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DYNAMICS OF THE GENERAL BIANCHI IX MODEL NEAR A COSMOLOGICAL SINGULARITY

Half a century ago, Belinsky and Khalatnikov proposed a generic solution of the Einstein equations near their cosmological singularity, basing on a generalization of the homogeneous model of Bianchi type IX. The consideration of the evolution of the most general non-diagonal case of this model is significantly simplified, if it is assumed that, when approaching the singularity $t = 0$, it reduces to the so-called asymptotic dynamics, at which inequality $\Gamma_1 \gg \Gamma_2 \gg \Gamma_3$ holds. It has been suggested that this inequality continues to be true from the moment of its first fulfilment up to the singularity of space-time. We analyze this assumption and show that it is incorrect in the general case. However, it is shown that in any case there exists a time t_0, after which this assumption becomes true. The value of t_0 is the smaller, the less is the degree of non-diagonality of the model. Some details of the behavior of the non-diagonal homogeneous model of Bianchi type IX are considered at the stage of asymptotic dynamics of approaching the singularity.

K e y w o r d s: general relativity, cosmology, singularity, general solution.

1. Introduction

The Belinskii–Khalatnikov–Lifshitz (BKL) conjecture is thought to be a generic solution to the Einstein equations near a spacelike cosmological singularity [1]. The generic solution means, roughly speaking, a solution that is stable against perturbations of the initial conditions defining the dynamics. The set of these conditions is of nonzero measure in the space of all possible initial conditions. The general solution should include the maximal possible number of "physically arbitrary" functions of all three space coordinates, i.e. four functiones for the vacuum space and eight ones for a space filled with matter fields [2, 3].

The considered problem is closely related to the general problem of the existence of space-time singularities. On one hand, the existence of such solutions may mean that there are some intrinsic problems in general relativity, as it is believed that a physical theory should be free of singularities. On the other hand, the singularity theorems argue that singularities are integral parts of general relativity (see [4] and references therein). It is commonly expected that the quantization of general relativity may "heal" the classical singularities.

Quantization of the BKL scenario should be preceded by the quantization of the Bianchi IX model. It seems to be a natural strategy as the BKL scenario was obtained by analyzing the dynamics of the Bianchi IX spacetime [5]. For analyses of the dynamics of the Bianchi IX universe carried out within the method of dynamical systems, we recommend [6] (diagonal case of the space 3-metric) and [7] (nondiagonal case). Quantization of gravitational systems is known to be quite complicated due to the nonlinearity of the dynamics and the existence of dynamical constraints (relations among degrees of freedom). One usually tries to identify special cases minimizing these difficulties. In the context of cosmological singularities, one tries to find the dynamics that addresses just the singularity problems without additional complications.

As a result, some hope has been associated with the simplified dynamics considered long time ago. The first results were published in 1971 by V.A. Belinskii, I.M. Khalatnikov, and M.P. Ryan in preprint [8]. However, it did not appear in the form of an article, but became a part of paper [9] and the basis for some further research and contains a number of important conclusions about the asymptotical dynamics of the system [10, 11].

We see that this work is related to a problem that arose half a century ago and has not yet been solved due to its complexity. During this time, the concepts of dark energy (DE) and the associated accelerated expansion of the Universe have been firmly established in the general theory of relativity. Convincing evidence for the existence of DE is the dependence of the redshift and photometric distance for type Ia supernova explosions. However, the existence of DE either in the form of a cosmological constant or in the form of a dynamic DE can affect the generic solution near singularities. This can be seen in the example of singularities, which are the source of not only gravitational, but also scalar fields [12,13]. In this article, we do not consider the effect of DE, limiting ourselves to the influence of matter and radiation. We can say that we consider the problem in the very form in which it was posed 50 years ago. Note that it remains relevant in some models of dynamic DE, in which its influence disappears near the singularity.

The aim of this paper is the study of some details of the asymptotical dynamics including the correctness of the assumption, on which the paper [9] is based. In the next section, we recall, following [8–10], the way for the dynamics of the general (non-diagonal) Bianchi IX universe to transform into the asymptotic dynamics. It is based on making a special assumption concerning some of the terms defining the dynamics. Afterwards, we study the asymptotical dynamics and investigate whether this assumption is correct in the sense that it can be satisfied in a continuous way in the evolution of the system from the moment of its imposition until approaching the singularity. This determines whether the simplified system of equations can represent the general dynamics in the asymptotic regime near the singularity.

The BKL solution was extended later to the timelike singularity [14], and merged to the general scenario including both singularities [15]. There are some differences between both cases, but the general dynamics near timelike and spacelike singularities is the same [16]. Here, we will consider only a generic solution near a spacelike cosmological singularity, but its result could be easy transferred to the case of a timelike one.

2. Exact Dynamics and Its Asymptotic Form

In what follows, we use the terminology and notation of Refs. [5, 9]. The general form of a line element of the nondiagonal Bianchi IX model in the synchronous reference system is

$$ds^2 = dt^2 - \gamma_{ab}(t)\mathbf{e}^a_\alpha \mathbf{e}^b_\beta dx^\alpha dx^\beta, \tag{1}$$

where the Latin indices $a, b, ...$ run from 1 to 3 and label frame vectors, and the Greek indices $\alpha, \beta, ...$ take values $1, 2, 3$ and concern the space coordinates, and γ_{ab} is a spatial metric.

The homogeneity of the Bianchi IX model means that the three independent differential 1-forms $e^a_\alpha dx^\alpha$ are invariant under the transformations of the isometry group of the Bianchi IX model, i.e. $e^a_\alpha(x)dx^\alpha = e^a_\alpha(x')dx'^\alpha$, so e^a_α have the same functional form in the old and new coordinate systems. Each $e^a_\alpha dx^\alpha$ cannot be presented in the form of a total differential of a function of coordinates. Three frame vectors $\mathbf{e}^a$ of the Bianchi type IX homogeneous space are presented in [3, 17] in the form

$$\begin{aligned} \mathbf{e}^{(1)} &= \mathbf{l} = (\sin z,\ -\cos z \sin x,\ 0),\\ \mathbf{e}^{(2)} &= \mathbf{m} = (\cos z,\ \sin z \sin x,\ 0),\\ \mathbf{e}^{(3)} &= \mathbf{n} = (0, \cos x, 1). \end{aligned} \tag{2}$$

The spatial metric $\gamma_{ab}(t)$ is a 3×3 matrix with all nonvanishing elements to be determined. We put

$$\boldsymbol{\gamma} = \mathbf{O}^{-1}\boldsymbol{\Gamma}\mathbf{O}, \tag{3}$$

where $\mathbf{O}(t)$ is the orthogonal matrix parametrized in the standard way by the Euler angles ψ, θ, φ and $\boldsymbol{\Gamma}(t) = \mathrm{diag}(\Gamma_1(t), \Gamma_2(t), \Gamma_3(t))$. So, the matrix γ_{ab} is comletely described by six time-dependent parameters $\psi, \theta, \varphi, \Gamma_1, \Gamma_2, \Gamma_3$.

We examine an approximate solution of the Einstein equations for the Bianchi IX model with the metric (1) near the singularity which corresponds to $t = 0$. We can restrict ourselves to the vacuum case, because the energy-momentum tensor components for a hydrodynamically moving matter give a negligible contribution to the dynamics as $t \to 0$. The influence of DE in the general case can be significant, but we assume that it can be neglected.

It was shown [5] that, near the cosmological singularity, the *general* form of the metric γ_{ab} should be considered. Consequently, one cannot diagonalize the metric for all values of time. After making use of the Bianchi identities, freedom in the rotation of the metric γ_{ab} and frame vectors e^a_α, one arrives at the system of equations specifying the dynamics of the nondiagonal Bianchi IX model.

They have simpler form, if we redefine the cosmological time variable t as follows: $dt = \sqrt{\gamma}\, d\tau$, where γ denotes the determinant of γ_{ab}. However, the equations including $\dot\psi, \dot\theta, \dot\varphi, \ddot\Gamma_1, \ddot\Gamma_2, \ddot\Gamma_3$, where "dot" denotes $d/d\tau$, are still quite complicated. The explicit form of these equations is presented in [9] [see Egs. (2.14)–(2.20)] or in [10] [see Eqs. (34)–(38)].

These equations include the constant C which is important for our purposes. So, we give its definition. One can get, from the Einstein equations [9], the condition for the tensor $\kappa_a^b = \gamma^{bc}\dot\gamma_{ac}$ in the form

$$\kappa_a^b \varepsilon^a{}_{bc} = C_c. \tag{4}$$

Here ε_{abc} is the Levi-Civita symbol, and C_c are three constants which can be considered as components of a three-dimensional vector. Choosing the third coordinate axis along the direction of this vector, we get

$$C_1 = C_2 = 0, \quad C_3 = C. \tag{5}$$

The diagonal case corresponds to $C = 0$ and to the absence of rotation, i.e. $\dot\varphi = \dot\theta = \dot\psi = 0$.

Belinsky, Khalatnikov, and Ryan [8] proposed a conjecture, which greatly simplifies the general form of the dynamics near the singularity, if one accepts that whenever an inequality $\Gamma_1 > \Gamma_2 > \Gamma_3$ holds true, it becomes a strong one:

$$\Gamma_1 \gg \Gamma_2 \gg \Gamma_3 \tag{6}$$

(the order of indices in (6) is unimportant and depends on the initial conditions). Our primary goal is to verify the validity of such a conjecture.

If condition (6) is satisfied, θ, φ, and ψ tend to some constant values θ_0, φ_0, and ψ_0 near the singularity. The set of equations is substantially reduced to (see, [8, 9] for more details)

$$\begin{aligned} \frac{d^2 \ln a}{d\tau^2} &= \frac{b}{a} - a^2, \\ \frac{d^2 \ln b}{d\tau^2} &= a^2 - \frac{b}{a} + \frac{c}{b}, \\ \frac{d^2 \ln c}{d\tau^2} &= a^2 - \frac{c}{b}, \end{aligned} \tag{7}$$

where a, b, c are functions of time τ, satisfying the constraint:

$$\begin{aligned} &\frac{d\ln a}{d\tau}\frac{d\ln b}{d\tau} + \frac{d\ln a}{d\tau}\frac{d\ln c}{d\tau} + \frac{d\ln b}{d\tau}\frac{d\ln c}{d\tau} = \\ &= a^2 + \frac{b}{a} + \frac{c}{b}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} &a := \Gamma_1, \quad b := \Gamma_2 C^2 \cos^2\theta_0, \\ &c := \Gamma_3 C^4 \sin^2\theta_0 \cos^2\theta_0 \sin^2\psi_0. \end{aligned} \tag{9}$$

We assume $\sin\theta_0 \neq 0$, $\cos\theta_0 \neq 0$ and $\sin\psi_0 \neq 0$. Note that the definition of b and c includes the parameter C and the diagonal case corresponds to $C = 0$. Thus, there is no simple reduction of this nondiagonal case to the diagonal one.

Belinsky, Khalatnikov, and Ryan [8, 9] argue that condition (6) is satisfied during the evolution of the system toward the cosmological singularity. Therefore, in the asymptotic regime near the singularity, the spatial 3-metric can be determined by the solution to Eqs. (7)–(8) without any loss of generality.

3. The Asymptotic Dynamics near the Singularity

Introducing new variables $\xi = \ln(a^2)$, $\eta = \ln(b/a)$ and $\zeta = \ln(c/b)$, we get Eqs. (7) in the form

$$\ddot\xi = 2(e^\eta - e^\xi),\ \ddot\eta = 2(e^\xi - e^\eta) + e^\zeta,\ \ddot\zeta = e^\eta - 2e^\zeta. \tag{10}$$

Naturally, the set of equations (10) is much simpler than the general one, but it requires condition (6) to be valid. Is this condition carried out all the time? From (6) and (9), we have the relations

$$\begin{aligned} &\Gamma_1/\Gamma_2 = C^2\cos^2\theta_0 (a/b) = C^2 \cos^2\theta_0 e^{-\eta} \gg 1, \\ &\Gamma_2/\Gamma_3 = C^2 \sin^2\theta_0 \sin^2\psi_0 e^{-\zeta} \gg 1. \end{aligned} \tag{11}$$

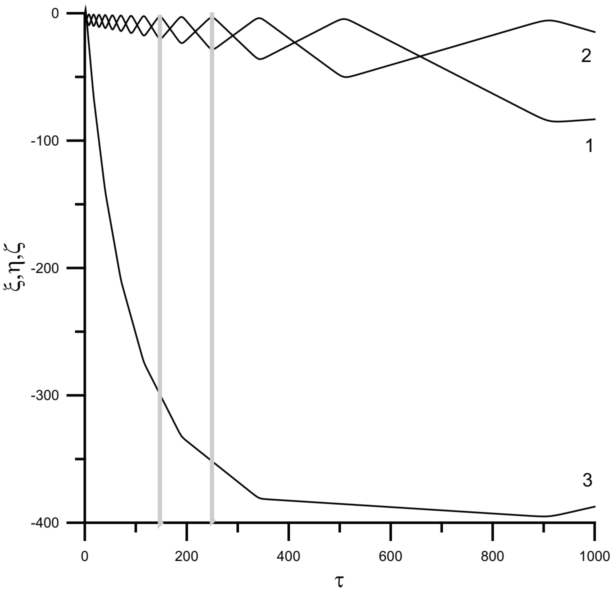

Fig. 1. An example of the dynamics of $\xi = \ln(a^2)$ (line *1*), $\eta = \ln(b/a)$ (line *2*) and $\zeta = \ln(c/b)$ (line *3*) as functions of τ. Two vertical grey lines correspond to two successive maxima of ξ

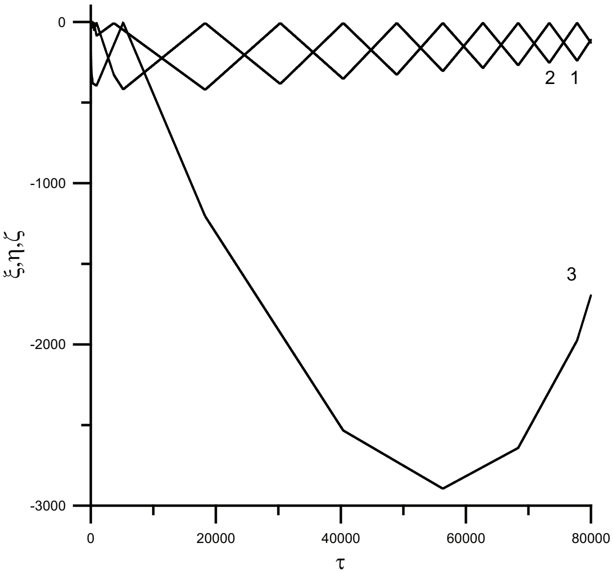

Fig. 2. The same as Fig. 1, but on a larger time interval

The only way to satisfy inequalities (11) over the entire range of η and ζ is to require

$$C \gg \cos^{-1}\theta_0 e^{\eta_{\max}/2},$$
$$C \gg \sin^{-1}\theta_0 \sin^{-1}\psi_0 e^{\zeta_{\max}/2}, \tag{12}$$

where $\eta_{\max}$ and $\zeta_{\max}$ are the maximal values of η and ζ. Inequality (12) sets the lower limit for the value of C.

We can also consider (12) as a restriction on the maximal values of the functions η and ζ. Let us choose certain values η_0 and ζ_0 depending on C, θ_0, and ψ_0 in such a way that inequality (11) is fulfilled at $\eta = \eta_0$, $\zeta = \zeta_0$ but violated at larger values of η or ζ. This reduces inequality (6) to the conditions

$$\eta < \eta_0, \quad \zeta < \zeta_0, \tag{13}$$

which have to be satisfied during the evolution until the singularity. So, we are interested in the dependence of η and ζ on τ.

In Fig. 1, we plot results of the numerical solution of (10) with rather arbitrarily chosen initial values and derivatives satisfying condition (8). This is only an illustration of the dynamics of functions ξ, η, and ζ. Next, we'll study it analytically. In the meantime, we note that the maxima of all functions are achieved at values close to zero. However, the local minimum of ζ is achieved at $\zeta \approx -400$. But this is not the global one. If we extend the plot to larger τ region, we obtain Fig. 2 with values of ζ below –1900. The value of $\exp(\zeta)$ varies by more than 850 orders in this plot. Should have our calculations continued, we would find a local minimum lower that the first one, and the function ζ would start to increase.

If we use the condition (13) alone and choose $\zeta_0 = -250$, then it starts to fulfil at $\tau \approx 100$ and continues to be valid until $\tau \approx 3000$, after which condition (6) is violated. This means that the simplified system of equations (7)–(8) cannot represent the general dynamics at $\tau > 100$.

Can these conditions be fulfilled until the singularity? We need some analitycal study to answer this question.

The functions $a(\tau), b(\tau)$, $c(\tau)$ and $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$ undergo complex oscillations which cannot be described analytically with all details. An evolution toward the singularity takes a finite interval of cosmological time t, but an infinite interval of the evolution parameter τ. An infinite number of oscillations occur during this time interval. They are separated by the so-called Kasner epochs, when the space-time is similar to the well-known Kasner metric [18]

$$\mathrm{d}s^2 = \mathrm{d}t^2 - t^{2p_1}\mathrm{d}x^2 - t^{2p_2}\mathrm{d}y^2 - t^{2p_3}\mathrm{d}z^2, \tag{14}$$

where Kasner indices p_i satisfy the conditions

$$p_1 + p_2 + p_3 = 1, \quad {p_1}^2 + {p_2}^2 + {p_3}^2 = 1. \tag{15}$$

Thus, one of them is negative and other two are positive. More precisely, they are subject to the inequalities

$$-\frac{1}{3} \leq p_{\min} \leq 0 \leq p_{\text{mid}} \leq \frac{2}{3} \leq p_{\max} \leq 1, \tag{16}$$

where the Kasner indices p_i are ranked in value. The index $p_{\max}$ is the largest one, $p_{\min}$ is the minimal one, which is always negative, and p_{mid} lies between them. Kasner epochs correspond to almost linear sections in Figs. 1 and 2.

Consider a region of space bounded by the borders with constant values of the space coordinates x^α. Its volume $dV = \gamma^{1/2}dx^1dx^2dx^3 \propto abc$ depends on τ. Let us introduce the function

$$Q(\tau) = \frac{d\ln V}{d\tau} = (\ln(a))^\cdot + (\ln(b))^\cdot + (\ln(c))^\cdot = \\ = \frac{3}{2}\dot{\xi} + 2\dot{\eta} + \dot{\zeta}. \tag{17}$$

It satisfies the equation

$$\dot{Q} = e^\xi. \tag{18}$$

The function $Q(\tau)$ for the functions ξ, η and ζ from Fig. 1 is plotted in Fig. 3. One can see that it looks like stairs with flat steps. We denote its value at steps by Λ_i, where i is the number of step, starting with 0. We do not *a priori* know the sign of Λ_0, which depends on initial values of $\dot{\xi}, \dot{\eta}, \dot{\zeta}$ at $\tau = 0$. We can choose the direction toward the singularity from two possibilities, namely the directions along or opposite to the τ increase. The natural choice is the direction in which V decreases, so we get $V \to 0$ at $t \to 0$. If $Q(0) > 0$ or in different words $\Lambda_0 > 0$, $\tau \to -\infty$. If $\Lambda_0 < 0$, $\tau \to \infty$.

Let's apply the approach successfully used in the papers [1, 3]. On many intervals of τ variation, the right-hand sides of the equations (10) and (18) can be neglected in comparison with the main terms in the left-hand sides. These are the so-called Kasner epochs. During each Kasner's epoch $\ln(a)$, $\ln(b)$ and $\ln(c)$ are linear functions of τ:

$$Q = \Lambda_i, \quad \xi = K_i\tau + \text{const}, \quad \eta = L_i\tau + \text{const}, \\ \zeta = M_i\tau + \text{const}, \quad \ln(a) = A_i\tau + \text{const}, \\ \ln(b) = B_i\tau + \text{const}, \quad \ln(c) = D_i\tau + \text{const}. \tag{19}$$

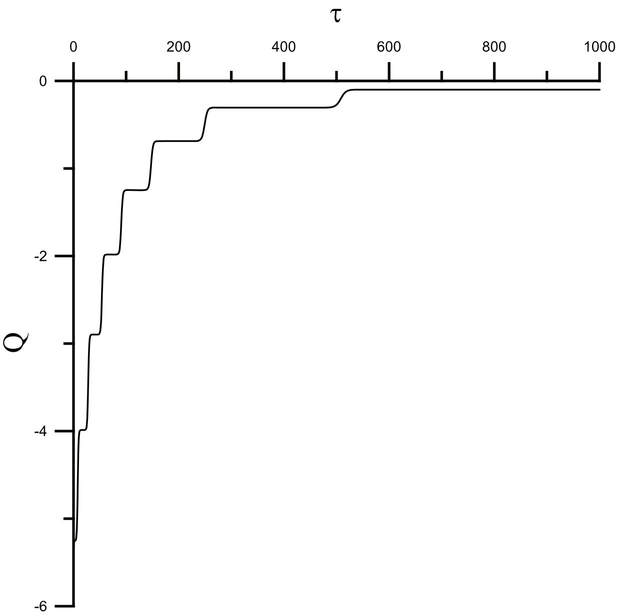

Fig. 3. The function $Q(\tau)$ for the functions ξ, η, ζ from Fig. 1

Here i be a number of maxima of functions $\xi(\tau), \eta(\tau), \zeta(\tau)$ between the current and the initial values of τ. This number is used as an index numbering consecutive Kasner epochs. Each of them is characterized by its own set of Kasner indices. The rule for changing the indices for adjacent Kasner epochs and other details are described in [9, 10].

It is clear that

$$K_i = 2A_i, \quad L_i = B_i - A_i, \quad M_i = D_i - B_i, \\ \Lambda_i = A_i + B_i + D_i = 1.5K_i + 2L_i + M_i. \tag{20}$$

For the Kasner epoch, the dependence of a, b, c on time is similar to metric (14), so for it $\tau = \ln(t) + \text{const}$. From (14) and (17) we can express

$$A_i = \Lambda_i\,(p_1)_i, \quad B_i = \Lambda_i\,(p_2)_i, \quad D_i = \Lambda_i\,(p_3)_i, \tag{21}$$

where $(p_1, p_2, p_3)_i$ is the i-th set of Kasner indices p_1, p_2, p_3 satisfying the conditions (15).

Kasner epochs are separated by transition epochs, in which one of the functions $\exp(\xi(\tau))$, $\exp(\eta(\tau))$, or $\exp(\zeta(\tau))$ entering the right-hand sides of the equations (10) and/or (18) increases so much that it can no longer be neglected. Indeed, among the terms on the right side of (10) in a general case there is one that increases faster than the terms on the left side when approaching the singularity. However, one can neglect the rest of the terms in the right-hand sides and solve the resulting system of equations. In all cases, the function that plays the role of perturbation in the right-hand sides first increases, reaches its maximum value, then begins to fall, and the solution passes into the next Kasner epoch. The change of epochs is clearly visible in Figs. 1 and 2. Transition epochs correspond to the areas of τ variation around the maxima of one of three functions ξ, η, ζ.

Having a basic understanding of the character of the asymptotic dynamics near the singularity we can consider the behavior of these functions during transition epochs. We consider three different cases separately. The approximate solutions near each maximum have the following forms, where $S, K, L, M = $ $=$ const.

Near a maximum of $\xi(\tau)$ at $\tau = T_\xi$ we get

$$\begin{aligned} &\xi = 2\ln(S) - 2U(\tau),\ \eta = 2U(\tau) + L\tau + \mathrm{const}, \\ &\zeta = M\tau + \mathrm{const},\ U(\tau) = \ln(\cosh(S(\tau - T_\xi)), \\ &Q = \mathrm{const} + S\tanh(S(\tau - T_\xi)), \end{aligned} \tag{22}$$

Near a maximum of $\eta(\tau)$ at $\tau = T_\eta$ we get

$$\begin{aligned} &\eta = 2\ln(S) - 2U(\tau),\ \xi = 2U(\tau) + K\tau + \mathrm{const}, \\ &\zeta = U(\tau) + M\tau + \mathrm{const}, \\ &U(\tau) = \ln(\cosh(S(\tau - T_\eta))),\ Q = \mathrm{const}, \end{aligned} \tag{23}$$

Near a maximum of $\zeta(\tau)$ at $\tau = T_\zeta$ we get

$$\begin{aligned} &\zeta = 2\ln(S) - 2U(\tau), \quad \xi = K\tau + \mathrm{const}, \\ &\eta = U(\tau) + L\tau + \mathrm{const}, \quad Q = \mathrm{const}, \\ &U(\tau) = \ln(\cosh(S(\tau - T_\zeta))). \end{aligned} \tag{24}$$

We started from the assumption that we can neglect all terms in the right-hand side of (10) except one containing a function of the variable which reaches a maximum. It is easy to verify that this condition is satisfied for each of the solutions (22)–(24) during the transition epoch. After its completion $U(\tau)$ reaches a linear asymptotic and the system switches to the next Kasner epoch. Matching the solutions (22)–(24) with the solutions (19) at both sides of the maximum we obtain the rules of changing of K_i, L_i, M_i, Λ_i on passing through transition epochs in the forms

$$\begin{aligned} &K_{i+1} = -K_i,\ L_{i+1} = L_i + 2K_i,\ M_{i+1} = M_i, \\ &\Lambda_{i+1} = \Lambda_i + K_i = \Lambda_i(1 + 2(p_1)_i), \end{aligned} \tag{25}$$

$$K_{i+1} = K_i + 2L_i,\ L_{i+1} = -L_i,\ M_{i+1} = M_i + L_i, \tag{26}$$

$$K_{i+1} = K_i,\ L_{i+1} = L_i + M_i,\ M_{i+1} = -M_i \tag{27}$$

for maxima of ξ, η and ζ, respectively. So, the function $Q(\tau)$ is constant except the regions of maxima of ξ, where it looks like a step function with the charecteristic tanh-like-like shape. One can see an example of it in Fig. 3. The value of $|Q|$ monotonically decreases to zero when approaching the singularity. This decrease is irregular. Intervals with almost constant $Q = \Lambda_i$ are connected with the sections with a sharp change of Q. These changes are caused by tending τ to maxima of ξ and are described by the hyperbolic tangent $\tanh(S(\tau - T_\xi))$. Its absolute value decreases at maxima of ξ, where $(p_1)_i < 0$, but the sign remains the same. The values of Λ do not change at maxima of η or ζ according to (18). After the n-th such maximum we obtain

$$\Lambda_n = \Lambda_0 \prod_{j=1}^{n} (1 + 2(p_1)_j) \xrightarrow[n\to\infty]{} 0. \tag{28}$$

We can obtain the values $\xi_{\max}$, $\eta_{\max}$ and $\zeta_{\max}$ for the i-th maximum

$$\begin{aligned} &\xi_{\max} = 2\ln(L_i/2) = 2\ln(\Lambda_i(p_1)_i), \\ &\eta_{\max} = 2\ln(L_i/2) = 2\ln(\Lambda_i(p_2 - p_1)_i/2), \\ &\zeta_{\max} = 2\ln(M_i/2) = 2\ln(\Lambda_i(p_3 - p_2)_i/2), \\ &\exp(\eta_{\max}/2)_i \propto \Lambda_i,\ \exp(\zeta_{\max}/2)_i \propto \Lambda_i \end{aligned} \tag{29}$$

and substitute them into (12). Taking into account the decreasing of $|\Lambda|$ at each maximum of ξ we see than the condition (12) is fulfilled near the singularity for any nonzero value of C at $\sin\theta_0 \neq 0$, $\cos\theta_0 \neq 0$ and $\sin\psi_0 \neq 0$.

In terms of inequalities (13) this can be formulated in the following way. The local maxima of η and ζ become smaller as τ increases, and sooner or later these inequalities begin to be satisfied for all subsequent values of τ (we assume that $\Lambda_0 < 0$ and the singularity corresponds to $\tau = \infty$ as in Figs. 1–3). Let us denote the value of τ where this happens as $\tau_0(C, \theta_0, \psi_0)$. The greater the value of C for the same θ_0, ψ_0, the smaller the value of τ_0. So, the inequality (6) is satisfied during the evolution of the set of equations (7, 8) of simplified asymptotical dynamics near the singularity and therefore this assumption is not contradictory in principle. However, as we just demonstrated, this conjecture is only valid if applied within the τ interval between τ_0 and the singularity.

4. Some Relations for the Asymptotical Dynamics near the Singularity

Let us estimate some properties and parameters of the asymptotical dynamics near the singularity. First of all we are interesting in a relationship between the cosmological time t, the new time coordinate τ and the volume $V \propto abc$. Unfortunately, a steplike form of the function $Q(\tau)$ complicates this problem.

Consider one step or plateau of this function. It is located between two adjacent peaks of ξ, which are separated by the interval $\Delta\tau_i$. We mark two such peaks in Fig. 1 by grey lines. During this interval the function ξ decreases from $\xi_{\max}$ to $\xi_{\min}$ and then increases almost to the initial value at the rate (19). We know the value of $\xi_{\max}$ from (29). If we consider the typical situation $\zeta \ll \xi, \eta$, then (10) gives us

$$\ddot{\xi} + \ddot{\eta} = 0, \quad \xi + \eta = R\tau + Y, \quad R, Y = \text{const.} \tag{30}$$

We can find the value of R from (19) and (21). A local minimum of ξ almost coincides with a local maximum of η given by (29). This gives us the value of $\xi_{\min}$. As a result, we get an estimation

$$\Delta\tau_i \approx$$
$$\approx \frac{2\ln(\Lambda_i(p_2+p_1)_i/2) - \Lambda_i(p_2+p_1)_i\tau - Y}{\Lambda_i(p_1)_i} \xrightarrow[\tau\to\infty]{}$$
$$\xrightarrow[\tau\to\infty]{} s_i\tau, \quad s_i = -1 - \left(\frac{p_2}{p_1}\right)_i. \tag{31}$$

Here $s_i = u_i^{-1}$, where $u > 0$ is a parameter used to write Kasner indices in parametric form [3]

$$\begin{aligned} p_1 &= \frac{-u}{1+u+u^2}, \\ p_2 &= \frac{1+u}{1+u+u^2}, \\ p_3 &= \frac{u(1+u)}{1+u+u^2}, \end{aligned} \tag{32}$$

which changes its value when one Kasner era replaces another [3, 9].

In (31) we deal with terms depending on τ and on Λ_i. The latter increases according to (28). Consider the case of very big τ_0, corresponding to small t. In a very rough approximation, $\ln(\Lambda_i)$ depends linearly on i, i.e. the number of maxima of ξ. Assuming that the term proportional to τ is the main one, we get the estimation (31) for very large values of τ. It yields that $\Delta\tau_i \propto \tau$ and increases with τ growth. One can see this in Figs. 1–3. So, our assumption proved correct and $\ln(\Lambda_i)$ grows slower than τ.

After the interval $\Delta\tau_i$ from (31) we have the next step and the value of Q changes by

$$\Delta Q = \Lambda_{i+1} - \Lambda_i = 2(p_1)_i\Lambda_i \tag{33}$$

according to (25). We remind that $(p_1)_i < 0$.

One could approximate the function Q by the sum of Heaviside step functions, but this leads to cumbersome expressions. However, it is possible to get the same qualitative result by using a very rough approximation by replacing each of the steps by the line with a slope $\Delta Q/\Delta\tau$ and λ_i by Q, which yields a much simpler differential equation

$$\dot{Q} = W_i\frac{Q}{\tau}, \quad W_i = \frac{2(p_1)_i}{s_i} = \left(\frac{-2u^2}{1+u+u^2}\right)_i. \tag{34}$$

So $-2 < W_i < 0$. We can very roughly suppose that $W_i = W = \text{const} < 0$ and get $(P, D = \text{const})$

$$Q \approx P\tau^W, \quad V \propto \exp(D\tau^{1+W}), \quad D = \frac{P}{1+W}. \tag{35}$$

One can regard W as a kind of a mean value of W_i, so $-2 < W < 0$. We see that, at $-1 < W < 0$, the volume V vanishes at singularity and at $-2 < W < -1$ it tends to some finite value. Only the first case has a physical meaning and we shall consider it. The interval of cosmological time t between a point with coordinate $\tau = \tau_1$ and the singularity at $\tau = \infty$ is equal to

$$\Delta t \propto \int_{\tau_1}^{\infty} \exp(D\tau^{1+W}/2)\, d\tau \tag{36}$$

with $D < 0$. This integral reduces to the incomplete gamma function and is finite at $-1 < W < 0$. So, the singularity corresponds to some finite cosmological time, and we can choose it as the origin of the coordinate t. Near the singularity i.e. for large τ_1 we have the asymptotic behaviour $V \propto t^2$.

Now, let us consider how the value of τ_0 after which the condition (6) is satisfied depends on value of C. According to (12) and (29) we can rewrite this condition in the form

$$C > C_0 = FQ(\tau_0). \tag{37}$$

Here F is a factor depending on θ_0, ψ_0 and also on what we mean by the term "much larger" in (6). So, near the singularity $Q(\tau_0) \propto \tau_0^W \propto C$ and $\tau_0 \propto C^{1/W}$. The value t_0 corresponding to τ_0 is estimated as

$$t_0 \propto \exp(ZC^\omega) \qquad (38)$$

with $\omega = 1 + W^{-1} < 0$ and $Z = \text{const} < 0$. This dependence is very strong for small values of C, i.e. for almost diagonal metrics.

5. Conclusions

The main result of our analysis is is that there exists an instant of time, τ_0, in the *asymptotic* evolution of the *general* dynamics of the Bianchi IX universe toward the singularity at $\tau \to +\infty$, such that, for $\tau > \tau_0$, the strong inequality (6) is satisfied [1]. Therefore, the asymptotic dynamics is *self-consistent*, but for $\tau > \tau_0$.

The value τ_0 depends on the constant C, which characterizes the non-diagonality of the metric. The smaller its value, the greater the value of τ_0. Taking the limit $C \to 0$, which leads to the diagonal case when applied to the exact dynamics of the Bianchi IX model, does not make sense at the level of the asymptotic dynamics.

We are aware that the considerations concerning the asymptotic dynamics might be devoid of any physical meaning because of possible quantum effects. All our analyses make the assumptions that the quantum effect can be neglected. The importance of quantum effects will be examined elsewhere.

Let us list the main findings of this paper:

1. There is an instant of time before the general cosmological singularity in Bianchi IX model, after which a fulfilment of the condition (6) guarantees that it will hold until the singularity is reached.

2. If this condition held before the said instant of time, it will be broken.

3. The smaller the value C, which characterizes the non-diagonality of the metric, the closer this instant of time is to the singularity. Note that the special case of the diagonal metric ($C = 0$) cannot be obtained from this solution as a limit case.

4. The asymptotical dynamics is thus applicable only within the time interval (38) between the said instant of time and the singularity.

5. There is no guarantee that a generic BKL solution will necessarily evolve into asymptotical dynamics, although its stochastic nature makes it a likely scenario.

It is sufficint to validate the correctness of the asymptotical dynamics in classic GRT. It is worth recalling that we are based on Einstein's equations and do not take into account the influence of DE, the scalar field, if any, and quantum effects near the singularity. In addition, let us note that the time interval t_0 could be of the same order or less than Planck unit of time if the metrics is "almost diagonal" and C is small enough. In this case this approximation is nonphysical. One must take this possibility into account when trying to quantize the Bianchi IX model.

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[1] One can choose the initial conditions for the dynamics in such a way that the singularity occurs at $\tau \to -\infty$, in which case, condition (6) is fulfilled, but for $\tau < \tau_0$.

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ДИНАМІКА ЗАГАЛЬНОЇ
МОДЕЛІ ТИПУ IX ЗА БІАНКІ
ПОБЛИЗУ КОСМОЛОГІЧНОЇ СИНГУЛЯРНОСТІ

Півстоліття тому Бєлінський та Халатников запропонували загальний розв'язок рівнянь Ейнштейна поблизу їхньої космологічної сингулярності, який базується на узагальненні однорідної моделі типу IX за Біанкі. Розгляд еволюції у найбільш загальному недіагональному випадку цієї моделі значно спрощується, якщо припустити, що при наближенні до сингулярності $t = 0$ вона зводиться до так званої асимптотичної динаміки, коли виконується нерівність $\Gamma_1 \gg \Gamma_2 \gg \Gamma_3$. Було висунуто припущення про те, що ця нерівність продовжує бути вірною від моменту її першого виконання аж до сингулярності простору-часу. У статті аналізується це припущення і показується, що це не так у загальному випадку. Однак показано, що завжди існує момент часу t_0, після досягнення якого це припущення стає вірним. Величина t_0 тим менша, чим менший ступінь недіагональності моделі. Розглянуто деякі деталі поведінки однорідної недіагональної моделі типу IX за Біанки на етапі асимптотичної динаміки наближення до сингулярності.

Ключові слова: загальна теорія відносності, космологія, сингулярність, загальний розв'язок.