q-1}{\ln q} D_q^{(z)} = \tilde{D}_q^{(z)}, \quad (2.25)$$

which is also reduced to the ordinary derivative in the limit $q \rightarrow 1$, we obtain an analog of the standard thermodynamic relation

$$N = z \tilde{D}_q^{(z)} \ln \Xi = \sum_j n_j. \quad (2.26)$$

It is of interest that, e.g., the authors of work [32] did not mention about the additional multiplier at the Jackson derivative.

In order to preserve the standard expressions for thermodynamic functions, the derivatives have to be replaced for the variables written in the exponential form ($z = e^{\beta \mu}$ or $y_j = e^{-\beta \varepsilon_j}$). Accordingly, when changing to the q -deformed thermodynamics, all the derivatives must be rewritten in terms of z or y_j , and, afterward, the Jackson derivative [31] has to be introduced. The expression for the internal energy can serve as an example. Instead of the standard expression

$$U = - \frac{\partial}{\partial \beta} \ln \Xi = \frac{\partial}{\partial \beta} \sum_j \ln(1 - z e^{-\beta \varepsilon_j}), \quad (2.27)$$

the following transformation has to be done:

$$U = \sum_j \frac{\partial y_j}{\partial \beta} \tilde{D}_q^{(y_j)} \ln(1 - z y_j), \quad (2.28)$$

where

$$\tilde{D}_q^{(y_j)} f(y_j) = \frac{f(q y_j) - f(y_j)}{y_j(q-1)}.$$

As a result, the formula for the internal energy in the deformed case reads

$$U = \sum_j \varepsilon_j \frac{1}{\ln q} \ln \frac{e^{\beta \varepsilon_j} z^{-1} - 1}{e^{\beta \varepsilon_j} z^{-1} - q} = \sum_j \varepsilon_j n_j, \quad (2.29)$$

i.e. it coincides with the expected one.

3. μ -Deformed Algebra and μ -Deformed Thermodynamics

A new way to construct thermodynamics was proposed while studying a μ -deformed oscillator [17, 29]. The corresponding algebra of creation-annihilation operators is given by the commutation relation

$$[a, a^\dagger] = \varphi(N) - \varphi(N+1), \quad (3.1)$$

with $aa^\dagger \equiv \varphi(N)$, where

$$\varphi(N) = \frac{N}{1 + \mu N} \quad (3.2)$$

is called the structural deformation function. As was in the previous case with the q -deformation, the particle number operator N is not equal to the product $a^\dagger a$. Instead, analogously to Eq. (2.2), we now have

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a. \quad (3.3)$$

Let us consider a μ -deformed analog of the Bose gas, a many-particle system with the Hamiltonian

$$H = \sum_j (\varepsilon_j - \nu) N_j, \tag{3.4}$$

where ε_j is the single-particle energy, N_j the particle number operator for the j -th state, and ν is the chemical potential (we denote it using a different letter to avoid ambiguities). The corresponding logarithm of the grand partition function $\Xi = \text{Sp } e^{-\beta H}$ equals

$$\ln \Xi = - \sum_j \ln(1 - ze^{-\beta\varepsilon_j}), \tag{3.5}$$

where $z = e^{\beta\nu}$ is the activity. The total number of particles is determined by the formula

$$N = z \frac{\partial}{\partial z} \ln \Xi. \tag{3.6}$$

Now, instead of the ordinary derivative $\frac{d}{dx}$ and the Jackson derivative (in the case of q -deformation), the μ -derivative is applied [29]:

$$\mathcal{D}_x^{(\mu)} x^n = [n]_\mu x^{n-1}. \tag{3.7}$$

Here, the μ -number is defined as follows:

$$[n]_\mu \equiv \frac{n}{1 + \mu n} = \varphi(n). \tag{3.8}$$

As $\mu \rightarrow 0$, the derivative $\mathcal{D}_x^{(\mu)}$ transforms into the ordinary derivative, whereas $[n]_{\mu \rightarrow 0} \rightarrow n$.

By replacing $\frac{\partial}{\partial z}$ by $\mathcal{D}_z^{(\mu)}$, we obtain

$$\begin{aligned} N^{(\mu)} &= z \mathcal{D}_z^{(\mu)} \ln \Xi = \\ &= -z \mathcal{D}_z^{(\mu)} \sum_j \ln(1 - ze^{-\beta\varepsilon_j}). \end{aligned} \tag{3.9}$$

Expanding the logarithm into a series and applying the derivative $\mathcal{D}_z^{(\mu)}$ to every term, we have

$$\begin{aligned} N^{(\mu)} &= z \mathcal{D}_z^{(\mu)} \sum_j \sum_{n=1}^{\infty} \frac{(ze^{-\beta\varepsilon_j})^n}{n} = \\ &= \sum_j \sum_{n=1}^{\infty} \frac{e^{-n\beta\varepsilon_j}}{n} [n]_\mu z^n. \end{aligned} \tag{3.10}$$

In the ground state, the energy $\varepsilon_0 = 0$ and $z \rightarrow 1$. Therefore, the term with $j = 0$ diverges, so it is written down separately,

$$N^{(\mu)} = \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} z^n + \sum_{j>0} \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} z^n e^{-n\beta\varepsilon_j}. \tag{3.11}$$

This (first) term

$$n_0^{(\mu)} = \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} z^n \tag{3.12}$$

describes the ground state occupation in the μ -deformed analog of the Bose condensate.

Now, let us consider an ideal D -dimensional Bose gas. We should change from the summation over the energy states to the integration over the energy with the density of states

$$g(\varepsilon) = \frac{V_D}{\Gamma(\frac{D}{2})} \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{D}{2}} \varepsilon^{\frac{D}{2}-1}, \tag{3.13}$$

where m is the particle mass, and V_D the volume in the D -dimensional space. For the deformed analog of the particle number, we have

$$\begin{aligned} N^{(\mu)} &= n_0^{(\mu)} + \frac{V_D}{\Gamma(\frac{D}{2})} \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{D}{2}} \times \\ &\times \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} z^n \int_0^{\infty} d\varepsilon \varepsilon^{\frac{D}{2}-1} e^{-n\beta\varepsilon}. \end{aligned}$$

After the integration, we obtain

$$N^{(\mu)} = n_0^{(\mu)} + \frac{V_D}{\lambda^D} \sum_{n=1}^{\infty} \frac{[n]_\mu}{n^{D/2+1}} z^n, \tag{3.14}$$

where

$$\lambda = \left(\frac{2\pi\beta\hbar^2}{m}\right)^{1/2} \tag{3.15}$$

is the de Broglie thermal wavelength.

In order to construct the μ -deformed thermodynamics further, let us define the deformed grand partition function $\Xi^{(\mu)}$ as follows:

$$\ln \Xi^{(\mu)} = \left(z \frac{\partial}{\partial z}\right)^{-1} N^{(\mu)}, \tag{3.16}$$

where the inverse operator means the integration,

$$\left(z \frac{\partial}{\partial z}\right)^{-1} f(z) = \int_0^z dz' \frac{f(z')}{z'}, \tag{3.17}$$

so that

$$\left(z \frac{\partial}{\partial z}\right)^{-1} z^n = \frac{z^n}{n}.$$

As a result, the quantity $\ln \Xi^{(\mu)}$ can be expanded in a series,

$$\ln \Xi^{(\mu)} = \sum_{n=1}^{\infty} \frac{[n]_{\mu}}{n^2} z^n + \frac{V_D}{\lambda^D} \sum_{n=1}^{\infty} \frac{[n]_{\mu}}{n^{D/2+2}} z^n. \quad (3.18)$$

Bearing in mind the polylogarithm definition

$$\text{Li}_{\alpha}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{\alpha}}, \quad (3.19)$$

it is convenient to introduce its μ -generalization

$$\text{Li}_{\alpha}^{(\mu)}(z) = \sum_{n=1}^{\infty} \frac{[n]_{\mu}}{n^{\alpha+1}} z^n. \quad (3.20)$$

As $[n]_{\mu} \rightarrow n$, the latter transforms into the conventional polylogarithm. Hence,

$$\ln \Xi^{(\mu)} = \text{Li}_1^{(\mu)}(z) + \frac{V_D}{\lambda^D} \text{Li}_{D/2+1}^{(\mu)}(z). \quad (3.21)$$

From whence, we obtain the equation of state

$$\frac{PV_D}{T} = \ln \Xi^{(\mu)} = \text{Li}_1^{(\mu)}(z) + \frac{V_D}{\lambda^D} \text{Li}_{D/2+1}^{(\mu)}(z). \quad (3.22)$$

This expression will be used in the next section.

4. Virial Expansion

In order to find a correspondence between various fractional statistics, we have to compare their thermodynamic functions. This operation can be done on the basis of the relevant virial expansions [34–36]. Afterward, using the calculated expressions for the virial coefficients, the correspondence between various fractional statistics can be established.

The virial coefficients are the coefficients in the virial expansion of the pressure in a many-particle system in the density (concentration) parameter:

$$\frac{P}{T} = \frac{N}{V_D} \left[1 + b_2 \frac{N}{V_D} \lambda^D + b_3 \left(\frac{N}{V_D} \lambda^D \right)^2 + \dots \right], \quad (4.1)$$

where P is the pressure, T the temperature, the ratio $\frac{N}{V_D}$ is the D -dimensional concentration, and λ the de Broglie thermal wavelength. The multipliers b_j are the dimensionless j -th virial coefficients. The first coefficient in the expansion corresponds to the ideal gas, the second one takes into account the (effective) pair interaction between the gas particles, the third one makes allowance for three-particle interactions, and so forth.

A first step in deriving expressions for the virial expansion coefficients is the cluster expansion of the grand partition function in a series in the activity z ,

$$\frac{1}{V_D} \ln \Xi = \sum_{\ell=1}^{\infty} B_{\ell} z^{\ell}, \quad (4.2)$$

where B_{ℓ} are the so-called cluster integrals. Then the virial coefficients can be found, by using the relations

$$b_2 \lambda^D = -\frac{B_2}{B_1^2}, \quad (4.3)$$

$$b_3 \lambda^{2D} = -2 \frac{B_3}{B_1^3} + 4 \frac{B_2^2}{B_1^4}. \quad (4.4)$$

From the equation of state (3.22) corresponding to the μ -deformed thermodynamics and accounting for the series expansion (3.19) of the μ -deformed polylogarithm, we immediately obtain the cluster integrals (4.2):

$$\begin{aligned} \frac{1}{V_D} \ln \Xi &= \frac{1}{V_D} \left(\text{Li}_1^{(\mu)}(z) + \frac{V_D}{\lambda^D} \text{Li}_{D/2+1}^{(\mu)}(z) \right) = \\ &= \frac{1}{V_D} \left(\frac{[1]_{\mu}}{1^2} z + \frac{[2]_{\mu}}{2^2} z^2 \frac{[3]_{\mu}}{3^2} z^3 + \dots \right) + \\ &+ \frac{1}{\lambda^D} \left(\frac{[1]_{\mu}}{1^{\frac{D}{2}+2}} z + \frac{[2]_{\mu}}{2^{\frac{D}{2}+2}} z^2 + \frac{[3]_{\mu}}{3^{\frac{D}{2}+2}} z^3 + \dots \right) = \\ &= B_1 z + B_2 z^2 + B_3 z^3 + \dots \end{aligned} \quad (4.5)$$

Whence

$$B_1 = \frac{[1]_{\mu}}{V_D} + \frac{1}{\lambda^D} \frac{[1]_{\mu}}{1}, \quad (4.6)$$

$$B_2 = \frac{[2]_{\mu}}{4V_D} + \frac{1}{\lambda^D} \frac{[2]_{\mu}}{2^{\frac{D}{2}+2}}, \quad (4.7)$$

$$B_3 = \frac{[3]_{\mu}}{9V_D} + \frac{1}{\lambda^D} \frac{[3]_{\mu}}{3^{\frac{D}{2}+2}}. \quad (4.8)$$

In the thermodynamic limit as $V_D \rightarrow \infty$ (in this case, $N \rightarrow \infty$ and $\frac{N}{V_D} = \text{const}$), we have

$$B_{\ell} = \frac{1}{\lambda^D} \frac{[\ell]_{\mu}}{\ell^{\frac{D}{2}+2}}. \quad (4.9)$$

Ultimately, after calculating the cluster integrals, we arrive at the following expressions for virial coefficients (4.3) and (4.4), respectively:

$$b_2^{(\mu)} = -\frac{[2]_{\mu}}{([1]_{\mu})^2} \frac{1}{2^{D/2+2}}, \quad (4.10)$$

$$b_3^{(\mu)} = -\frac{2 \cdot [3]_\mu}{([1]_\mu)^3} \frac{1}{3^{D/2+2}} + \left(\frac{2 \cdot [2]_\mu}{([1]_\mu)^2} \frac{1}{2^{D/2+2}} \right)^2. \quad (4.11)$$

At $D = 3$, those formulas coincide with the results of work [29].

Now, let us calculate the cluster integrals for q -deformations. They can be easily found on the basis of the following expression for the occupation numbers n_j :

$$\frac{N}{V_D} = \frac{1}{V_D} \sum_j n_j = \sum_{\ell=1}^{\infty} \ell B_\ell z^\ell. \quad (4.12)$$

Let us change from the summation over the energy levels to the integration over the energy with the density of states [37]

$$g(\varepsilon) = \frac{V_D}{\Gamma(\frac{D}{2})} \left(\frac{m}{2\pi\hbar^2} \right)^{\frac{D}{2}} \varepsilon^{\frac{D}{2}-1}. \quad (4.13)$$

The distribution function n_j is known [see Eq. (2.19)]:

$$n_j = \frac{1}{\ln q} \ln \frac{e^{\beta\varepsilon_j} z^{-1} - 1}{e^{\beta\varepsilon_j} z^{-1} - q}.$$

Therefore,

$$\begin{aligned} \frac{N}{V_D} &= \frac{1}{V_D} \int_0^\infty d\varepsilon g(\varepsilon) n(\varepsilon) = \\ &= \frac{1}{\Gamma(\frac{D}{2})} \left(\frac{m}{2\pi\hbar^2} \right)^{\frac{D}{2}} \int_0^\infty d\varepsilon \varepsilon^{\frac{D}{2}-1} \ln \frac{1 - ze^{-\beta\varepsilon}}{1 - qze^{-\beta\varepsilon}}. \end{aligned} \quad (4.14)$$

Taking the series expansion of a logarithm into account, we obtain, with an accuracy to the z^3 -term, that

$$\begin{aligned} \frac{N}{V_D} &= -\frac{1}{\lambda^D} \frac{1}{\ln q} \left(z + \frac{1}{4} \frac{z^2}{2^{D/2-1}} + \frac{1}{9} \frac{z^3}{3^{D/2-1}} - \right. \\ &\left. - qz - \frac{1}{4} \frac{q^2 z^2}{2^{D/2-1}} + \frac{1}{9} \frac{q^3 z^3}{3^{D/2-1}} \right) = \\ &= B_1 z + 2B_2 z^2 + 3B_3 z^3, \end{aligned} \quad (4.15)$$

From whence, knowing the cluster integrals, we find the following expressions for the virial coefficients (4.3) and (4.4):

$$b_2^{(q)} = -\frac{\ln q}{2^{D/2+2}} \frac{q+1}{q-1}, \quad (4.16)$$

$$b_3^{(q)} = -\frac{2 \ln^2 q}{9 \cdot 3^{D/2}} \frac{q^3 - 1}{(q-1)^3} + \frac{\ln^2 q}{2^{D+2}} \frac{(q+1)^2}{(q-1)^2}. \quad (4.17)$$

5. Calculation Results for Virial Coefficients

Figure 1 illustrates the results of calculations for the virial coefficients $b_2^{(q)}$ (q -deformation, panel *a*) and $b_2^{(\mu)}$ (μ -deformation, panel *b*) in the case of space dimension $D = 3$. The corresponding general views for $2 \leq D \leq 4$ are shown in Fig. 2.

It is easy to verify that, in the limits $q \rightarrow 1$ and $\mu \rightarrow 0$, we obtain expressions for the ideal Bose gas, namely,

$$b_2^{(q)} = -\frac{\ln q}{2^{D/2+2}} \frac{q+1}{q-1} \Big|_{q \rightarrow 1} = -\frac{1}{2^{D/2+1}}, \quad (5.1)$$

$$b_2^{(\mu)} = -\frac{[2]_\mu}{([1]_\mu)^2} \frac{1}{2^{D/2+2}} \Big|_{\mu \rightarrow 0} = -\frac{1}{2^{D/2+1}}. \quad (5.2)$$

Hence, if the space dimension amounts, e.g., to $D = 2$, the second virial coefficients expectedly equal [34]

$$b_2^{(q)} = -\frac{1}{4}, \quad b_2^{(\mu)} = -\frac{1}{4}.$$

By comparing the expressions for the second virial coefficient, we can observe an approximate correspondence between the q - and μ -deformed Bose gases. For this purpose, we solve the equation

$$b_2^{(q)} = b_2^{(\mu)}. \quad (5.3)$$

Note that the solution of this equation does not depend on the space dimension D , because the corresponding multipliers mutually compensate each other. The obtained dependence is rather cumbersome,

$$\begin{aligned} \mu(q) &= \frac{1}{2(q-1)} \left\{ 2(1-q) + (1+q) \ln q - \right. \\ &\left. - \sqrt{2(1-q^2) \ln q + (1+q)^2 \ln^2 q} \right\}, \end{aligned} \quad (5.4)$$

and it is convenient to plot it (see Fig. 3).

The results of calculations for the third virial coefficient in the two-dimensional case are shown in Fig. 4. In the limiting cases $q \rightarrow 1$ and $\mu \rightarrow 0$, which correspond to the ordinary Bose gas, we obtain the result $b_3^{(q \rightarrow 1)} = b_3^{(\mu \rightarrow 0)} = \frac{1}{36}$, as it has to be [34].

It is of interest to see how different is the third virial coefficient for different deformations. For this purpose, let us substitute the parameter μ in the expression for $b_3^{(\mu)}$ by the dependence $\mu(q)$ [Eq. (5.3)] and equate the result to $b_3^{(q)}$. The results obtained for

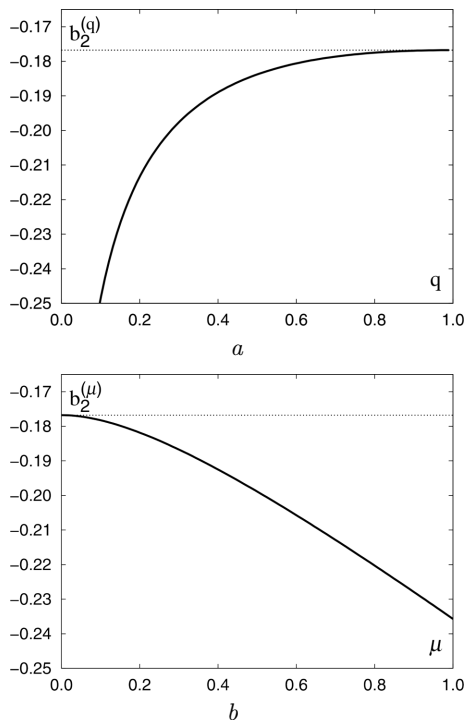


Fig. 1. Second virial coefficient b_2 in the case of space dimension $D = 3$. Plot a corresponds to the q -deformation, and plot b to the μ -deformation

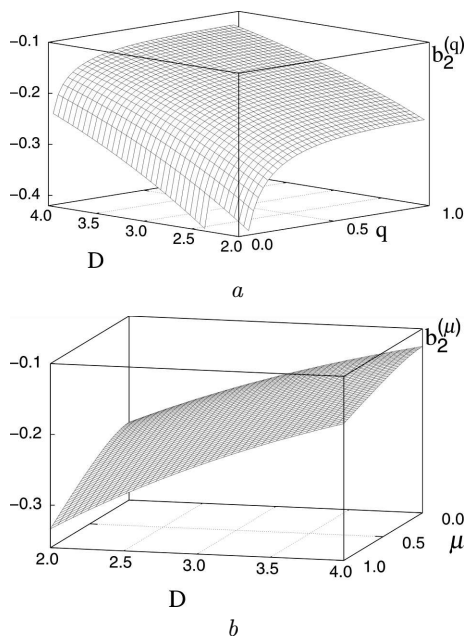


Fig. 2. Second virial coefficient b_2 for various space dimensions $2 \leq D \leq 4$. Plot a corresponds to the q -deformation, and plot b to the μ -deformation

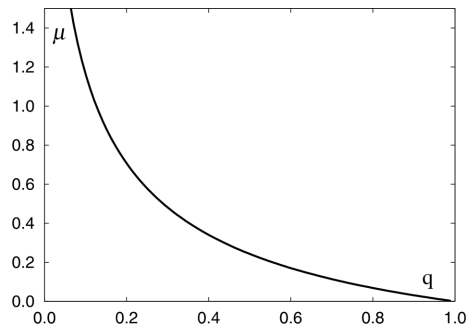


Fig. 3. Solution $\mu(q)$ [formula (5.4)] of Eq. (5.3)

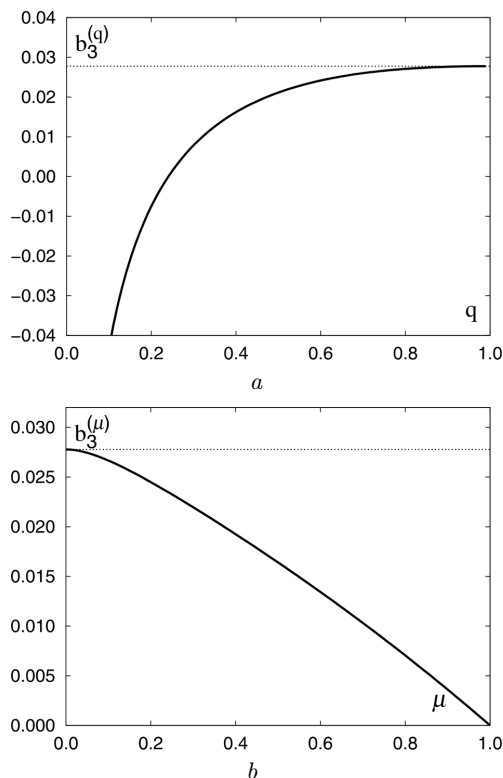


Fig. 4. Third virial coefficient b_3 in the case of space dimension $D = 2$. Plot a corresponds to the q -deformation, and plot b to the μ -deformation

various space dimensions are shown in Fig. 5. One can easily see that the difference between the virial coefficients decreases as the space dimension D increases.

Hence, on the basis of the second virial coefficient, we established an approximate correspondence between the q - and μ -deformed Bose gases, which is independent of the space dimension. However, this dependence manifests itself already at the level of the third virial coefficient. Therefore, in order to con-

struct a more accurate correspondence (say, for the μ -deformed Bose gas), two-parameter statistics can be used. Alternatively, instead of q -deformation, the (p, q) -deformation can be considered [38]. In the next section, we will use one of the two-parameter statistics, which was proposed in works [35, 39].

6. Two-Parameter Model as an Approximation of μ -Deformed Bose Gas

Let us model the thermodynamics of the μ -deformed Bose gas with the help of the non-additive Polychronakos statistics [35, 39]. In the latter, the occupation numbers look like

$$n_j^{\text{NAPS}} = \frac{1}{z^{-1} e_Q^{\beta \varepsilon_j} - \gamma}, \tag{6.1}$$

where e_Q^x is the Tsallis Q -exponential function¹ [40],

$$e_Q^x = \begin{cases} e^x, & \text{if } Q = 1, \\ [1 + (1 - q)x]^{1/(1-q)} & \text{if } Q \neq 1 \text{ and } 1 + (1 - Q)x > 0, \\ 0^{1/(1-q)} & \text{if } Q \neq 1 \text{ and } 1 + (1 - Q)x \leq 0. \end{cases} \tag{6.2}$$

The cluster integrals are calculated from relation (4.12). We obtain

$$\begin{aligned} \frac{N}{V_D} &= \frac{1}{V_D} \sum_j n_j^{\text{NAPS}} = \frac{1}{V_D} \int_0^\infty d\varepsilon \frac{g(\varepsilon)}{z^{-1} e_Q^{\beta \varepsilon} - \gamma} = \\ &= \frac{1}{\Gamma(\frac{D}{2})} \left(\frac{m}{2\pi\hbar^2} \right)^{\frac{D}{2}} \int_0^\infty d\varepsilon \varepsilon^{\frac{D}{2}-1} \sum_{\ell=1}^\infty \frac{\gamma^{\ell-1}}{\left(e_Q^{\beta \varepsilon} \right)^\ell} z^\ell = \\ &= \sum_{\ell=1}^\infty \ell B_\ell z^\ell. \end{aligned} \tag{6.3}$$

When operating with the Tsallis Q -exponential functions, one should bear in mind that the rules applicable to ordinary exponential functions are not valid for their Tsallis counterparts. For instance, $(e_Q^x)^y \neq e_Q^{xy}$ and $e_Q^{x+y} \neq e_Q^x e_Q^y$ [41]. Furthermore, if the parameter $Q > 1$, the interval of nonzero values for the function e_Q^x is confined to the right by the argument value $x_0 = 1/(Q - 1)$ [see definition (6.2)].

¹ The lowercase letter q is usually used in this notation. However, instead, we use the capital letter Q here to distinguish it from the q -deformation parameter.

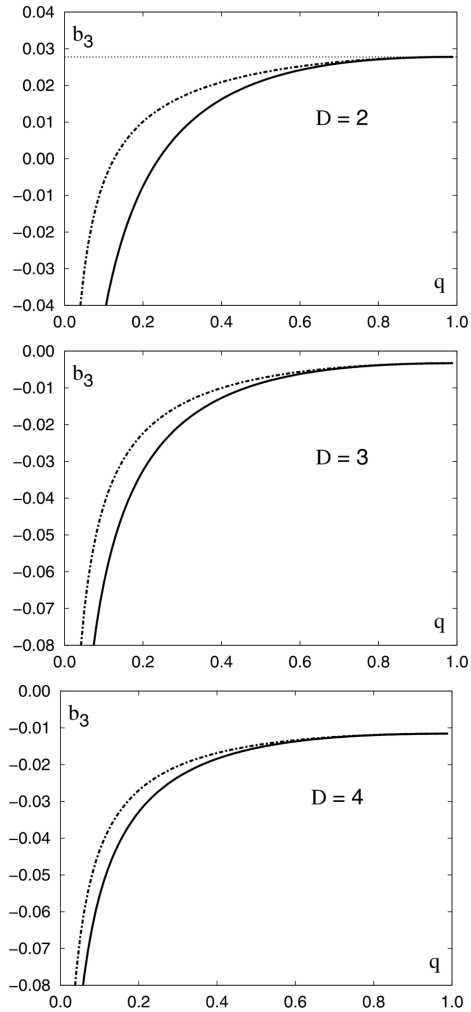


Fig. 5. Third virial coefficient for various space dimensions D : $b_3^{(q)}$ (solid curves) and $b_3^{(\mu)}$ (dash-dotted curves). The parameter $\mu(q)$ is described by Eq. (5.4)

After simple calculations, we obtain the following result for the cluster integrals:

$$B_\ell = \frac{1}{\lambda^D} \frac{\gamma^{\ell-1}}{\ell} \{\ell; D/2\}_Q, \tag{6.4}$$

where the notation $\{\ell; s\}_Q$ was introduced; namely,

$$\begin{aligned} \{\ell; s\}_Q &= \ell^{-s} \quad \text{if } Q = 1, \\ \{\ell; s\}_Q &= \frac{B\left(s, \frac{\ell}{1-Q} - s\right)}{(1-Q)^s \Gamma(s)} \quad \text{if } Q < 1, \\ \{\ell; s\}_Q &= \frac{B\left(s, \frac{\ell}{Q-1} + 1\right)}{(Q-1)^s \Gamma(s)} \quad \text{if } Q > 1. \end{aligned} \tag{6.5}$$

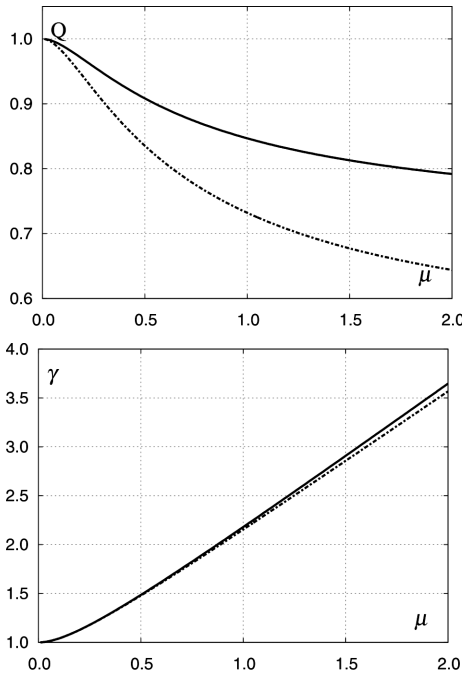


Fig. 6. Dependences of the parameters of the non-additive Polychronakos statistics on the parameter of the μ -deformed Bose gas. The space dimension $D = 3$ (solid curves) and 2 (dash-dotted curves)

Here, $B(x, y)$ is the Euler beta-function. Now, the virial coefficients are functions of two parameters, Q and γ :

$$b_2^{\text{NAPS}}(Q, \gamma) = -\frac{\gamma \{2; D/2\}_Q}{2 \{1; D/2\}_Q^2}, \quad (6.6)$$

$$b_3^{\text{NAPS}}(Q, \gamma) = -\frac{2\gamma^2 \{3; D/2\}_Q}{3 \{1; D/2\}_Q^3} + \gamma^2 \frac{\{2; D/2\}_Q^2}{\{1; D/2\}_Q^4}. \quad (6.7)$$

By solving the system of equations

$$\begin{cases} b_2^{\text{NAPS}}(Q, \gamma) = b_2^{(\mu)}, \\ b_3^{\text{NAPS}}(Q, \gamma) = b_3^{(\mu)}, \end{cases} \quad (6.8)$$

we obtain the dependences of the parameters Q and γ on μ . They are shown in Fig. 6.

Thus, the thermodynamics of the μ -deformed Bose gas with an accuracy up to the third virial coefficient can be described with the help of the non-additive Polychronakos statistics, the parameters $Q(\mu)$ and $\gamma(\mu)$ of which are determined from the system of equations (6.8).

7. Conclusions

To summarize, two approaches modifying the algebra of creation-annihilation operators and, accordingly, two different ways for constructing the thermodynamics of systems corresponding to the so-called q - and μ -deformations are considered in this work. It is shown how analytical expressions can be derived for the thermodynamic functions of the ideal D -dimensional q - and μ -deformed Bose gases. In order to make the reading convenient and the presentation comprehensive, the main expressions obtained by other authors are also given, with some of them being also generalized to the D -dimensional case.

By comparing the results obtained for the second virial coefficient, an approximate correspondence between the q - and μ -deformed Bose gases is revealed, which turns out to be independent of the space dimension. However, this dependence manifests itself for the third virial coefficient. Therefore, two-parameter deformations should be used to construct more accurate correspondences. As an example, the non-additive Polychronakos statistics is considered. Its parameters are used to calculate the dependences $Q(\mu)$ and $\gamma(\mu)$, which reproduce the thermodynamics of the μ -deformed Bose gas up to the third virial coefficient inclusive.

This work was partially sponsored in the framework of the project FF-83F (the state registration number 0119U002203) of the Ministry of Education and Science of Ukraine.

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Received 12.23.19.

Translated from Ukrainian by O.I. Voitenko

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ІДЕАЛЬНИЙ БОЗЕ-ГАЗ У ДЕЯКИХ
ДЕФОРМОВАНИХ ТЕРМОДИНАМІКАХ.
ЗВ'ЯЗОК МІЖ ПАРАМЕТРАМИ ДЕФОРМАЦІЙ

Резюме

У роботі розглянуто два підходи до побудови термодинаміки в межах q - та μ -формалізму, що відповідають певним деформаціям алгебри операторів породження–знищення. Шляхом зіставлення отриманих результатів для другого віріального коефіцієнта встановлено наближену відповідність між ідеальними q - та μ -деформованими бозе-газами, що не залежить від вимірності простору. Розбіжності виникають лише на рівні третього віріального коефіцієнта. Показано, як за допомогою двопараметричної неадитивної статистики Поліхронакоса можна змулювати μ -деформований бозе-газ з точністю до третього віріального коефіцієнта включно.