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B.E. GRINYUK, I.V. SIMENOG
Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine
(14-b, Metrolohychna Str., Kyiv 03143, Ukraine; e-mail: bgrinyuk@bitp.kiev.ua)

MODEL-INDEPENDENT SOLUTION OF *nd*-SCATTERING PROBLEM IN THE QUARTET STATE

A model-independent description of the phase shift of the elastic nd-scattering in the quartet state is grounded, and explicit solutions for the low-energy scattering parameters (the quartet scattering length $a_{3/2}$ and effective range $r_{3/2}$) are obtained in the form of asymptotically exact expansions in terms of the ratio of the experimental two-nucleon low-energy scattering parameters.

K e y w o r d s: *nd*-scattering, quartet state, model-independent solution.

1. Introduction

A great number of calculations of the quartet *nd*-scattering low-energy parameters, in particular, the quartet scattering length, revealed an independence of the result (from qualitative point of view) on the model of nuclear forces used (see, for example, the earliest calculations [1, 2]), i.e., one has the so-called "model-independence" of the three-nucleon system properties in the quartet state. This fact can be explained [3, 4] due to the smallness of the radius of nuclear forces as compared to the two-particle scattering length and the role of Pauli principle suppressing the role of nuclear forces in this problem. Qualitatively, the value of the quartet scattering length can be explained already within the Skornyakov–Ter-Martirosyan (STM) approximation of the three-particle Faddeev equations in the zero-range of nuclear forces limit. In the present paper, we find more accurate exact solution of the problem in the form of an expansion in small parameter originating from the smallness of the radius of nuclear forces up to the fourth order in the parameter.

2. Statement of the Problem

We are going to ground the "model-independent" description [4] of the elastic *nd*-scattering phase shift in the state with total spin $S = 3/2$. For simplicity, we start with the Faddeev equations for the case of a model separable potential. It can be shown that the result will be the same in the general case. The Faddeev equation for the scattering amplitude (at the energies below the break up threshold) can be written in the real form:

$$\mathcal{F}(p,k)\,\frac{u^2\left(-\alpha^2-\frac{3}{4}\left(p^2-k^2\right)\right)}{\left(p^2-k^2\right)\,f\left(-\alpha^2-\frac{3}{4}\left(p^2-k^2\right)\right)}+ \frac{2}{\pi}\,⨍_0^\infty dp'\,\frac{p'^2}{p'^2-k^2}\,U(p,p')\,\mathcal{F}(p',k)=-U\left(p,k\right), \quad (1)$$

where

$$U(p,p')=\frac{1}{4\pi}\int d\Omega\,\frac{u\Big(\big(\mathbf{p}+\frac{\mathbf{p}'}{2}\big)^2\Big)\,u\Big(\big(\mathbf{p}'+\frac{\mathbf{p}}{2}\big)^2\Big)}{p^2+p'^2+(\mathbf{p},\mathbf{p}')+\alpha^2-\frac{3}{4}k^2} \quad (2)$$

and $u\left(k^2\right)$ is a formfactor of the separable interaction potential (in the momentum representation) $\hat{V}\left(k,k'\right)=\lambda\left|u\left(k^2\right)\right\rangle\!\left\langle u\left(k'^{\,2}\right)\right|$. The crossed integral in (1) means the principal value of the integral in a vicinity of the point $p'=k$. The expression

$$S\left(p^2-k^2\right)=\frac{u^2\left(-\alpha^2-\frac{3}{4}\left(p^2-k^2\right)\right)}{\left(p^2-k^2\right)\,f\left(-\alpha^2-\frac{3}{4}\left(p^2-k^2\right)\right)}$$

on the left-hand side of Eq. (1) has no singularities.

If the function $\mathcal{F}\left(k,p\right)$ is found from Eq. (1), then the phase shift is determined by the relation

$$\left(k\cot\delta_{3/2}(k)\right)^{-1}=\mathcal{F}(k,k). \quad (3)$$

In Eq. (1), the triplet two-particle scattering amplitude $f\left(k^2\right)=\left(k\cot\delta(k)-ik\right)^{-1}$ continued onto negative energies is indicated explicitly. The function $\mathcal{F}(p,k)$ in Eq. (1) is real.

3. Model-Independent Equation for the Quartet *nd*-Scattering Amplitude

We assume the formfactor of the separable potential to have such a form that, at small radius of forces, there exists an expansion

$$u\left(k^2\right) = 1 + u^{[1]}(0)\; k^2R^2 + \frac{1}{2}u^{[2]}(0)\; \left(k^2R^2\right)^2 + ... \equiv$$
$$\equiv \sum_{n=0}^{\infty} \frac{1}{n!}\; u^{[n]}(0)\; \left(k^2R^2\right)^n, \qquad (4)$$

where $u^{[n]}\left(k^2\right)$ denotes the n-th derivative with respect to k^2, and $u^{[0]}(0)$ is assumed to be 1.

If one keeps only the main term of the expansion for $u\left(k^2\right)$, then, instead of (1), there appears the model-independent (i.e., independent of a specific model of forces) equation [4]

$$F(p,k)\, \frac{1}{\left(p^2-k^2\right)\, f\left(-\alpha^2 - \frac{3}{4}\left(p^2-k^2\right)\right)} +$$
$$+ \frac{2}{\pi}\fint_0^{\infty} dp' \frac{p'^2}{p'^2 - k^2}\, U_0(p,p')\, F(p',k) = -U_0(p,k), \qquad (5)$$

which contains the two-nucleon amplitude (at negative energies), and which is shown below to be a good approximation to Eq. (1). Here, we denoted

$$U_0(p,p') = \lim_{R\to 0} U(p,p') =$$
$$= \frac{1}{4\pi}\int d\Omega \frac{1}{p^2 + p'^2 + (\mathbf{p},\mathbf{p}') + \alpha^2 - \frac{3}{4}k^2} =$$
$$= \frac{1}{2pp'} \ln\left(\frac{p^2 + p'^2 + pp' + \alpha^2 - \frac{3}{4}k^2}{p^2 + p'^2 - pp' + \alpha^2 - \frac{3}{4}k^2}\right). \qquad (6)$$

In Eq. (5), the principal value of the integral is taken in a vicinity of $p' = k$, as well as near the points, where the function

$$C\left(p^2-k^2\right) \equiv \frac{1}{\left(p^2-k^2\right)\, f\left(-\alpha^2 - \frac{3}{4}\left(p^2-k^2\right)\right)} =$$
$$= \frac{1}{p^2-k^2}\left(\tilde{k}\cot\delta\left(\tilde{k}\right) - i\tilde{k}\right)\Big|_{\tilde{k}=+i\sqrt{\alpha^2+\frac{3}{4}\left(p^2-k^2\right)}} \qquad (7)$$

may become zero. Thus, the solution of Eq. (5) may have a pole singularity.

To find the quartet scattering length $a_{3/2}$ and effective range $r_{3/2}$, we obtain the expressions in terms of the solution of Eq. (5) at the zero energy. We denote $\tilde{F}(p) \equiv F(p,0)$, and we further measure momenta in units of α. Thus, we have:

$$a_{3/2} = -\tilde{F}(0),$$
$$a_{3/2}^2\, r_{3/2} = -\frac{3}{2}\tilde{F}(0)\left(1 + \frac{8}{3}\frac{C^{[1]}(0)}{C(0)}\right) - 4\tilde{F}^{[1]}(0) -$$
$$-\frac{4}{\pi}\int_0^{\infty} \frac{dp}{p^2}\left(\frac{C\left(p^2\right)}{C(0)}\left(1 + \frac{3}{8}p^2\right)\tilde{F}^2(p) - \tilde{F}^2(0)\right) -$$
$$-\frac{4}{\pi}\int_0^{\infty} dp\left(1 + \frac{3}{4}p^2\right)\tilde{F}^2(p)\,\frac{C^{[1]}\left(p^2\right)}{C(0)}, \qquad (8)$$

where $C^{[1]}\left(p^2\right)$ and $\tilde{F}^{[1]}\left(p^2\right)$ denote the first derivatives with respect to p^2 taken from $C\left(p^2\right)$ and $\tilde{F}\left(p^2\right)$, respectively. We assume that $C\left(p^2\right) \neq 0$ at finite p, and, thus, the solution $\tilde{F}(p)$ obtained from Eq. (5) has no poles. Otherwise, the last integral from expression (8) should be regularized by extracting the main singularity from the integrand.

Consider the properties of the solution of Eq, (5) in the limit of the zero range of forces, when Eq. (5) transforms to the STM one. From the low-energy expansion, one has

$$C\left(p^2-k^2\right) =$$
$$= \frac{3}{4}\left(\frac{1}{1+\sqrt{1+\frac{3}{4}\left(p^2-k^2\right)}} - \frac{1}{2}\alpha r_0 - \left(2 + \frac{3}{4}\left(p^2-k^2\right)\right)\mathfrak{f}\right) +$$
$$+ \frac{1}{p^2-k^2}\sum_{n=3}^{\infty}\left(\left(1 + \frac{3}{4}\left(p^2-k^2\right)\right)^n - 1\right) X_n, \qquad (9)$$

where we took into account that $1/f(-1) = 0$. Here, r_0, $\mathfrak{f} \equiv (\alpha r_0)^3 P$, X_n – are the effective range and the form parameter *etc.*, respectively.

Equation (5) in the limit of the zero range of forces is solvable under the condition

$$1 + \frac{2}{\pi}\fint_0^{\infty} dp\, \frac{p^2}{p^2-k^2}\, F_0(p,k) = 0, \qquad (10)$$

the solution $F_0(p,k)$ having a power asymptotics at large momenta $\sim 1/p^{2+\varepsilon}$ [4], where $\varepsilon = 0.1662219...$ is determined from the secular equation

$$L(2+\varepsilon) \equiv \frac{1}{(2+\varepsilon)\sqrt{3}}\,\frac{\cos\left(\frac{(1-\varepsilon)\pi}{6}\right)}{\cos\left(\frac{\varepsilon\pi}{2}\right)} = 1. \qquad (11)$$

Equation (11) contains only real roots, and the main ones of them are $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon = 0.1662219\ldots$. The analysis of the STM equation at $p \to \infty$ reveals the main term of the asymptotics to be $d_0/p^{2+\varepsilon}$, while the more slowly decreasing term $\sim 1/p^2$ is absent. From a qualitative point of view, condition (10) and the law of asymptotical decrease $\sim 1/p^{2+\varepsilon}$ mean that, due to the Pauli principle, an effective potential of the neutron interaction with the deuteron at small distances $\rho \sim R$ looks like a centrifugal barrier ε/ρ^2, and the wave function for such a potential approaches zero at small distances as $\sim \rho^{1+\varepsilon}$.

A correction term of the first order in the radius of forces $F_1(p,k)$ to the STM equation can be found as an iteration of Eq. (5) with respect to r_0 with the account for expansion (9). This function obeys the following condition:

$$\fint_0^\infty dp \, \frac{p^2}{p^2 - k^2} \, F_1(p,k) = 0, \tag{12}$$

and it decreases at $p \to \infty$ as $F_1(p,k) \to d_1/p^{1+\varepsilon}$, where

$$\frac{d_1}{d_0} = \frac{\frac{\sqrt{3}}{4}}{1 - L(1+\varepsilon)} = -0.127413\ldots. \tag{13}$$

Corrections of higher orders in the radius of forces have the following asymptotics: of the second order, $F_2(p,k) \sim 1/p^\varepsilon$; of the third order, $F_3(p,k) \sim p^{1-\varepsilon}$; and a similar correction function of the fourth order does not exist due to a too slow decrease of the inhomogeneous term of the corresponding equation.

Now, let us discuss the accuracy of the quartet phase shift which can be obtained within the model-independent equation (5) (i.e., let us estimate the order in the radius R of forces of the correction terms to the phase shift derived from Eq. (5)). Using expansion (4) and the perturbation theory in the radius of forces, we obtain from Eq. (1) for a correction term to $F(k,k)$ which is proportional to $u^{[1]}(0)$, the following expression:

$$\delta_{[1]} F(k,k) = -2\, u^{[1]}(0)\, R^2 A^2(k), \tag{14}$$

where

$$A(k) \equiv 1 + \frac{2}{\pi} \fint_0^\infty dp \, \frac{p^2}{p^2 - k^2} \, F(p,k). \tag{15}$$

The correction term proportional to $u^{[2]}(0)$ equals

$$\delta_{[2]} F(k,k) = 2u^{[2]}(0) R^4 A(k) \times \\ \times \left(\left(1 - \frac{5}{4}k^2\right) A(k) - \frac{1}{\pi} \fint_0^\infty dp p^2 F(p,k) \right). \tag{16}$$

The rest correction terms (in particular, proportional to $\left(u^{[1]}(0)\right)^2$ and $u^{[3]}(0)$), we do not present here. They have a somewhat more complicated structure, but similar to (14) and (16).

Using conditions (10), (12), and the law of decrease of the functions $F_0(p,k)$ and $F_1(p,k)$ at $p \to \infty$, one has that $A(k)$ defined by (15) is of the order of $R^{1+\varepsilon}$ at $R \to 0$. Then the correction term (14) at small radius of forces is of the order of $\mathcal{O}\left(R^{4+2\varepsilon}\right)$. It can be shown that the same order of smallness is inherent in the correction term (16), as well as the corrections proportional to $\left(u^{[1]}(0)\right)^2$, $u^{[3]}(0)$ *etc.* Thus, we have ultimately that $F(k,k)$ found from the model-independent equation (5) (but not (!) the total function $F(p,k)$ at $p \neq k$), and, consequently, the phase shift $\delta_{3/2}(k)$ may have corrections depending on the details of the interaction potential only of the order of $\mathcal{O}\left(R^{4+2\varepsilon}\right)$. We omit a detailed cumbersome consideration of the same important statement proved by ourselves in the case of separable potentials of an arbitrary rank [4], as well as in the case of local potentials of the general type (acting in the s-state). The latter is demonstrated using the Faddeev equations and some identities for integral equations. But we omit these space consuming sophisticated transformations and calculations and restrict ourselves with a qualitative explanation of this result. We only note that, from the very first view, it might seem that the model dependence should reveal itself in the phase shift $\delta_{3/2}(k)$ already in terms of the order of R^2, because the two-nucleon amplitude is known to be model-independent up to and including the first order in the radius of forces. But the effect of an additional suppressing of the role of interaction potential details lies in the fact that the contribution of the potential into the phase shift is proportional also to a probability density to find the nucleons together at short distances, and this probability density is small due to the Pauli principle. More exactly, since the wave function of the system in an effective potential ε/R^2 behaves itself at short distances as $\sim R^{1+\varepsilon}$, this probability density given by the wave function absolute value squared appears to be proportional to $R^{2+2\varepsilon}$. Together with the smallness $\sim R^2$ of the model-dependent effects originating from the two-nucleon amplitude, one has the general result $\mathcal{O}\left(R^{4+2\varepsilon}\right)$ for the order of smallness of the effects depending on the model of interaction.

4. Model-Independent Explicit Solution for the Quartet *nd*-Scattering Length and Effective Range

Now, we find the solution of Eq. (5) to determine the quartet scattering length and effective range. In expansion (9), we consider only the scattering length a, the effective range r_0, and the "form parameter" $\mathfrak{f}$ (of the order of $\sim R^3$). The account for the higher order terms from expansion (9) could lead us to a contribution to the answer of higher orders than $\mathcal{O}\left(R^{4+2\varepsilon}\right)$.

The iteration solution of Eq. (5) in the small parameter $r_0/(2a)$ (further, we use the triplet experimental parameters a_t and r_{0t}) gives for the quartet scattering length $a_{3/2}$:

$$\frac{a_{3/2}}{a_t} = C_{\mathrm{STM}} + C_1\,\frac{r_{0t}}{2a_t} + C_2\left(\frac{r_{0t}}{2a_t}\right)^2 + C_3\left(\frac{r_{0t}}{2a_t}\right)^3 + C_4\left(\frac{r_{0t}}{2a_t}\right)^4 + C_{4+\varepsilon}\left(\frac{r_{0t}}{2a_t}\right)^{4+\varepsilon} + \tilde{C}_3\,\mathfrak{f}_t + \tilde{C}_4\,\mathfrak{f}_t\,\frac{r_{0t}}{2a_t} + \mathcal{O}\big(R^{4+2\varepsilon}\big), \qquad (17)$$

where the coefficients (for the integer powers of the small parameter)

$$\begin{aligned} &C_{\mathrm{STM}} \cong 1.179066, \quad C_1 \cong -0.071901, \\ &C_2 \cong -0.02979, \quad C_3 \cong 0.15319, \\ &C_4 \cong 0.9279, \quad \tilde{C}_3 \cong 0.425, \quad \tilde{C}_4 \cong 0.685. \end{aligned} \qquad (18)$$

One should also account for the known relation between the two-particle parameters

$$\mathfrak{f}_t = 1 - \frac{1}{\alpha a_t} - \frac{1}{2}\,\alpha r_{0t} + \mathcal{O}\left(R^5\right). \qquad (19)$$

The asymptotic expansion (17) is a series in integer powers of the radius of forces only up to the fourth power of the small parameter, and then, due to the appearance of divergencies of the integrals over dp from $\tilde{F}_n(p)$ at large momenta for $n \geq 4$ (where $\tilde{F}_n(p) \equiv F_n(p,0)$), the power series transforms into an expansion in $\sim R^{\varepsilon}$. The model-independent constants C_n are expressed in terms of solutions $\tilde{F}_n(p)$ (for $n = 0, 1, 2, 3$) of the limit equations similar to the STM one. The constant C_{STM} is the well-known STM constant, the values C_1 and C_2 were determined first in [5]. Note an important detail: although the solution $\tilde{F}_4$ does not exist, nonetheless, the constants C_4 and $\tilde{C}_4$ can be completely determined.

For the problem under consideration, it is a happy coincidence that expansion (17) contains a rather small parameter $r_{0t}/(2a_t) \cong 0.16$ (also $\mathfrak{f}_t$ is assumed to be near zero), and coefficients (18) of expansion (17) are not large. As a result, the exact solution (17) gives not only a correct qualitative result for the quartet scattering length, but even a rather accurate quantitative answer for this value. If one uses the experimental two-particle values $a_t = 5.424$ fm and $r_{0t} = 1.759$ fm, then, already in the zero range of forces (STM approximation), one has $a_{3/2} \approx C_{\mathrm{STM}}\,a_t \cong 6.395$ fm. The account for the terms up to the fourth order in (17) results in $a_{3/2} \cong 6.34$ fm, and this result coincides with the calculations with the use of the Faddeev equations with pair potentials fitted in such a way that to reproduce simultaneously the correct experimental values of the deuteron binding energy, the triplet scattering length, and the triplet effective range. The account for small errors in experimental values of a_t and r_{0t} results in the ultimate theoretical estimation of $a_{3/2}$ to be 6.34 ± 0.01 fm. This value is within the experimental gap: $a_{3/2} = 6.35 \pm 0.02$ fm.

The consideration of the quartet effective range on the basis of Eq. (5) using (8) results in the expansion

$$\frac{r_{3/2}}{2a_t} = D_0 + D_1\,\frac{r_{0t}}{2a_t} + D_2\left(\frac{r_{0t}}{2a_t}\right)^2 + \ldots, \qquad (20)$$

where the model-independent constants appear to be

$$D_0 \cong -0.01914, \quad D_1 \cong 1.05581, \quad D_2 \cong 0.19753. \qquad (21)$$

It should be noted that, unlike the STM approximation for the quartet scattering length, the approximation of the zero range of forces for the quartet effective range is insufficient even from qualitative point of view, since, within this approximation, $r_{3/2}$ is small and negative. Thus, the account for at least the first correction term $\sim r_{0t}/(2a_t)$ is very important. Within approximation (20), we have $r_{3/2} \approx 1.71$ fm. This result is in a qualitative concordance with the numerical calculations of $r_{3/2}$ carried out with the use of Faddeev equations ($r_{3/2} = 1.75 \pm 0.01$ fm [5]) (the gap in the $r_{3/2}$ value is present due to small experimental errors in a_t and r_{0t}). To achieve a better (quantitative) coincidence with the result obtained within the Faddeev equations formalism, one has to find the next terms in expansion (20).

5. Conclusions

To summarize, we note that the model-independent solutions are obtained for the low-energy parameters of the quartet *nd*-scattering phase shift in the form of explicit expressions in terms of the two-nucleon triplet low-energy parameters. These explicit formulae for the three-nucleon parameters have the form of expansions in powers of the experimental two-nucleon values ratio to be the correction terms to the STM approximation of the zero range of forces. It is explained why the quartet *nd*-scattering phase shift is model-independent to the fourth power in the radius of forces.

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1. A.G. Sitenko, V.F. Kharchenko, N.M. Petrov. The separable representation of the two-particle *t*-matrix and the three-nucleon problem. *Phys. Lett. B* **28** (5), 308 (1968).
2. V.F. Kharchenko, N.M. Petrov. Treatment of the three-nucleon problem based on the separable expansion of the two-particle *t*-matrix. *Nucl. Phys. A* **137** (2), 417 (1969).
3. V. Efimov. Qualitative treatment of three-nucleon properties. *Nucl. Phys. A* **362** (1), 45 (1981).
4. I.V. Simenog. The Problem of a model-independent description of the few nucleon interactions. Preprint ITP-84-11E, Kyiv, 1984.
5. B.E. Grinyuk, I.V. Simenog, A.I. Sitnichenko. Model-independent description of quartet *nd*-scattering at low energies. *Sov. J. Nucl. Phys.* (Engl. Transl.) **39** (2), 402 (1984).

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Б.Є. Гринюк, І.В. Сименог

МОДЕЛЬНО-НЕЗАЛЕЖНИЙ РОЗВ'ЯЗОК
ЗАДАЧІ *nd*-РОЗСІЯННЯ В КВАРТЕТНОМУ СТАНІ

Обґрунтовано точність модельно-незалежного опису фази пружного *nd*-розсіяння в квартетному стані і отримано явні розв'язки для низькоенергетичних параметрів розсіяння (квартетної довжини розсіяння $a_{3/2}$ та ефективного радіуса $r_{3/2}$) у вигляді асимптотично точних розкладів через відношення експериментальних двонуклонних низькоенергетичних параметрів.

Ключові слова: *nd*-розсіяння, квартетний стан, модельно-незалежний розв'язок.