https://doi.org/10.15407/ujpe67.2.117

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MIRROR SYMMETRY AS AN OPERATOR ALGEBRA IN THE NONCOMMUTATIVE SPACE-TIME GEOMETRY

The analysis of the geometric and algebraic properties of mirror mappings allowed the latter to be used as the operator algebra of a noncommutative geometry. The coordinates of the noncommutative geometry are auto- or cross-correlation coordinates in the mirror-mapped spaces. A particular case of the six-dimensional Kähler manifold which is mapped on the noncommutative geometry with the vector Clifford algebra Cl_4 has been considered. This mapping corresponds to a tetraquark composed from two quark–anti-quark pairs with the charges $\pm\frac{2}{3}q$ taken from different generations.

Keywords: mirror symmetry, noncommutative geometry, Clifford algebra, correlation.

1. Introduction

Recent decades have been marked by substantial advances reached in the application of geometric methods to physics. First of all, it concerns the appearance of the string theory of space-time [1, 2]. In the framework of this theory, the mirror symmetry of the geometric parameters of the Calabi–Yau spaces was discovered [3], which considerably simplified the obtaining of the solutions of the approximate equations of the theory in the mirror approach, whereas the physical content of the results remained the same. Despite the impressive results, the string theory still remains incomplete, because nobody knows which of the vast number of possible six(or more)-dimensional spaces corresponds to the physics of our space-time. However, the fathers of string theory do not lose their optimism and hope for that the further progress is possible by means of replacing the conventional geometry with a new apparatus, the noncommutative geometry.

The fundamentals of the noncommutative geometry have been developed by Alain Connes [4] for quantum fields. He used an operational algebra to describe the geometry. However, in this approach, some difficulties concerning the primordial state description still remain unresolved [5].

The aim of this work was to study the methods by means of which the mirror symmetry can map a six-dimensional complex space into a noncommutative geometry, as well as to find which physical processes can correspond to this procedure. The results of the work can be applied in nuclear physics, quantum mechanics, quantum electrodynamics, string theory, gravitation theory, and astrophysics.

2. Geometry and Algebra of Mirror Mapping

2.1. Fundamentals of the geometry

Mathematics gives us examples of the mirror symmetry. Positive and negative numbers form mirror pairs of the sinverse symmetry with respect to zero. A complex number z and the conjugate one z^* form a mirror pair with respect to the real axis. A combination of the inverse and permutation symmetries with respect to the unit circle is demonstrated by the mirror pair z and $1/z^*$.

In geometry, historically, let us proceed *a posteriori.* Figure 1 demonstrates mirror mappings (mirror symmetries) of a plane located in a coordinate frame with the orthogonal basis vectors **1** and **2**. The orientation φ of the plane "mirror" is the multiple of a rotation by an angle of 45°, i.e., $\varphi = k\pi/4$, where $k = 0, 1, 2, 3$. If $\varphi \geq \pi$, the reflection in the "mirror" has no sense. The discrete orientation of the "mirrors" and the pronounced symmetries of the initial and reflected basis vectors are interrelated and correspond to different properties of the mirror mapping, the in-

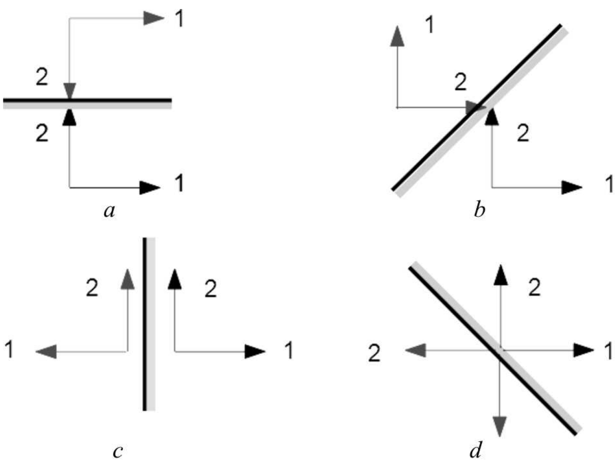

Fig. 1. Mirror mappings of oriented planes

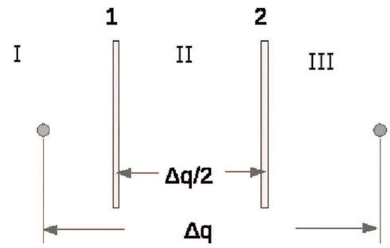

Fig. 2. "Translation" by means of two mirrors

verse and permutation symmetries. In addition, the reflected mirror symmetries (see Figs. 1, *c* and *d*) are opposite to the symmetries shown in Figs. 1, *a* and *b*. Note that, for opposite symmetries, the orientations of the "mirrors" are orthogonal in the phase space of their rotation.

There is another, rather unusual geometric symmetry, which can be called "translation". This symmetry consists of a kit of two plane mirrors separated by the "distance" $\triangle q/2$ (Fig. 2). It has some specific features. The sizes of zones I–III, where this symmetry is valid, are identical and equal to $\triangle q/2$. An arbitrary point in zone I can be transferred by the distance $\triangle q$ to turn out in zone III. An unusual character of this symmetry consists in the dimension of the physical quantity $\triangle q$. Indeed, let us define a 2-dimensional unit vector in zone I. In polar coordinates, it is described by a single parameter $\triangle q = \varphi$. Assuming that $\varphi = \alpha\pi/2$ ($\alpha = 0, 1, 2, 3$) in zone II, we obtain another geometric representation for Figs. 1, *a* to *d*.

This is a rather trivial result, which illustrates that the same result can be achieved by rotating either the "mirror" or the reflected coordinate system (by rotating a unit vector in the fixed coordinate system). However, there is a significant difference between those rotations: the vector rotates by the angle $\triangle\varphi$, whereas the "mirrors" must be rotated by the angle $\triangle\varphi/2$. The situation becomes complicated, if the coordinates (r, φ) are determined in zone I, and they can be reduced to the previous ones, if the first mirror is "normalized" by $1/r^2$. However, a manifold with the coordinates $(1/r, \varphi)$ is mapped in zone III in this case. This example describes one of the variants of the appearance of the so-called T-duality in the string theory, which was considered in work [6]. Finally, one can imagine a scenario with generalized "mirrors", the distance between which is measured in action units h, where h is the Planck constant. In this case, Fig. 2 is a good illustration of the uncertainty relation.

2.2. Properties of mirror-image operators

The symmetry of vectors in Figs. 1, *a* and *b* can be expressed via the mirror-image operators $\mathbf{s}_1$ and $\mathbf{s}_2$. If the notation in the form of ket and bra vectors, $\mathbf{1} \mapsto |+\rangle$ and $\mathbf{2} \mapsto |-\rangle$, are used for the vectors in Fig. 1, then, immediately from Figs. 1, *a* and *b* follows the action of these operators on the indicated vectors:

$$\begin{aligned} &\mathbf{s}_1|+\rangle = +1\,|+\rangle, \quad &\mathbf{s}_1|-\rangle = -1\,|-\rangle, \\ &\mathbf{s}_2|+\rangle = +1|-\rangle, \quad &\mathbf{s}_2|-\rangle = +1|+\rangle. \end{aligned} \tag{1}$$

The basis vectors $|+\rangle$ and $|-\rangle$ are eigenvectors of the operator $\mathbf{s}_1$ with the eigenvalues $\lambda_\pm = \pm 1$. For normalized orthogonal basis vectors, their scalar product is defined as $\langle\pm|\pm\rangle = 1$ and $\langle\pm|\mp\rangle = 0$.

From the last expression, it is possible to directly obtain a matrix representation for the Hermitian operators of mirror mappings,

$$\mathbf{s}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{s}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

These operators form a rectangular basis of the vector Clifford algebra Cl_2, where the scalar product is defined as the anticommutator so that

$$\mathbf{s}_1 \cdot \mathbf{s}_2 = \frac{1}{2}[\mathbf{s}_1, \mathbf{s}_2]_+ = \frac{1}{2}(\mathbf{s}_1\mathbf{s}_2 + \mathbf{s}_2\mathbf{s}_1) = 0,$$

and the external (vector) product as the commutator so that

$$\mathbf{s}_1 \wedge \mathbf{s}_2 = \frac{1}{2}[\mathbf{s}_1, \mathbf{s}_2] = \frac{1}{2}(\mathbf{s}_1\mathbf{s}_2 - \mathbf{s}_2\mathbf{s}_1) = \mathbf{s}_1\mathbf{s}_2.$$

In the vector algebra Cl_2, the quantity $\mathbf{s}_1\mathbf{s}_2$ is a bivector or an axial vector, and the operators $\pm\mathbf{s}_1\mathbf{s}_2$ determine J-transformations, i.e., rotations by angles $\pm 90°$. In particular, $(\mathbf{s}_1\mathbf{s}_2)\mathbf{s}_1 = -\mathbf{s}_2$ is the left turn, and $(\mathbf{s}_2\mathbf{s}_1)\mathbf{s}_1 = \mathbf{s}_2$ the right one. This operator can also be represented in the form $\mathbf{s}_1\mathbf{s}_2 = i\mathbf{s}_3$, where $\mathbf{s}_3$ complements the basis vectors in Cl_2 to the vector algebra Cl_3. The complete algebra Cl_3 has eight linearly independent basis elements, additionally including the scalar $\mathbf{1}$, three bivectors $\boldsymbol{b}_k = i\mathbf{s}_k = \epsilon_{kij}\mathbf{s}_i\mathbf{s}$, where ϵ_{kij} is the Levi-Civita symbol, and the pseudoscalar $\mathbf{I} = \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3 = i\mathbf{1}$. The Cl_3 algebra possesses a new symmetry, $i\mathbf{1}$, which, for example, transforms the real plane into the purely imaginary one.

Combinations with new symmetries generate a new kind of symmetry, projectors $\mathbf{P}_k$. Let us form the operator $\mathbf{P}_{k\pm} = \frac{1}{2}(\mathbf{1} \pm \mathbf{s}_k)$ with the following properties: $(\mathbf{P}_{k\pm})^2 = \mathbf{P}_{k\pm}$ and $\mathbf{P}_{k+}\mathbf{P}_{k-} = \mathbf{0}$. The result of its action, for example, on the basis vectors looks like $\mathbf{P}_{1+}(|+\rangle + |-\rangle) = |+\rangle$. To put it differently, in the "mirror-world", the projector $\mathbf{P}$ distinguishes (removes) one dimension from the plane.

It should be noted that the combinations of the scalar $\mathbf{1}$ with the mirror-image operators in the algebra Cl_2 made it possible to determine all three known systems of complex numbers depending on the properties of the squared imaginary unit: $\epsilon^2 = 0, \pm 1$ [7]. On the other hand, the geometrical mirror symmetries in Fig. 1 indicate the existence of mirror symmetries generated by the operators $\mathbf{s}_i$ (the ordinary "mirror") and $-\mathbf{s}_i$ (the "anti-mirror"). Here, we may naturally add $\mathbf{1} \mapsto \mathbf{s}_0$ (the "identical mirror") and $i\mathbf{1}$ (the "imaginary mirror"), as well as the bivectors $\boldsymbol{b}_k$, as the elements of J-transformation.

The formalization of mirror mappings using the vector Clifford algebra made it possible to considerably extend our understanding of mirror symmetry. That is why it is expedient to consider the algebra Cl_3 as a generalized formulation of mirror symmetry and consider its properties and the operations in it as a consequence of the extended law of mirror symmetry.

The operator notations $-\mathbf{s}_1$ and $-\mathbf{s}_2$ are not accidental, because the spectral mapping

$$F_1(\psi) = \mathbf{V}\left(\frac{\psi}{2}\right)\mathbf{s}_1\mathbf{V}^{-1}\left(\frac{\psi}{2}\right) \tag{3}$$

is defined for the matrix representation of the diagonal operator $\mathbf{s}_1$, where $\mathbf{V}\left(\frac{\psi}{2}\right)$ is the corresponding rotation matrix and $\mathbf{V}\left(\frac{\psi}{2}\right)\mathbf{V}\left(\frac{\psi}{2}\right) = 1$. For the discrete rotation angles $\psi = \alpha\pi/2$, where $\alpha = 0, 1, 2, 3$, the "spectrum" of $\mathbf{s}_1$ has the form

$$F_1 \Rightarrow \{\mathbf{s}_1, \mathbf{s}_2, -\mathbf{s}_1, -\mathbf{s}_2\}. \tag{4}$$

From whence, it follows that the discreteness of the rotation angle $\triangle\psi = \pi/2$ as an element of J-transformation is associated with the conservation condition for the inverse and permutation symmetries.

The mirror symmetry implies the presence of an initial space and a mapped one (a "mirror world"). The initial space is constructed on the eigenvectors $\varphi^k_{1,2}$ of the mirror-image operators $\mathbf{s}_k$, which, in turn, belong to the three-dimensional Kähler manifold with the components (ξ, η), where $\xi = x + iz$ and $\eta = y - iz$ [7]. Let us introduce a multi-dimensional vector space $\mathcal{H}(x^0, x^1, ...)$ and a vector $|R\rangle$ in it, which is projected onto the plane of eigenvectors of the operators $\mathbf{s}_k$,

$$|R\rangle \Rightarrow |r^k\rangle = r^k_1|\varphi^k_1\rangle + r^k_2|\varphi^k_2\rangle, \tag{5}$$

where $|r^k\rangle$ is the k-representation of the vector $|R\rangle$, and $r^k_n = \langle\varphi^k_n|R\rangle$ are the covariant coordinates of $|R\rangle$ in the basis $|\varphi^k_n\rangle$ of eigenvectors of the operators $\mathbf{s}_k$. For complex coordinates, the Hermitian operators $\mathbf{s}_k$ give real values for the quantities like $G^k_k = \langle r^k|\mathbf{s}_k|r^k\rangle = |r^k_1|^2 + |r^k_2|^2$. The latter expression is the scalar product of the vector $|r^k\rangle$ and the vector $\langle r^k|$ belonging to the conjugate space and mirror-mapped by the operator $\mathbf{s}_k$, $\langle r^k|_{\mathrm{ref}} = \langle r^k|\mathbf{s}_r$. This expression corresponds to the mutual correlation (cross-correlation) of vectors at the mirror mapping.

3. Spectral Representation of Mirror-Image Operators and the Quantization in the "Mirror World"

3.1. Spectral-representation operators

Under the conservation condition for the inverse and permutation symmetries for mirror mappings, the spectral representation of the mirror mapping operator $\mathbf{s}_1$ (4) has to be considered discrete. In this case, the spectral components of $\mathbf{s}_{1,2}$ and $-\mathbf{s}_{1,2}$, which are positive and negative "spectral frequencies", have to be separated in some way. Figure 1 suggests that such a possibility does exist if each pair $(\mathbf{s}_{1,2}, -\mathbf{s}_{1,2})$ of the orthogonal basis vectors generates a (hyper) plane and serves as a basis in a new vector space; in other words, if $(\mathbf{s}_{1,2}, -\mathbf{s}_{1,2}) \Rightarrow \mathbf{e}_{1,2}$.

The discreteness of the "spectrum" can be introduced in the following manner. In expression (3) for the "spectrum" of the diagonal operator $\mathbf{s}_1$, the rotation matrices can be expressed in terms of quaternion and the result can be reduced to the form

$$F_1\{\mathbf{s}_1\} = \exp(-i\mathbf{s}_3\psi)\mathbf{s}_1 = \mathcal{F}_3(\psi)\mathbf{s}_1, \tag{6}$$

where the bivector $\mathbf{s}_1\mathbf{s}_2 = i\mathbf{s}_3$ is defined as a quaternionic imaginary unit, and $\mathcal{F}_3(\psi)$ denotes a spectral transformation operator. Let us express the latter in the form of an integral which performs the required discretization and reflects the action of the $\pm\mathbf{s}$ operators in various spaces,

$$\mathcal{F}_3 = \frac{1}{2\pi}\int\limits_0^\infty d\psi\, \Theta_\alpha(\psi) \otimes \exp(-i\mathbf{s}_3\psi), \tag{7}$$

where $\Theta_\alpha(\psi)$ has the structure of the matrices $\mathbf{s}_\alpha$ $(\alpha = 0, 1, 2, 3)$ consisting of Dirac delta functions $\delta(\psi)$, and $\otimes$ denotes the Kronecker product. Using the operator $\mathbf{s}_0$ in the matrix representation, we obtain one of possible discrete representations for $\Theta(\psi)$ in the form

$$\Theta_0(\psi) = \frac{1}{2}\begin{pmatrix}\delta(\psi - n2\pi) & 0\\ 0 & \delta(\psi - \pi - n2\pi)\end{pmatrix} + \\ + \frac{1}{2}\begin{pmatrix}\delta(\psi - \frac{\pi}{2} - n2\pi) & 0\\ 0 & \delta(\psi - \frac{3\pi}{2} - n2\pi)\end{pmatrix}, \tag{8}$$

where the periodicity of the quaternionic variable is taken into account and $n = 0, 1, 2, \ldots$. For $\Theta_\alpha(\psi)$, the normalization condition looks like

$$N\{\mathbf{s}_0\} = \frac{1}{2\pi}\int\limits_0^\infty d\psi \Theta_0(\psi) \otimes \mathbf{s}_0 = \begin{pmatrix}\mathbf{s}_0 & 0\\ 0 & \mathbf{s}_0\end{pmatrix} = \mathbf{e}_0. \tag{9}$$

Then, from formula (7), we obtain

$$\mathcal{F}_3(n) = \frac{1}{2}\left[\begin{pmatrix}\mathbf{s}_0 & 0\\ 0 & -\mathbf{s}_0\end{pmatrix} + \begin{pmatrix}-\mathbf{s}_1\mathbf{s}_2 & 0\\ 0 & \mathbf{s}_1\mathbf{s}_2\end{pmatrix}\right] \otimes \\ \otimes \exp(-i\mathbf{s}_3\, n\, 2\pi). \tag{10}$$

In this expression, the quantity $\exp(-i\mathbf{s}_3 n2\pi) \equiv \mathbf{s}_0$ and does not affect the form of the matrices. But it shows that a discrete series of operators $\mathcal{F}$ in the $\mathbf{s}_1\mathbf{s}_2$-bivector coordinate is formed. The action of the operator $\mathcal{F}_3$ on $\mathbf{s}_1$ is defined as $\mathcal{F}_3 \otimes \mathbf{s}_1$ and brings about the basis vectors $\mathcal{F}_3 \otimes \mathbf{s}_1 = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)$ and $\mathcal{F}_3 \otimes \mathbf{s}_2 = \frac{1}{2}(-\mathbf{e}_1 + \mathbf{e}_2)$, where the new basis vectors are

$$\mathbf{e}_1 = \begin{pmatrix}\mathbf{s}_1 & 0\\ 0 & -\mathbf{s}_1\end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix}\mathbf{s}_2 & 0\\ 0 & -\mathbf{s}_2\end{pmatrix}. \tag{11}$$

Expressions (8) and (10) illustrate an example of geometric discretization along the bivector axis.

Spectral representation (7) of $\Theta_\alpha(\psi)$ with the matrix $\mathbf{s}_0$ is a discrete analog of the mathematical definition of the spectrum of diagonal matrices and the geometric interpretation of the orthogonality concept for the mirror-symmetry operators $\mathbf{s}_{1,2}$ and $-\mathbf{s}_{1,2}$. In integral (7), the matrix $\Theta_\alpha(\psi)$ can be generalized to all matrices of the basis vectors $\mathbf{s}_\alpha$ in the Cl_3 algebra but the spectral representation is realized only for even α-values, i.e., $\alpha = 0$ and 2. This fact makes it possible to obtain various spectral representations for 4×4 matrices, including those that are applied in quantum mechanics (the Majorana and Dirac representations). Really, for $\Theta_2(\psi)$, the above operations give

$$\gamma_1 = \begin{pmatrix}0 & \mathbf{s}_1\\ -\mathbf{s}_1 & 0\end{pmatrix}, \quad \gamma_2 = \begin{pmatrix}0 & \mathbf{s}_2\\ -\mathbf{s}_2 & 0\end{pmatrix}, \tag{12}$$

where $\gamma_{1,2}$ are the Dirac matrices in the indexing of mirror-symmetry operators. The normalization integral produces a new vector,

$$N_2\{\mathbf{s}_0\} = \frac{1}{2\pi}\int\limits_0^\infty d\psi \Theta_2(\psi) \otimes \mathbf{s}_0 = \begin{pmatrix}0 & \mathbf{s}_0\\ \mathbf{s}_0 & 0\end{pmatrix} = \mathbf{e}_4. \tag{13}$$

The matrices $\gamma_{1,2}$ are bivectors because $(\gamma_{1,2})^2 = -\mathbf{e}_0$. They can be expressed in terms of the new vector $\mathbf{e}_4$ in the form $\gamma_1 = \mathbf{e}_1\mathbf{e}_4$ and $\gamma_2 = \mathbf{e}_2\mathbf{e}_4$.

The quaternion $Q_3 = \exp(-i\mathbf{s}_3\psi)$ determines rotations in the plane $\{\mathbf{s}_1, \mathbf{s}_2\}$ of the basis vectors $\mathbf{s}_1$ and $\mathbf{s}_2$. In work [7], it was shown that the quaternions $Q_1 = \exp(-i\mathbf{s}_1\psi)$ and $Q_2 = \exp(-i\mathbf{s}_2\psi)$ determine rotations in the planes $\{\mathbf{s}_2, \mathbf{s}_3\}$ and $\{\mathbf{s}_3, \mathbf{s}_1\}$, respectively. The application of these rotations in Eq. (7) allows one to obtain additional spectral representations for the mirror-symmetry operators:

$$\mathcal{F}_1 \otimes \mathbf{s}_2 = \frac{1}{2}(\mathbf{e}_2 + \mathbf{e}_3),$$

$$\mathcal{F}_1 \otimes \mathbf{s}_3 = \frac{1}{2}(-\mathbf{e}_2 + \mathbf{e}_3),$$

$$\mathcal{F}_2 \otimes \mathbf{s}_3 = \frac{1}{2}(\mathbf{e}_3 + \mathbf{e}_1),$$

$$\mathcal{F}_2 \otimes \mathbf{s}_1 = \frac{1}{2}(-\mathbf{e}_3 + \mathbf{e}_1).$$

The operators $\mathbf{e}_\alpha$ ($\alpha = 0, 1, 2, 3, 4$) form the basis vectors of the Cl_4 algebra in noncommutative geometry.

3.2. Spectral representation in terms of physical variables

Integral (7), which implements the discrete spectral representation of mirror-symmetry operators, is based on the mathematical and geometric properties of these operators. As a result of application of the spectral representation, there appeared an additional geometric discretization along the bivector axes, which is associated with the periodicity of rotation processes. These discrete properties of noncommutative geometry require that possible physical processes ensuring their emergence should be analyzed. Physical variables appear when factorizing the phase $\psi = \omega t = 2\pi t/T$, where T is the rotation period, or, in the other notation, $\psi = 2\pi Et/h = 2\pi S/h$, where E is the energy of the rotation process, $2\pi E/h = \omega$, S has the action meaning, and the normalizing constant h is assumed to be equal to the Planck constant. In such cases, the integration variables in Eq. (7) change to t or S, and the quantities $1/T$ and $1/h$ transform into the factors $(1/T)^{-1}$ and $(1/h)^{-1}$, respectively, in the operators $\mathbf{e}_\alpha$. By analogy with the quantum-mechanical operators of coordinate, $\mathbf{q}$, and momentum, $\mathbf{p}$, these two types of factorization lead to spectral representations that can be called the "frequency" or the "quantum" representation, respectively.

There are conditions under which those two representations are equivalent. Let us assume that the rotation process itself is a physical process when the exponential factor in Eq. (7) is substituted by the operator $\mathbf{R} = E\exp(-i\mathbf{s}_k\omega t)$, where E is the energy of the process $\mathbf{R}$. In the course of this process, the parameters E and ω vary within certain limits. The condition that the spectral representation is independent of the $\mathbf{R}$-process parameters can be formulated from Eq. (7), in particular, in the form

$$(E_1T_1 - E_2T_2)(\mathbf{e}_1 + \mathbf{e}_2) = 0. \tag{14}$$

Whence it follows that $E_1T_1 = E_2T_2 = \mathrm{const}$. If we put this constant equal to the Planck constant h, then it follows that $E = h\nu$ and $\mathbf{R}$ is a non-classical rotation process. As a basic integral for the spectral representation in terms of physical variables, we adopt the following expression with ωt:

$$\mathcal{F}_k = 2\pi \int_0^\infty dt\Theta_{0,2}(t) \otimes \exp(-i\mathbf{s}_k\omega t). \tag{15}$$

It should be noted that when factorizing the phase $\psi = 2\pi S/h$ and integrating over the variable S, the arguments of the delta-functions take the form $S - \frac{1}{4}mh - nh$, where $m = 0, 1, 2, 3$. This expression shows that $\mathbf{s}_k$ and $-\mathbf{s}_k$ in terms of action differ from each other by half the action quantum, $\frac{1}{2}h$. This fact confirms a possibility of the physical realization of Fig. 2 with $\triangle q/2 = h/2$.

4. Correlation Representation of Mirror Mappings in Noncommutative Geometry

Let us introduce autocorrelation operators $\mathbf{G}^k = |r^k\rangle\langle r^k|$. For normalized vectors $|r^k\rangle$, they have the projector property $(\mathbf{G}^k)^2 = \mathbf{G}^k$. Let us also define the operators $\mathbf{K}^k_{rl...} = \mathbf{G}^k\mathbf{s}_r\mathbf{s}_l...$, which correspond to the mutual correlation (cross-correlation) between the vector $|r^k\rangle$ and the vector $\langle r^k|$ located in the conjugate space and mirror-mapped by the operators $\mathbf{s}_r\mathbf{s}_l...$, i.e., $\langle r^k|_{\mathrm{ref}} = \langle r^k|\mathbf{s}_r\mathbf{s}_l...$. The operators $\mathbf{K}^k_{rl...}$ possess the following property:

$$\mathbf{K}^k_{rl...}|r^k\rangle = p^k_{rl...}|r^k\rangle, \quad p^k_{rl...} = \langle r^k|\mathbf{s}_r\mathbf{s}_l...|r^k\rangle, \tag{16}$$

where $p^k_{rl...}$ are the eigenvalues of the cross-correlation operator $\mathbf{K}^k_{rl...}$. These numbers are scalar products of vectors in the initial and mirror spaces. This condition guarantees the existence of the operator expression $p^k_{rl...}\mathbf{s}_r\mathbf{s}_l...$ and the representation of the cross-correlation operator eigenvalues up to constant factors $\lambda_{rl...}$ that are eigenvalues of the operators $\mathbf{s}_r\mathbf{s}_l...$. It is this circumstance that makes it possible to change from the operator representation of $\mathbf{s}_r$ to a pure vector interpretation of $\mathbf{s}_r$ as basic vectors in noncommutative geometry. From the definition of the operator $\mathbf{G}^k$ and expression (5), we obtain

$$\mathbf{G}^k = |r^k_1|^2\mathbf{p}^k_{11} + |r^k_2|^2\mathbf{p}^k_{22} + r^k_1 r^{k*}_2\mathbf{p}^k_{12} + r^k_2 r^{k*}_1\mathbf{p}^k_{21}, \tag{17}$$

where $\mathbf{p}^k_{nm} = |\varphi^k_n\rangle\langle\varphi^{k*}_m|$, and the asterisk (*) means the complex conjugation. The operators $\mathbf{p}^k_{nm}$ have the property $(\mathbf{p}^k_{nm})^2 = \mathbf{p}^k_{nm}\delta_{nm}$, where δ_{nm} is the Kronecker symbol. In other words, the operators $\mathbf{p}^k_{nn}$ determine projectors and $(\mathbf{p}^k_{nm})^2 = 0$ $(n \neq m)$. However, the commutator of the operators $\mathbf{p}^k_{12}$ and $\mathbf{p}^k_{21}$

equals $[\mathbf{p}_{12}^k, \mathbf{p}_{21}^k] = \mathbf{p}_{11}^k - \mathbf{p}_{22}^k$. If we express the projectors $\mathbf{p}_{nn}^k$ in terms of the mirror-symmetry operators $\mathbf{s}_k$ in the standard form,

$$\mathbf{p}_{11}^k = \mathbf{p}_+^k = \frac{1}{2}(\mathbf{1} + \mathbf{s}_k), \tag{18}$$

$$\mathbf{p}_{22}^k = \mathbf{p}_-^k = \frac{1}{2}(\mathbf{1} - \mathbf{s}_k), \tag{19}$$

then $[\mathbf{p}_{12}^k, \mathbf{p}_{21}^k] = \mathbf{s}_k$.

In the Cl_3 algebra, the operators $\mathbf{p}_{nm}^k$ can be represented using two pairs,

$$\mathbf{p}_{12}^2 = \frac{1}{2}(\mathbf{s}_1 + i\mathbf{s}_3), \quad \mathbf{p}_{21}^2 = \frac{1}{2}(\mathbf{s}_1 - i\mathbf{s}_3) \tag{20}$$

and

$$\mathbf{p}_{12}^1 = \frac{1}{2}(\mathbf{s}_2 + i\mathbf{s}_3), \quad \mathbf{p}_{21}^1 = \frac{1}{2}(\mathbf{s}_2 - i\mathbf{s}_3). \tag{21}$$

The former generates the $k = 2$-representation for the operator $-\mathbf{s}_2$, and the latter the $k = 1$-representation for the operator $\mathbf{s}_1$. An analysis shows that the general expression for the representation $\mathbf{G}^k$ in the k-basis looks like

$$\mathbf{G}^k = \frac{1}{2}\varepsilon_{klr}[|r_1^k|^2(\mathbf{1} + \mathbf{s}_k) + |r_2^k|^2(\mathbf{1} - \mathbf{s}_k) + \\ + r_1^k r_2^{k*}(\mathbf{s}_l + i\mathbf{s}_r) + r_2^k r_1^{k*}(\mathbf{s}_l - i\mathbf{s}_r)], \tag{22}$$

where ε_{klr} is the Levi–Civita symbol for the permutations (123), (231), and (312). Therefore, the formula for $\mathbf{G}^1$ in Eq. (22) has the form

$$\mathbf{G}^1 = g_0^1\mathbf{s}_0 + g_1^1\mathbf{s}_1 + g_2^1\mathbf{s}_2 + g_3^1\mathbf{s}_3, \tag{23}$$

where the expansion coefficients are as follows:

$$\begin{aligned} g_0^1 &= \frac{1}{2}(|r_1^k|^2 + |r_2^k|^2), \quad g_1^1 = \frac{1}{2}(|r_1^k|^2 - |r_2^k|^2), \\ g_2^1 &= \frac{1}{2}(r_1^k r_2^{k*} + r_2^k r_1^{k*}), \quad g_3^1 = \frac{i}{2}(r_1^k r_2^{k*} - r_2^k r_1^{k*}), \end{aligned} \tag{24}$$

and i is the imaginary unit. The expansion coefficients g_α^1 are real numbers, and $\mathbf{G}^1$ looks like

$$\mathbf{G}^1 = |r^1\rangle\langle r^1| = \begin{pmatrix} |r_1|^2 & r_1 r_2^* \\ r_2 r_1^* & |r_2|^2 \end{pmatrix}. \tag{25}$$

In optics, $\mathbf{G}^1$ is called the coherence matrix for quasi-monochromatic radiation, and the parameters $2g_\alpha$ are called the Stokes parameters describing the polarization of electromagnetic waves. Expansion (23) is complete because four $\mathbf{G}$ components are represented by four linear combinations of four linearly independent matrices $\mathbf{s}_\alpha$. If we turn to expression (16) for the cross-correlation operator $\mathbf{K}_{\alpha.}^k$ and sum up a construction of the type $p_\alpha^k \mathbf{s}_\alpha$, where $p_\alpha = 2g_\alpha$, over α, then, to an accuracy of a factor of 2, we will obtain a representation for the autocorrelation operator $\mathbf{G}^1$ in the initial space. Thus, expressions (22) and (23) reflect the correlation properties of a two-dimensional vector space with complex coordinates mirror-mapped onto a four-dimensional basis of a vector space with noncommutative geometry. This 4-basis can be considered as hyperbolic hypercomplex numbers and the conjugation of the form $\bar{\mathbf{s}}_0 = \mathbf{s}_0$ and $\bar{\mathbf{s}}_k = -\mathbf{s}_k$ (the Clifford conjugation) can be introduced. By analogy with complex numbers, for which $z_1 z_2 = (\bar{z}_1 \cdot z_2) + i[\bar{z}_1 \times z_2]$ for this 4-basis, the metric tensor $\mu_{\alpha\beta}$ is defined as a scalar product in the form

$$\mu_{\alpha\beta} = (\bar{\mathbf{s}}_\alpha \cdot \mathbf{s}_\beta) = \frac{1}{2}(\bar{\mathbf{s}}_\alpha \mathbf{s}_\beta + \bar{\mathbf{s}}_\beta \mathbf{s}_\alpha), \tag{26}$$

where $\alpha, \beta = 0, 1, 2, 3$. The tensor $\mu_{\alpha\beta}$ is diagonal with the signature $\mathrm{diag}\,(\mu_{\alpha\beta}) = = (1, -1, -1, -1)$, which is characteristic of the Riemannian (Minkowski) space. The external product is also defined as

$$(\bar{\mathbf{s}}_\alpha \wedge \mathbf{s}_\beta) = \frac{1}{2}(\bar{\mathbf{s}}_\alpha \mathbf{s}_\beta - \bar{\mathbf{s}}_\beta \mathbf{s}_\alpha). \tag{27}$$

Formulas (26) and (27) written for the 4-basis are also valid for 4-vectors.

5. Spectral Transformations at Mirror Mapping

5.1. Tangent spaces at mirror mapping

Let us analyze the tangent space – more precisely, the tangent subspace – for the vector $|r^k\rangle$ in the initial space and its components at mirror mapping. The components of the vector $|r^k\rangle$ are separated using the projectors $\mathbf{P}_{k\pm} = \frac{1}{2}(\mathbf{1} \pm \mathbf{s}_k)$ according to the formulas

$$\{|r^k\rangle\}_+ = \mathbf{P}_{k+}|r^k\rangle = r_1^k|+\rangle_k,$$
$$\{|r^k\rangle\}_- = \mathbf{P}_{k-}|r^k\rangle = r_2^k|-\rangle_k.$$

The basis vectors of the tangent space are formed on the basis of the action on the components $\{|r^k\rangle\}_\pm$ of the covariant 4-gradient with differentiation with respect to the coordinates x^α in the basis of the operators $\mathbf{s}_\alpha$.

Let us define the 4-gradient as $\bar{\nabla} = \partial_0 \mathbf{s}_0 - \boldsymbol{\nabla}$, where $\boldsymbol{\nabla} = \partial_k \mathbf{s}_k$ is the vector part (here, summation over the repeated index is implied). Since $\mathbf{P}_{k\pm} = (\mathbf{P}_{k\pm})^2$, the action of 4-gradient on the components $\{|r^k\rangle\}_\pm$ gives rise to the formation of the operator of tangent subspace, $\mathbf{T}_k$, according to the rule $\mathbf{T}_{k\pm} = \nabla \mathbf{P}_{k\pm}$. Then the process of obtaining a tangent subspace for the components of the vector $|r^k\rangle$ can be described by the expression $\mathbf{T}_{k\pm}\{|r^k\rangle\}_\pm$. Let us put, for example, $k = 1$. As a result, for $\mathbf{T}_{k+}\{|r^k\rangle\}_+$, we obtain

$$\mathbf{T}_{1+} r_1^1 |+\rangle_1 = (\partial_0 + \partial_1) r_1^1 |+\rangle_1 + \\ + \frac{1}{2}(\partial_2 + i\partial_3) r_1^1 \mathbf{s}_2 |+\rangle_1 - \frac{1}{2} i(\partial_2 + i\partial_3) r_1^1 \mathbf{s}_3 |+\rangle_1. \quad (28)$$

The corresponding expression for $\mathbf{T}_{k-}\{|r^k\rangle\}_-$ differs only in signs.

The solution r_1^1 of the equation $\mathbf{T}_{1+} r_1^1 |+\rangle_1 = 0$ can be obtained in the factorized form, $r_1^1 = r_1^1(x^2, x^3) w_1^1(x^0, x^1)$. For the equation $\mathbf{T}_{1-} r_2^1 |-\rangle_1 = 0$, the solution corresponds to the complex conjugate expression $r_2^{1*} = r_2^{1*}(x^2, x^3) w_2^{1*}(x^0, x^1)$. The derivatives $\partial_2 \pm i\partial_3 = 0$ determine the Cauchy–Riemann condition for the complex functions $r_1^1(x^2, x^3)$ and $r_2^{1*}(x^2, x^3)$, and the functions w_1^1 and w_2^{1*} determine waves in physical space-time coordinates $x^0 = ct$ and $x^k = (x, y, z)$. These waves propagate in opposite directions along the coordinate axis x^1. Really, the functions w_1^1 and w_2^{1*} can be written in terms of their Fourier transforms $w_1^1(t, x^1) = \mathcal{F}\{\omega_1 t - k_1 x^1\}$ and $w_2^{1*}(t, x^1) = \mathcal{F}^*\{\omega_2 t + k_2 x^1\}$. Then the corresponding equations look like

$$\begin{aligned} &(\partial_0 + \partial_1)\mathcal{F}\{\omega_1 t - k_1 x^1\} = (\omega_1/c - k_1) = 0, \\ &(\partial_0 - \partial_1)\mathcal{F}^*\{\omega_2 t + k_2 x^1\} = (\omega_2/c - k_2) = 0. \end{aligned} \quad (29)$$

We require that there should be such functions w_1 and w_2^* that satisfy Eqs. (29) and which derivatives are nonzero. The solutions w_1 and w_2^* describe the wave processes: $w_1^1(x^0, x^1) = w_1^1(\omega_1(t - x^1/c))$ and $w_2^{1*}(x^0, x^1) = w_2^{1*}(\omega_2(t + x^1/c))$. It is easy to see that the expansion in the eigenvectors of other operators has a structure similar to that of the above solutions. For this purpose, one should only cyclically permutate the coordinate numbers in the results obtained for the vector components. Since the structure of the differential operator in Eq. (28) corresponds to the Cl_2 algebra, we have three modifications of the Cl_2 algebras to describe the tangent subspaces of the "mirror-world", which depend on the propagation direction of the wave process. This direction is determined by the choice of the basis consisting of the eigenfunctions of the operators $\mathbf{s}_k$, in which the vector $|r^k\rangle$ is represented in the initial space.

It should be noted that the obtained results describe a rather wide class of wave processes; in particular, provided the corresponding form of the function w, it can be the motion of a particle with the velocity c. In this case, the physical content of the variable ω substantially changes.

5.2. Formation of the basis of noncommutative geometry in "mirror world-2"

The tangent space in "mirror world-1", which was built for our four-dimensional space-time, showed the existence of three wave processes. These waves propagate in three orthogonal directions described by the noncommutative geometry with the Cl_2 algebra. For analysis, let us take the variant of the tangent subspace for the vector $|r^k\rangle$ in the form of plane waves and satisfying the condition $\langle r^k | r^k \rangle = E$:

$$|r^k\rangle = \sqrt{\frac{E}{2}} \exp(ikx^k/2)[\exp(-i\omega t/2)|+\rangle_k + \\ + \exp(i\omega t/2)|-\rangle_k]. \quad (30)$$

This expression describes a vector with the rotational frequency $\omega/2$ in various bases of the eigenvectors of the operators $\mathbf{s}_k$. From expression (22), it follows that, for vectors (30), the autocorrelation operators $\mathbf{G}_k$ in "mirror world-1" have the form

$$\mathbf{G}_{ik} = \frac{1}{2} E\,[\mathbf{1} + \exp(-i\mathbf{s}_i \omega t)\mathbf{s}_k], \quad (31)$$

where $(ik) = (12), (23), (31)$, and $\mathbf{1} = \mathbf{s}_0$. Substituting the exponential factor in Eq. (15) by $\mathbf{G} = \mathbf{G}_{12} + \mathbf{G}_{23} + \mathbf{G}_{31}$ for $\alpha = 0$, we obtain

$$\mathcal{F}_0\{\mathbf{G}\} = \frac{ET}{2}(3\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3), \quad (32)$$

where the periodic dependence on the coordinates of the bivectors $i\mathbf{s}_k$ is omitted, because it does not change expressions for the formed basis vectors,

$$\mathbf{e}_\alpha = \begin{pmatrix} \mathbf{s}_\alpha & 0 \\ 0 & -\mathbf{s}_\alpha \end{pmatrix}, \quad \alpha = 0, 1, 2, 3. \quad (33)$$

On the basis of the results obtained, one can build the model of the space in "mirror world-2" in the following manner. There is a discrete S-space of action in the form of three-dimensional space $\{n_1, n_2, n_3\}$ composed of cubic cells. All cell edges have the length ET and they are the coordinate axes of the bivectors $i\mathbf{s}_k$. The lattice sites are described by the noncommutative geometry with basis vectors (33). For Θ_2 in Eq. (15), the system of basis vectors $\mathcal{F}_2\{\mathbf{G}\}$ can be obtained in the form

$$\mathcal{F}_2\{\mathbf{G}\} = \frac{ET}{2}(3\mathbf{e}_4 + \gamma_1 + \gamma_2 + \gamma_3), \tag{34}$$

where γ_k are the Dirac matrices in terms of the mirror-symmetry operators,

$$\gamma_k = \begin{pmatrix} 0 & \mathbf{s}_k \\ -\mathbf{s}_k & 0 \end{pmatrix}. \tag{35}$$

For a nonclassical rotation process, we should put $ET = h$ in expressions (32) and (34).

6. Noncommutative Geometry with Higher Dimensions

6.1. Passage to 4- or 5-dimensional noncommutative geometry

Joining the "mirror" ($\mathbf{s}_k$) and "anti-mirror" ($-\mathbf{s}_k$) operators into a single mirror mapping operator $\mathbf{e}_k$ increases the dimension of the initial mirror-image space. Let us introduce two complex spaces, $\mathcal{H}_a$ and $\mathcal{H}_b$, that do not intersect, but possess a common point 0. In the basic representation of $|r^k\rangle$ $(k = 1)$, the projections of the vectors in these spaces can be written in the form

$$\begin{aligned} |r_a\rangle &= a(f_1|+\rangle + f_2|-\rangle), \\ |r_b\rangle &= b(f_3|+\rangle + f_4|-\rangle), \end{aligned} \tag{36}$$

where a and b are constant factors. Let us join these spaces, $\mathcal{H}_a \oplus \mathcal{H}_b$, and construct a correlation operator $\mathbf{G}$, which, in this case, is represented by the 4×4 matrix

$$\mathbf{G} = \begin{pmatrix} |a|^2\mathbf{G}_{11} & ab^*\mathbf{G}_{12} \\ ba^*\mathbf{G}_{21} & |b|^2\mathbf{G}_{22} \end{pmatrix}, \tag{37}$$

where $\mathbf{G}_{kr}$ are 2×2 matrices. The diagonal operators $\mathbf{G}_{11}$ and $\mathbf{G}_{22}$ describe the correlations between the components of the vectors in each of the spaces $\mathcal{H}_a$ and $\mathcal{H}_b$, whereas $\mathbf{G}_{12}$ and $\mathbf{G}_{21}$ the correlation (interaction) of those components between $\mathcal{H}_a$ and $\mathcal{H}_b$. Hence, not only the dimension of the initial mirror-image space increases. In order to deal with the noncommutative geometry, it follows from expression (37) that the basis vectors (31) must be supplemented with symmetry operators with nondiagonal elements. Making use of the vector $\mathbf{e}_4$ from Eq. (34), one can obtain the extension of the noncommutative geometry Cl_3 to the Cl_4 algebra. These algebras form the S-space of action in "mirror world-2". For a nonclassical S-space, certain commutation relations are satisfied. In the case of Cl_3 algebra, these relations for bivectors and a 3-vector look like

$$\begin{aligned} \mathbf{e}'_k \wedge \mathbf{e}'_l &= \frac{1}{2}[\mathbf{e}'_k, \mathbf{e}'_l] = h^2\mathbf{e}_k\mathbf{e}_l, \\ \mathbf{e}'_r \cdot (\mathbf{e}'_k \wedge \mathbf{e}'_l) &= h^3\epsilon_{klr}\mathbf{e}_r\mathbf{e}_k\mathbf{e}_l, \end{aligned} \tag{38}$$

where ϵ_{klr} is the Levi-Civita symbol and $\mathbf{e}'_k = h\mathbf{e}_k$.

6.2. Passage to 6-dimensional mirror-image space

Let us consider separately the following nonclassical case. Let us couple the components of the vectors in the spaces $\mathcal{H}_a$ and $\mathcal{H}_b$ by means of the J-transformation: i.e., in Eq. (36), $f_3 = f_2$ and $f_4 = -f_1$, as well as $a = \sqrt{\frac{2q}{3h}}$ and $b = \sqrt{-\frac{2q}{3h}} = ia$, where q is a real constant associated with the elementary charge. As a result of the operation $\mathcal{H}_a \oplus \mathcal{H}_b$, we obtain a six-dimensional Kähler manifold of the initial mirror-image space for "mirror world-2". In this case, the correlation blocks in Eq. (37) read

$$\begin{aligned} \mathbf{G}_{11} &= \alpha \begin{pmatrix} |f_1|^2 & f_1f_2^* \\ f_2f_1^* & |f_2|^2 \end{pmatrix}, \\ \mathbf{G}_{12} &= \beta^* \begin{pmatrix} f_1f_2^* & -|f_1|^2 \\ |f_2|^2 & -f_2f_1^* \end{pmatrix}, \\ \mathbf{G}_{21} &= \beta \begin{pmatrix} f_2f_1^* & |f_2|^2 \\ -|f_1|^2 & -f_1f_2^* \end{pmatrix}, \\ \mathbf{G}_{22} &= \alpha \begin{pmatrix} |f_2|^2 & -f_2f_1^* \\ -f_1f_2^* & |f_1|^2 \end{pmatrix}, \end{aligned} \tag{39}$$

where $\alpha = \frac{2q}{3h}$ and $\beta = i\frac{2q}{3h}$.

The correlation $\mathbf{G}$ of the six-dimensional space in "mirror world-2" can be mapped to the noncommutative geometry with the help of the cross-correlation $\mathbf{K}$ of "mirror world-2" and "mirror world-3", which is performed using the basis vectors $\mathbf{e}_\alpha$ [Eq. (33)] and the bivectors γ [Eq. (35)] together with commutation relations (38). As a result, six vectors of the noncommutative geometry can be written in "mirror world-3". They look like

$$\Pi = p_0\mathbf{e}_0 + p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_4\mathbf{e}_4 + p_{14} ih\gamma_1 + p_{24} ih\gamma_2. \quad (40)$$

Here, the p_α-values are determined using formula (16):

$$p_0 = 2q(|f_1|^2 + f_2|^2), \quad p_1 = \frac{2}{3}q(|f_1|^2 - f_2|^2),$$
$$p_2 = \frac{2}{3}q(f_1 f_2^* + f_2 f_1^*), \quad p_4 = \frac{2}{3}q(f_1 f_2^* - f_2 f_1^*),$$
$$p_{14} = \frac{2}{3}q(f_1 f_2^* + f_2 f_1^*), \quad p_{24} = \frac{2}{3}(-q)(|f_1|^2 - f_2|^2).$$

All p_α-components are real-valued. The coordinate $p_3 = 0$. It corresponds to the sum of oppositely oriented axial vectors, which can be related to magnetic moments.

7. Conclusions

This article is not aimed at solving the problems in the string theory or noncommutative geometry. Instead, the attempt is made to find a common ground, where they can meet. For this purpose, the purely physical *a posteriori* method based on one of fundamental symmetries, mirror symmetry, was chosen. The latter allowed the concept of "initial state", which can acquire a real physical meaning, to be introduced into the theory.

Mirror mappings and conservation conditions for mirror symmetries – inverse and permutation ones – were taken as the basis of the research method. Its formalization with the help of the algebra of geometric constructions of mirror reflections on the plane made it possible to introduce 2×2-matrix mirror-image operators, which satisfy the Clifford algebra Cl_3 and form the basis of the noncommutative geometry. Components in this vector basis are auto-correlations or cross-correlations of the components of vectors in the initial and mirror-mapped space. The conservation conditions for mirror symmetries brought about the appearance of nJ-transformation, i.e., discrete rotations by angles $\pm n\,90°$ $(n = 0, 1, 2, ...)$, for the basis vectors. On the basis of this transformation, a kind of continuum integral [8] was introduced, which, with the help of mirror mapping operation, extended the representation of matrix operators to 4×4 matrices and the initial space of the mirror mapping (complex, in the general case) from 4 to 8 dimensions.

A physical implementation of the described mathematical results had to be sought in the rotation processes appearing at mirror mappings. The corresponding solution is obtained in terms of physical space-time coordinates for three tangent subspaces in "mirror world-1". In "mirror world-2", the application of continuum integral to the sum of subspaces made it possible to increase the dimension of the basis of the noncommutative geometry to the Cl_4 algebra, which includes Dirac matrices as bivectors of a vector algebra. For this result of transformation not to depend on the energy and frequency of the rotation process, the process must be nonclassical, i.e., the relation $E = h\nu$ must be satisfied.

The model of a four-component complex vector representing a six-dimensional complex space is created in "mirror world-2". The mapping of this model on six vectors of the noncommutative geometry with real-valued coordinates is constructed in "mirror world-3". These coordinates are cross-correlations between the components of the vectors from "mirror world-2" and the vectors mirror-mapped in "mirror world-3". From the physical point of view, the presented model can be regarded as a tetraquark composed of two quark-antiquark pairs with the charges $\pm\frac{2}{3}q$ from different quark generations. A way to associate the considered model with electromagnetic fields is illustrated in Appendix.

The author is grateful to Yuliya Parko and Andriy Stepenko (the "Zhytt'elyub" Foundation) for their help in preparing this paper.

APPENDIX
Tangent Space of Noncommutative Geometry of the Cl_3 Algebra

In expression (40), let us apply the correspondence principle as $h \to 0$. In the obtained 4-vector, let us denote its components as $p_0 \to A_0$, and the covariant components as $p_k \to -A_k$. Using the Minkowski space metric and in terms of new notations, we obtain

$$\mathbf{W} = A^0\mathbf{e}_0 + A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3. \quad (41)$$

For the 4-gradient, we have

$$\nabla = \partial_\alpha \mathbf{e}^\alpha = \partial_0\mathbf{e}_0 - \sum_{k=1}^{3} \partial_k\mathbf{e}_k. \quad (42)$$

The tangent space is constructed as the internal and external products, which are represented by the anticommutator and commutator, respectively,

$$\nabla \mathbf{W} = \nabla\, \mathbf{W} + \nabla \wedge \mathbf{W} = \frac{1}{2}[\nabla, \bar{\mathbf{W}}]_- + \frac{1}{2}[\nabla, \bar{\mathbf{W}}]. \tag{43}$$

Whence it follows that

$$\nabla\, \mathbf{W} = \partial_\alpha A^\alpha \mathbf{e}_0,$$

$$\nabla \wedge \mathbf{W} = -\partial_0 \mathbf{A} - \text{grad } A^0 + \text{rot } \mathbf{A} = \mathbf{P} + \mathbf{B}, \tag{44}$$

where $\mathbf{A} = A^k \mathbf{e}_k$, grad $= \sum_{k=1}^{k=3} \partial_k \mathbf{e}_k$, $\mathbf{P}$ is a polar vector, and $\mathbf{B} =$ rot $\mathbf{A}$ is a bivector with the components $\partial_k A^r - \partial_r A^k$ $(k, r = 1, 2, 3)$.

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Received 06.10.21.
Translated from Ukrainian by O.I. Voitenko

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ДЗЕРКАЛЬНА СИМЕТРІЯ ЯК АЛГЕБРА ОПЕРАТОРІВ ДЛЯ НЕКОМУТАТИВНОЇ ГЕОМЕТРІЇ ПРОСТОРУ-ЧАСУ

Аналіз геометричних і алгебраїчних властивостей дзеркальних відображень дозволив використовувати їх як операторну алгебру некомутативної геометрії. Координатами некомутативної геометрії є авто- або крос-кореляції координат дзеркально відображених просторів. Розглянуто окремий випадок шестивимірного келерова многовиду, який відображається на некомутативну геометрію з векторною алгеброю Кліффорда Cl_4. Цей випадок відображення відповідає тетракварку у складі двох пар кварк-антикварк з різних поколінь і зарядами $\pm\frac{2}{3}q$.

Ключові слова: дзеркальна симетрія, некомутативна геометрія, алгебра Кліффорда, кореляція.