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RELATIONSHIP BETWEEN THE PARAMETERS OF THE SECOND VIRIAL COEFFICIENT OF NON-ABELIAN ANYONS AND THE TWO-PARAMETRIC FRACTIONAL STATISTICS

A relationship between the parameters of the second virial coefficient for the system of non-Abelian anyons and two-parametric modifications of the Haldane–Wu and Polychronakos fractional statistics has been demonstrated. Parameters that can approximately describe non-Abelian anyons using the indicated statistics types are calculated. The limit at which the non-additivity/incompleteness parameter q tends to unity is considered.

K e y w o r d s: virial coefficient, non-Abelian anyons, non-additive/incomplete two-parametric statistics, Haldane–Wu fractional statistics, Polychronakos fractional statistics.

1. Introduction

In 1977, Leinaas and Myrheim proved that the conventional classification of particles into bosons and fermions is not applicable in the two-dimensional space [1]. Frank Wilczek has proposed to call such particles as anyons, because, during the permutation of two particles, the phase of the wave function can change by an arbitrary factor, not only 0 or π [2]. Mathematically, such a statistics corresponds to the braid group rather than the permutation one.

Anyons are used when describing the fractional quantum Hall effect observed in two-dimensional electron systems at low temperatures and in strong magnetic fields [3–6]. On the basis of anyons, a topological quantum computer was proposed to be constructed. Because of its topological nature, it has to be much more tolerant to the interference and errors, than the "ordinary" quantum computer [7, 8]. Note that there is some experimental evidence for the existence of excitations corresponding to anyons [9–11].

Particular attention should be paid to the following example. As far back as in 2016, a collaboration of physicists proposed the theoretical description of an anyon collider [24]. Already in April 2020, a group of French scientists experimentally proved the existence of such particles [25]. The relevant study demonstrated the fractional Abelian statistics in the quantum Hall effect mode at the filling factor $\nu = 1/3$. It was done by measuring the correlation current characteristics that emerge as a result of collisions between the anyons at the beam splitting.

But anyons are not the most exotic kind of particles. In 1991, Gregory Moore and Nicholas Read proposed a new type of particle called nonabelions [12]. The non-Abelian statistics means that if we have some identical particles, then the permutations in various pairs do not necessarily commute with one another. This event can occur, when the permutation of two particles changes not only the phase factor, but also the wave function of the state in general. In principle, such situations are quite realizable in the braid group [13]: from the viewpoint of physics, it is only necessary that the ground state of the quasiparticle system has to be degenerate, rather than single.

There is the evidence of that just such an exotic situation is realized in a quite specific system – an electron liquid in the fractional quantum Hall effect mode with a filling factor of 5/2. In 2005, it was shown experimentally that elementary excitations have the fractional charge $e/4$ at this filling factor value [14], as well as the charge $e/2$ [15].

Interest in such interferometric experiments [16] has grown in recent years, which is associated with a possibility to construct a quantum computer that would operate by manipulating non-Abelian quasiparticles. The work of a quantum computer can be described by a unitary transformation, whereas, in the topological quantum computer, it consists of matrices describing the "braiding" of quasiparticles.

In this paper, a relationship between the parameters of the second virial coefficient of the system of non-Abelian anyons and various types of fractional statistics is found. The obtained results can be used for the efficient development of thermodynamics for non-Abelian anyons that would be consistent with other, mathematically simpler statistics.

2. Non-Abelian Anyons

Non-Abelian anyons that are studied in this work are non-Abelian Chern–Simons (NACS) particles. These are point-like particles interacting by means of the non-Abelian topological Aaronov–Bohm effect. They carry non-Abelian charges and non-Abelian magnetic fluxes. As a result, they acquire fractional spins and obey the braid statistics, similarly to Abelian anyons [17].

The thermodynamic properties of the lowest Landau level were studied for non-Abelian anyons in a strong magnetic field, and it was shown that the corresponding virial coefficients do not depend on the statistical parameters [18]. As compared with the Abelian case, the thermodynamics of a system of free non-Abelian anyons is much more difficult to study, because all available results were obtained for the boundary conditions corresponding to the hard-core model [19]. At the same time, in the case of non-Abelian anyons with a soft core, there are no exact results even for the second virial coefficient. Accordingly, we can only approximately calculate the thermodynamics of a system of non-Abelian anyons. In work [17], a relationship between the second virial coefficient for non-Abelian anyons and various hard-core parameters was demonstrated. In this work, we consider the hard-core case.

The Hamiltonian of free NACS particles looks like [17]

$$H_N = -\sum_{\alpha=1}^{N} \frac{1}{\mu_\alpha}(\nabla_{\bar{z}_\alpha}\nabla_{z_\alpha} + \nabla_{z_\alpha}\nabla_{\bar{z}_\alpha}),$$
$$\nabla_{z_\alpha} = \frac{\partial}{\partial z_\alpha} + \frac{1}{2\pi\kappa}\sum_{\beta\neq\alpha}^{N} \hat{Q}^a_\alpha \hat{Q}^a_\beta \frac{1}{z_\alpha - z_\beta}, \qquad (2.1)$$
$$\nabla_{\bar{z}_\alpha} = \frac{\partial}{\partial \bar{z}_\alpha},$$

where the subscript $\alpha = 1, ..., N$ enumerates the particles, whereas $z_\alpha = x_\alpha + iy_\alpha$ and $\bar{z}_\alpha = x_\alpha - iy_\alpha$ are their spatial coordinates. The parameter of the theory κ is such that $4\pi\kappa$ is an integer number. The operators $\hat{Q}^a$ are the so-called isovectors in the isospin l representation; by their nature, they are angular momentum operators. The l-values are quantized and vary within the set of all integer and semiinteger numbers, with $l = 1/2$ being the minimum possible nontrivial value (the value $l = 0$ corresponds to a system of free bosons) [17]. Then, in whole, the virial coefficients depend on the magnitude of the isospin l quantum number and the parameter κ.

The statistical mechanics of NACS particles can be studied by introducing the grand partition function Ξ determined in terms of the Hamiltonian H_N for N-particle partition functions Z_N and the fugacity z:

$$\Xi = \sum_{N=0}^{\infty} z^N Z_N = \sum_{N=0}^{\infty} z^N \mathrm{Tr} e^{-\beta H_N}. \qquad (2.2)$$

Note the validity of the following cluster expansion:

$$\Xi = \exp\left\{A \sum_{n=0}^{\infty} \mathcal{B}_n z^n\right\}, \qquad (2.3)$$

where A is the gas area (of course, this is an equivalent of the volume V in the three-dimensional problem), and $\mathcal{B}_n$ are cluster integrals of the n-th order. In particular,

$$\mathcal{B}_1 = \frac{1}{A} Z_1, \quad \mathcal{B}_2 = \frac{1}{A}\left\{Z_2 - \frac{Z_1^2}{2}\right\}. \qquad (2.4)$$

The virial expansion of the equation of state in the powers of the density $\rho = N/A$ brings about

$$p = \rho T\left[1 + b_2(\rho\lambda_T^2) + b_3(\rho\lambda_T^2)^2 + ...\right], \qquad (2.5)$$

where

$$\lambda_T = \sqrt{\frac{2\pi\hbar^2}{mT}}, \qquad (2.6)$$

is the de Broglie wavelength of the particle with the mass m, and b_n are dimensionless virial coefficients of the n-th order. The second virial coefficient looks like

$$b_2 = -\frac{\mathcal{B}_2}{\mathcal{B}_1^2} = A\left\{\frac{1}{2} - \frac{Z_2}{Z_1^2}\right\}. \qquad (2.7)$$

Let us take the expression for the second virial coefficient from work [17], in which the calculation procedure of b_2 was shown in detail:

$$b_2(\kappa, l) = -\frac{1}{4(2l+1)}\left[1 + \frac{1}{(2l+1)}\sum_{j=0}^{2l}(2j+1)\times\right.$$
$$\times\Big\{\big(1+(-1)^{j+2l}\big)(\gamma_j^2 - 2\gamma_j) +$$
$$\left.+\big(1-(-1)^{j+2l}\big)[(\gamma_j+1) \bmod 2 - 1]^2\Big\}\right]. \quad (2.8)$$

Here, l is the isospin, and the parameters γ_j and ω_j satisfy the following relations:

$$\begin{aligned}\gamma_j &= \omega_j \bmod 2,\\ \omega_j &= \frac{1}{4\pi\kappa}[j(j+1) - 2l(l+1)].\end{aligned} \quad (2.9)$$

In Figure, the dependence of the second virial coefficient of non-Abelian anyons on the parameters k and l is illustrated. It should be noted that although the indicated parameters vary within the set of integer and semiinteger values, respectively, in order to improve the visual perception, the points on the graph belonging to specific values of the parameters k and l were connected.

3. Two-Parametric Fractional Statistics

From the expression for the occupation numbers, we can determine cluster integrals, in terms of which the virial coefficients are expressed. The distribution function is related to the cluster integrals as follows:

$$\frac{N}{A} = \frac{1}{A}\sum_j G_j n_j = \sum_{\ell=1}^{\infty} \ell \mathcal{B}_\ell z^\ell, \quad (3.1)$$

where G_j is the degeneracy of the j-th energy level ε_j, and z is the fugacity.

Let us consider free particles in a two-dimensional space and change in expression (3.1) from the summation over j to the integration over the energy with the density of states [26]

$$g(\varepsilon) = \frac{Am}{2\pi\hbar^2} \quad (3.2)$$

Then we expand the obtained result in a power series in z.

Let us analyze a few two-parametric statistical models for anyons [20].

3.1. Polychronakos statistics

In the Polychronakos statistics, the average occupation numbers equal

$$n_j^{\mathrm{P}} = \frac{1}{z^{-1}X(\varepsilon_j) + \gamma}, \quad (3.3)$$

where $X(\varepsilon_j) = e^{\varepsilon_j/T}$, and $\gamma = -\gamma'$ is a statistical parameter.

3.1.1. Incomplete Polychronakos statistics

By modifying the ordinary Polychronakos statistics so that $X(\varepsilon_j) = e^{q\varepsilon_j/T}$, where q is the deformation parameter, a two-parametric dependence can be obtained. This modification is called the incomplete Polychronakos statistics. In this case, the second virial coefficient takes the form

$$b_2^{\mathrm{IPS}} = -\frac{\gamma}{4}q. \quad (3.4)$$

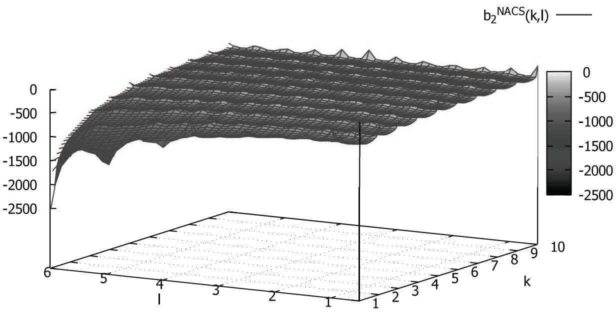
3.1.2. Non-additive Polychronakos statistics

Let us modify the ordinary Polychronakos statistics by means of the Tsallis q-exponent [21],

$$e_q^x = \begin{cases}\exp(x), & \text{if } q = 1,\\ [1+(1-q)x]^{1/(1-q)}, & \text{if } q \neq 1 \text{ and } 1+(1-q)x > 0,\\ 0^{1/(1-q)}, & \text{if } q \neq 1 \text{ and } 1+(1-q)x \le 0.\end{cases}$$

Depending on the non-additivity parameter q, the last option can be rewritten as follows:

$$0^{1/(1-q)} = \begin{cases}0, & \text{if } q < 1,\\ \infty, & \text{if } q > 1.\end{cases}$$



Dependence of the second virial coefficient of non-Abelian anyons on the parameters l and k

Table 1. **Calculated values of the parameters q of the fractional statistics for various anyonic k and l. For every value of the parameter l, the left column corresponds to $q_{\rm IPS} = q_{\rm IHWS}$, and the right one to $q_{\rm NAPS} = q_{\rm NAHWS}$**

k	$l = 1/2$		$l = 1$		$l = 3/2$		$l = 2$	
1	1.0000	1.0000	89.000	45.478	136.000	68.986	1753.0	877.50
2	0.2500	0.4215	13.000	7.3807	31,000	16.443	397.00	199.50
3	0.3333	0.5000	9.8889	5.7973	12.667	7.2116	158.33	80.154
4	0.4375	0.5897	5.0000	3.2656	6.6250	4.1171	101.00	51.481
5	0.5200	0.6562	3.8800	2.6673	4.0000	2.7322	89.000	45.478
10	0.7300	0.8136	2.1200	1.6880	1.0000	1.0000	19.400	10.614
20	0.8575	0.9034	1.4800	1.3064	0.6250	0.7367	3.4000	2.4064
50	0.9412	0.9605	1.1728	1.1132	0.7600	0.8351	1.6720	1.4234
100	0.9703	0.9801	1.0832	1.0550	0.8650	0.9085	1.2880	1.1867
1000	0.9970	0.9980	1.0080	1.0054	0.9852	0.9901	1,0245	1.0163

Table 2. **Parameters γ and g are coupled with the isospin l**

l	$\gamma_{\rm IPS} = \gamma_{\rm NAPS}$	$g_{\rm IHWS} = g_{\rm NAHWS}$
1/2	1/2	3/8
1	1/3	5/12
3/2	1/4	7/16
2	1/5	9/20

A detailed description of the q-exponent was also given in works [22, 23]. The definitions for $q \neq 1$ are often combined using the notation $[u]_+ \equiv \equiv \max(0, u)$ [22].

In this case, if $X(\varepsilon_j) = e_q^{\varepsilon_j/T}$, we obtain the non-additive Polychronakos statistics. Then the second virial coefficient can be obtained in the following form:

$$b_2^{\rm NAPS} = -\frac{\gamma}{4}\frac{2q^2}{1+q}. \qquad (3.5)$$

3.2. Haldane–Wu statistics

The average occupation numbers in this statistics are as follows:

$$n_j^{\rm HW} = \frac{1}{w\left(z^{-1}X(\varepsilon_j)\right) + g}, \qquad (3.6)$$

where $X(\varepsilon_j) = e^{\varepsilon_j/T}$, g is the parameter of statistical interaction, and the function $w(x)$ satisfies the transcendental equation

$$w^g(x)\left[1 + w(x)\right]^{1-g} = x. \qquad (3.7)$$

In the case $g = 0$, the solution of this equation is $w(x) = x - 1$, i.e. the Bose distribution. If $g = 1$, we obtain $w(x) = x$, i.e. the Fermi distribution. In the limit of large argument values for the function $w(x)$, the expression for the occupation numbers can be rewritten in the form

$$n_j^{\rm HW} = \frac{1}{w(x) + g} = \frac{1}{e^{(\varepsilon_j - \mu)/T} + (2g-1)}. \qquad (3.8)$$

3.2.1. Incomplete Haldane–Wu statistics

For the incomplete Haldane–Wu statistics, in the assumption that $X(\varepsilon_j) = e^{q\varepsilon_j/T}$, the second virial coefficient looks like

$$b_2^{\rm IHWS} = \frac{2g-1}{4}q. \qquad (3.9)$$

3.2.2. Non-additive Haldane–Wu statistics

In the non-additive Haldane–Wu statistics with $X(\varepsilon_j) = e_q^{\varepsilon_j/T}$, the second virial coefficient depends on the non-additive parameter as follows:

$$b_2^{\rm NAHWS} = \frac{2g-1}{4}\frac{2q^2}{(1+q)}. \qquad (3.10)$$

4. Results

By equating the corresponding factors in the expressions for the second viral coefficient of non-Abelian anyons (2.8) and the model systems (3.4), (3.5), (3.9), and (3.10), we can obtain a relationship for the parameters of a fractional statistics in the case where the given statistics makes it possible to describe non-Abelian anyons.

The factor $\gamma/4$ or $(2g-1)/4$ is convenient to equate to $\frac{1}{4(2l+1)}$, because this expression is the main term in the expansion, i.e. the parameters γ and g are coupled with the isospin l. Accordingly, the parameters q will be expressed in terms of the parameters l and κ.

The obtained results are summarized in Table 1. To avoid cumbersomeness, some constant parameters are given in Table 2.

Note that the calculation of $\gamma_j = \omega_j \bmod 2$ is not defined well for fractional and negative numbers. Therefore, the following calculation method was used in this work:

$$\gamma_j = \omega_j \bmod 2 =$$
$$= (\lfloor\omega_j\rfloor \bmod 2) + (\omega_j - \lfloor\omega_j\rfloor),$$

where the notation $\lfloor x\rfloor$ means the largest integer not exceeding x.

Note also that if the values of the parameter k are large, the second factor in the expression for the second virial coefficient (2.8) has to approach unity, like the parameter of statistics incompleteness (non-additivity) q, which can be seen from Table 1.

As an example, let us consider the non-additive Haldane–Wu statistics. At large k-values, the value of ω_j is small, as well as the value of γ_j. Therefore, the quantity q can be represented as $q \approx 1 + \Delta q$, and

$$\frac{2q^2}{1+q} = \frac{2(1+\Delta q)^2}{1+(1+\Delta q)} \simeq \frac{2(1+2\Delta q)}{2+\Delta q} =$$
$$= \frac{2(1+2\Delta q)}{2(1+\frac{\Delta q}{2})} \simeq 2(1+2\Delta q)\frac{1}{2}\left(1-\frac{\Delta q}{2}\right) \simeq$$
$$\simeq 1 + 2\Delta q - \frac{\Delta q}{2} \to 1.$$

Let us take into account that, in the limit of large κ, the values of ω_j and γ_j are small. Therefore, in this limit, the expression for Δq in terms of the parameters κ and l can be simplified to the form

$$\frac{3}{2}\Delta q = \frac{1}{4\pi\kappa(2l+1)}\sum_{j=0}^{2l}(2j+1) \times$$
$$\times [2l(l+1) - j(j+1)]\left(1+(-1)^{j+2l}\right). \tag{4.1}$$

5. Conclusions

In this work, the relationship between the parameters of the second virial coefficient of non-Abelian anyons, on the one hand, and the two-parametric incomplete and non-additive modifications of the Haldane–Wu and Polychronacos statistics is demonstrated. The expressions connecting the parameters of those fractional statistics with the parameters of non-Abelian anyons on the basis of the second virial coefficient are obtained. Numerical values of the parameters at which non-Abelian anyons can be approximately described with the use of the fractional statistics of the indicated types are calculated.

Note that the two-parametric statistics can be used to model non-Abelian anyons with a soft core. The corresponding analytic expressions will be very cumbersome, and the relationships between the parameters can be found only numerically.

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ЗВ'ЯЗОК ПАРАМЕТРІВ ДРУГОГО ВІРІАЛЬНОГО КОЕФІЦІЄНТА НЕАБЕЛЕВИХ ЕНІОНІВ З ДВОПАРАМЕТРИЧНИМИ ДРОБОВИМИ СТАТИСТИКАМИ

У цій роботі показано зв'язок між параметрами другого віріального коефіцієнта для системи неабелевих еніонів та двопараметричними модифікаціями дробових статистик Голдейна–Ву та Поліхронакоса. Розраховано параметри, для яких неабелеві еніони можуть описуватись даними типами статистик. Розглянуто границю, в якій параметр неадитивності/неповноти q прямує до одиниці.

Ключові слова: віріальний коефіцієнт, неабелеві еніони, неадитивна/неповна двопараметрична статистика, дробова статистика Голдейна–Ву, дробова статистика Поліхронакоса.