https://doi.org/10.15407/ujpe68.2.81

A.G. KUMELA*, A.B. GEMTA, A.K. HORDOFA, T.A. DESTA, M. DANGISH, H.D. MEKONNEN

Department of Applied Physics, School of Applied Sciences,
Adama Science and Technology University
*(Adama, P.O. Box 1888, Ethiopia; *e-mail: alemayehu.getahun@astu.edu.et)*

THE QUANTUM FEATURES OF CORRELATED PHOTONS WITH THE EFFECT OF PHASE FLUCTUATIONS

We theoretically investigate the effect of phase fluctuations on correlated photons resulting from nondegenerate three-level atoms under the cavity radiation. The photon statistics, photon number correlation, and entanglement properties of the system have been calculated employing the dynamical equation of the system. It is shown that, for the sub-Poissonian photon statistics, the degree of correlation increases with the atomic pumping rate, and the entanglement varies with phase fluctuations, rather than with the atomic pumping rate. The proposed system is well suitable for the quantum information processing.

Keywords: quantum features, correlated photons, nondegenerate three-level laser, entanglement.

1. Introduction

For the last fifty years, photon correlation experiments have been at the forefront of quantum optics [1, 2]. In particular, photon-pair creation has been shown in a range of photonic chip platforms, such as crystalline and amorphous silicon nanowires, to be employed as a heralded single-photon source or as a time-bin entangled source [3–5]. In addition, the quantum futures of degenerate and nondegenerate three-level lasers with different pumpings and cavity modes have been investigated and yielded many novel results with unexpected phenomena [6, 7]. Electromagnetically induced transparency (EIT) [8], lasing without inversion (LWI) [9], and spontaneous emission quenching via quantum interference [10] are some instances. In this regard, the two-photon quantum optical device produces a strongly correlated light with some nonclassical features such as the squeezing and entanglement [11, 12]. Those can be significantly affected by phase fluctuations and the dephasing [13].

Moreover, the existence of the entanglement can be studied through certain experimental and mathematical criteria [14–16]. For example, the famous positive-partial-transpose (PPT) is the experimental criterion [17]. In which the necessary condition for the joint density matrix ρ of two systems A and B to be detachable. Some other mathematical criteria are the Hillery–Zubairy one, logarithmic negativity, and relative entropy of the entanglement (REE) [18]. The Hillery–Zubairy criterion is based on uncertainties solely based on the Cauchy–Schwarz inequality and the properties of separability. REE is a measure based

Citation: Kumela A.G., Gemta A.B., Hordofa A.K., Desta T.A., Dangish M., Mekonnen H.D. The quantum features of correlated photons with the effect of phase fluctuation. *Ukr. J. Phys.* **68**, No. 2, 81 (2023). https://doi.org/10.15407/ujpe68.2.81.

Цитування: Кумела А.Г., Гемта А.Б., Хордофа А.К., Деста Т.А., Дангиш М., Меконнен Х.Д. Квантові риси корельованих фотонів з ефектом флуктуацій фази. *Укр. фіз. журн.* **68**, №2, 81 (2023).

on the distance of the state to the closest separable state ($S(\rho \parallel \delta = \mathrm{Tr}(\rho \log \rho - \rho \log \delta))$) [19].

Recently, several pioneers have reported on the effects of a dephasing and phase fluctuations on various quantum features. For instance, S. Tesfa [13] investigated the effect of phase fluctuations and the dephasing on the dynamics of the entanglement generated from a coherently pumped correlated emission laser. It was also found that the time evolution of the entanglement is significantly reliant on phase fluctuations and the dephasing at the early stages of the lasing process. In addition, F.X. Sun *et al.* [20] developed the theory of phase control coherence, entanglement, and quantum steering for an optomechanical system composed of a partially transmitting dielectric membrane and driven by short laser pulses. It was shown that the perfect coherence among the field modes excludes the possibility of the modes being entangled. Further, S. Qamar and Ms. Zubairy [21] investigated the influence of phase fluctuations on the time evolution of the entanglement generation in a three-level correlated emission laser (CEL) with injected coherence. The fluctuations in the phase ϕ modify the coherence between the two corresponding atomic levels. In addition, the strong influence of phase fluctuations on the entanglement generation was observed using the injected coherence CEL system. However, the effect of a dephasing resulting from the involved quantum phenomena such as vacuum fluctuations and the atomic broadening along with the phase fluctuations corresponding to the incapability of preparing the atoms initially in the perfect 50 : 50 probability are the main issues that are not addressed in the literature.

In this research, we study the non-degenerate three-level laser that is pumped from the cavity radiation mode as a partially coherent superposition of upper and lower atomic states having a certain phase associated with the corresponding atomic levels. The fluctuations in this randomly distributed phase play a key role in modifying the atomic coherence. The effect of phase fluctuations was investigated using the DGCZ criterion with the logarithmic negativity and the HZ criterion with the violation of the Cauchy–Schwarz (CS) inequality.

2. The Model and Dynamical Equations

The Hamiltonian that describes the interaction of a nondegenerate three-level atom with a two-mode cavity radiation of light modes $\hat{a}_1$ and $\hat{a}_2$ can be expressed in the rotating-wave approximation and the interaction picture as [22]

$$\hat{H}_I = ig\big[|3\rangle\langle 2|\hat{a}_1 - \hat{a}_1^\dagger|2\rangle\langle 3| + |2\rangle\langle 1|\hat{a}_2 - \hat{a}_2^\dagger|1\rangle\langle 2|\big], \quad (1)$$

where g is a coupling constant, which is taken to be the same for both transitions, whereas $\hat{a}_1(\hat{a}_2)$ are the annihilation operators for the modes of a cavity radiation. In writing Eq. (1), we have considered $\hbar = 1$ for simplicity only.

In this study, we take the initial state of a three-level atom to be

$$|\psi_A(0)\rangle = C_3(0)|3\rangle + C_1(0)e^{i\varphi}|1\rangle. \quad (2)$$

Here, $C_3(0)$ and $C_1(0)e^{i\varphi}$ are probability amplitudes for the atom initially in the upper and lower energy levels, respectively, and φ is an arbitrary phase difference between the two states. Note that this term does not include the fluctuations originating from the instability and broadening of the pumped laser. Since φ can be randomly distributed about a fixed mean phase φ_0, and the contribution of every phase change does not seem realistic. Taking the phase fluctuations as a Gaussian random process [23, 24] and using a deviation of the phase fluctuation instead of the actual phase have led to

$$\langle \exp(\pm i\delta\varphi)\rangle = \exp(-\langle \delta\varphi^2/2\rangle). \quad (3)$$

The Gaussian random process, $\langle \delta\varphi\rangle$ is zero for $\langle \delta\varphi^2/2\rangle = \theta$ which represents a deviation of the phase fluctuation from φ_0, taken as zero for convenience, which is generally designated simply as a phase fluctuation. Hence, the initial density operator for a single atom has the form

$$\hat{\rho}_A(0) = \rho_{33}^{(0)}|3\rangle\langle 3| + \rho_{31}^{(0)}|3\rangle\langle 1| + \rho_{13}^{(0)}|1\rangle\langle 3| + \rho_{11}^{(0)}|1\rangle\langle 1|, \quad (4)$$

where $\rho_{33}^{(0)} = |C_3|^2$ and $\rho_{11}^{(0)} = |C_1|^2$, are the initial populations of the atom corresponding to the upper and lower levels, respectively, and $\rho_{31}^{(0)} = |\rho_{31}|e^{i(\varphi_0+\delta\varphi)}$ is the initial partial coherence. This states that the three-level atom is initially prepared in a coherent superposition of the top and bottom levels [25].

Thus, we apply the linear and adiabatic approximation schemes [26] in the good cavity limit that the equation of the density operator evolution for the cavity modes. In the absence of a damping through the coupled mirror, [6], has the form:

$$\dot{\hat{\rho}}_1(t) = \frac{A\rho_{33}^{(0)}}{2}\left[2\hat{a}_1^\dagger\hat{\rho}\hat{a}_1 - \hat{a}_1\hat{a}_1^\dagger\hat{\rho} - \hat{\rho}\hat{a}_1\hat{a}_1^\dagger\right] + \\ + \frac{A\rho_{11}^{(0)}}{2}\left[2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{a}_2\hat{\rho} - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2\right] - \\ - \frac{A\rho_{31}^{(0)}}{2}\left[2\hat{a}_2\hat{\rho}\hat{a}_1 - \hat{\rho}\hat{a}_1\hat{a}_2 - \hat{a}_1\hat{a}_2\hat{\rho}\right]e^{i\varphi} - \\ - \frac{A\rho_{13}^{(0)}}{2}\left[2\hat{a}_1^\dagger\hat{\rho}\hat{a}_2^\dagger - \hat{\rho}\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2^\dagger\hat{\rho}\right]e^{-i\varphi}, \quad (5)$$

where $A = \frac{2g^2 r_a}{\gamma^2}$ is the linear gain coefficient [27], and, for convenience, we have set $\rho_{13}^{(0)} = \rho_{31}^{(0)*}$.

Then we consider a system coupled with a two-mode vacuum reservoir. The density operator which is extracted from the vacuum reservoir by the partial trace operation is [6]

$$\dot{\hat{\rho}}_2(t) = \frac{\kappa}{2}[2\hat{a}_1\hat{\rho}\hat{a}_1^\dagger - \hat{a}_1^\dagger\hat{a}_1\hat{\rho} - \hat{\rho}\hat{a}_1^\dagger\hat{a}_1] + \\ + \frac{\kappa}{2}[2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{a}_2\hat{\rho} - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2]. \quad (6)$$

Using Eqs. (5) and (6), the master equation for the system takes the form:

$$\dot{\hat{\rho}}(t) = \frac{\kappa}{2}[2\hat{a}_1\hat{\rho}\hat{a}_1^\dagger - \hat{a}_1^\dagger\hat{a}_1\hat{\rho} - \hat{\rho}\hat{a}_1^\dagger\hat{a}_1] + \\ + \frac{1}{2}A\rho_{33}^{(0)}[2\hat{a}_1^\dagger\hat{\rho}\hat{a}_1 - \hat{a}_1\hat{a}_1^\dagger\hat{\rho} - \hat{\rho}\hat{a}_1\hat{a}_1^\dagger] + \\ + \frac{1}{2}(A\rho_{11}^{(0)} + \kappa)[2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{a}_2\hat{\rho} - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2] + \\ + \frac{A\rho_{31}^{(0)}}{2}\left[2\hat{a}_2\hat{\rho}\hat{a}_1 - \hat{\rho}\hat{a}_1\hat{a}_2 - \hat{a}_1\hat{a}_2\hat{\rho}\right]e^{i\varphi} + \\ + \frac{A\rho_{13}^{(0)}}{2}\left[2\hat{a}_1^\dagger\hat{\rho}\hat{a}_2^\dagger - \hat{\rho}\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2^\dagger\hat{\rho}\right]e^{-i\varphi}. \quad (7)$$

The above master equation can be used to derive a time variation for the expectation values of various system operators. The terms proportional to $\rho_{33}^{(0)}$ and $\rho_{11}^{(0)}$ describe the gain of a cavity light for the mode a_1 and a loss for the mode a_2, respectively. The terms proportional to $\rho_{31}^{(0)}$ are related to the correlation of the generated radiation that indicates the existence of quantum features. These terms are responsible for the squeezing obtained in the cascade laser system. Furthermore, the terms proportional to κ describe the damping of cavity modes due to their coupling with a two-mode vacuum reservoir via a single-port mirror.

This proves that it is useful to introduce a new parameter that relates the probabilities of the atom to be in the upper and lower levels [28]. We define the parameter η such that $\rho_{33}^{(0)} = \frac{1-\eta}{2}$ with $-1 < \eta < 1$. For three-level atoms initially in a coherent superposition of the top and bottom levels, one obtains: $\rho_{11}^{(0)} = \frac{1+\eta}{2}$ and, in view of the relation $|\rho_{31}^{(0)}|^2 = \rho_{33}^{(0)}\rho_{11}^{(0)}$, one easily finds $\rho_{31}^{(0)} = \frac{1}{2}e^{-\theta}\sqrt{1-\eta^2}$.

Employing the master equation (7), the evolution of the two-mode cavity radiation in terms of c-number variables associated with the normal ordering, $\alpha_1(t)$ and $\alpha_2(t)$ can be expressed in the form [29]

$$\frac{d}{dt}\alpha_1(t) = -\Gamma_+\alpha_1(t) - \xi_+\alpha_2^*(t) + f_1(t), \\ \frac{d}{dt}\alpha_2(t) = -\Gamma_-\alpha_2(t) - \xi_-\alpha_1^*(t) + f_2(t), \quad (8)$$

where

$$\Gamma_\pm = \frac{\kappa}{2} - \frac{A}{4}(\eta \pm 1), \quad \xi_\pm = \pm\frac{A}{4}e^{-\theta}\sqrt{1-\eta^2}; \quad (9)$$

$f_1(t)$ and $f_2^*(t)$ are noise forces whose properties remain to be determined, $\alpha_1(t)$ and $\alpha_2(t)$ are the c-number variables corresponding to the cavity-mode operators $\hat{a}_1$ and $\hat{a}_2$.

Following the procedure described in [30], we obtain:

$$\alpha_1(t) = A_+(t)\alpha_1(0) + B_+(t)\alpha_2^*(0) + F_+(t) + W_+(t), \quad (10)$$

$$\alpha_2(t) = A_-(t)\alpha_2(0) + B_-(t)\alpha_1^*(0) + F_-(t) + W_-(t), \quad (11)$$

where

$$A_\pm(t) = \frac{1}{2}\left[(1 \pm p)e^{-\lambda_- t} + (1 \mp p)e^{-\lambda_+ t}\right],$$

$$B_\pm(t) = \frac{q_\pm}{2}\left[e^{-\lambda_+ t} - e^{-\lambda_- t}\right], \quad (12)$$

$$F_\pm(t) = \frac{1}{2}\int\limits_0^t\Big[(1 \pm p)e^{-\lambda_-(t-t')} + \\ + (1 \mp p)e^{-\lambda_+(t-t')}\Big]f_1(t')dt', \quad (13)$$

$$W_{\pm}(t) = \frac{q_+}{2}\int_0^t \left[e^{-\lambda_+(t-t^{'})} - e^{-\lambda_-(t-t^{'})}\right] f_2^*(t^{'})dt^{'},$$
$$W_{-}(t) = \frac{q_-}{2}\int_0^t \left[e^{-\lambda_+(t-t^{'})} - e^{-\lambda_-(t-t^{'})}\right] f_1^*(t^{'})dt^{'},$$
(14)

with

$$p = \frac{1}{\eta}, \quad q_{\pm} = \frac{\pm e^{-\theta}\sqrt{1-\eta^2}}{\eta}, \quad \lambda_{\pm} = \frac{\kappa}{2} + \frac{A}{4}(\eta \pm 1). \tag{15}$$

The correlation properties of the noise forces $f_1(t)$ and $f_2(t)$ associated with the normal ordering, satisfy the relations

$$\begin{aligned}&\langle f_1(t)\rangle = \langle f_2(t)\rangle = \langle f_1(t^{'})f_1(t)\rangle = \\ &= \langle f_2(t)f_2(t^{'})\rangle = 0, \\ &\langle f_1^*(t)f_2(t^{'})\rangle = \langle f_2(t^{'})f_2^*(t)\rangle = \\ &= \langle f_2^*(t^{'})f_1(t)\rangle = 0,\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\langle f_1(t^{'})f_1^*(t)\rangle = \frac{A(1-\eta)}{2}\delta(t-t^{'}), \\ &\langle f_2(t^{'})f_1(t)\rangle = -\frac{\xi_-}{2}\delta(t-t^{'}).\end{aligned} \tag{17}$$

3. Quadrature Fluctuations

This section deals with the effects of phase fluctuations and the dephasing on the degree of squeezing by applying quadrature operators. A two-mode light is, generally, said to be in the squeezed state for when the quadrature variances satisfy $\Delta c_{\pm}^2 \leq 1$, and the uncertainty relation holds [31]

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2), \tag{18}$$

where $\hat{a}_1$ and $\hat{a}_2$ represent the separate modes. With this consideration, the squeezing properties of the cavity radiation can be studied by applying the quadrature operators defined by [32]

$$\begin{aligned}&\hat{c}_+ = (\hat{c}^\dagger + \hat{c}), \\ &\hat{c}_- = i(\hat{c}^\dagger - \hat{c}),\end{aligned} \tag{19}$$

where the corresponding variance can be readily obtained in terms of c-number variables associated with the normal ordering using Eqs. (18) and (19) as

$$\begin{aligned}\Delta c_{\pm}^2 &= 1 + \langle\alpha_1^*(t)\alpha_1(t)\rangle + \langle\alpha_2^*(t)\alpha_2(t)\rangle + \langle\alpha_1^*(t)\alpha_2(t)\rangle + \\ &+ \langle\alpha_1(t)\alpha_2^*(t)\rangle \pm \big\{\langle\alpha_1(t)\alpha_2(t)\rangle + \langle\alpha_1^*(t)\alpha_2^*(t)\rangle + \\ &+ \frac{1}{2}[\langle\alpha_1^2(t)\rangle + \langle\alpha_2^2(t)\rangle + \langle\alpha_1^{*2}(t)\rangle + \langle\alpha_2^{*2}(t)\rangle]\big\}.\end{aligned} \tag{20}$$

It is necessary to determine the various correlations described in Eq. (20) by using Eqs. (16) and (17). In line with this, assuming the cavity to be initially in a two-mode vacuum state, and the noise force at time t is not statistically related to the cavity mode variables at earlier times, we can readily verify that

$$\langle\alpha_1^2\rangle = \langle\alpha_2^2\rangle = \langle\alpha_1\alpha_2^*\rangle = \langle\alpha_1^*\alpha_2\rangle = 0. \tag{21}$$

It is possible to express the variance of the quadrature operators (20) in terms of the c-number variables associated with the normal ordering as [29]

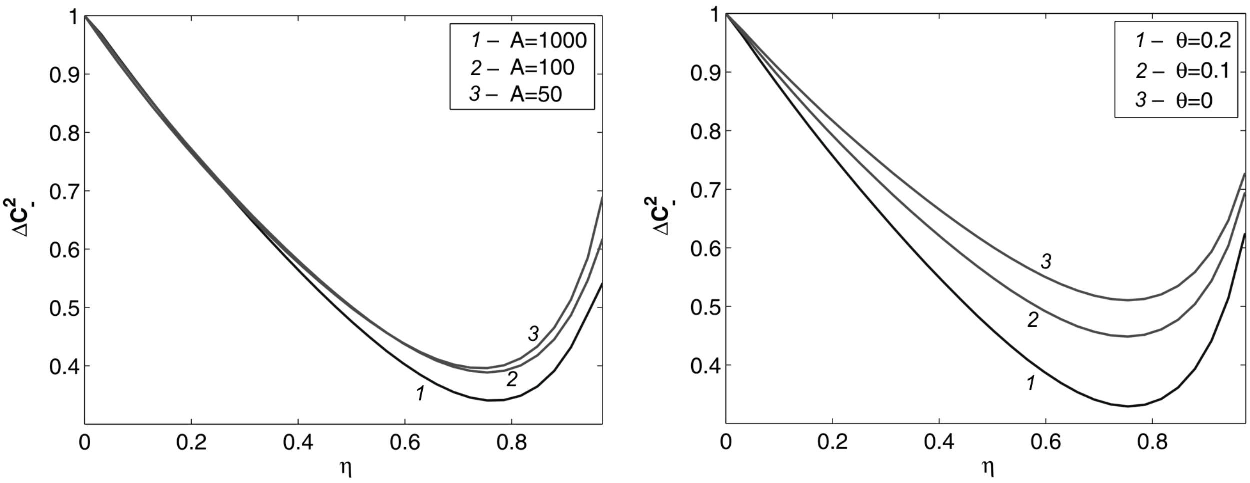
$$\Delta c_{\pm}^2 = 1 + \langle\alpha_1^*(t)\alpha_1(t)\rangle + \langle\alpha_2^*(t)\alpha_2(t)\rangle \pm 2\langle\alpha_1(t)\alpha_2(t)\rangle, \tag{22}$$

where

$$\begin{aligned}\langle\alpha_1^*\alpha_1\rangle &= -\frac{Ae^{-\theta\sqrt{1-\eta^2}}}{16\eta^2}\times \\ &\times\left[\left(\frac{(\eta^2-1)}{2k+A(\eta+1)}\frac{(\eta^2-1)}{2k+A(\eta-11)}\right) + \frac{2(\eta^2+1)}{2k+A\eta}\right] + \\ &+ \frac{Ae^{-\theta}\sqrt{1-\eta^2}(1-\eta)}{4\eta^2}\times \\ &\times\left[\frac{2}{2k+A\eta} + \frac{\eta-1}{2k+A(\eta+1)} + \frac{\eta+1}{2k+A(\eta-1)}\right] - \\ &- \frac{Ae^{-\theta}\sqrt{1-\eta^2}(1-\eta^2)}{16\eta^2}\times \\ &\times\left[\frac{-2}{2k+A\eta} + \frac{1}{2k+A(\eta+1)} + \frac{1}{2k+A(\eta-1)}\right],\end{aligned} \tag{23}$$

$$\begin{aligned}\langle\alpha_2^*\alpha_2\rangle &= \frac{Ae^{-\theta\sqrt{1-\eta^2}}}{16\eta^2}\times \\ &\times\left[\left(\frac{(\eta^2-1)}{2k+A(\eta+1)} + \frac{(\eta^2-1)}{2k+A(\eta-11)}\right) + \frac{2(\eta^2+1)}{2k+A\eta}\right] - \\ &- \frac{Ae^{-\theta}\sqrt{1-\eta^2}(1-\eta)}{4\eta^2}\times \\ &\times\left[\frac{2}{2k+A\eta} + \frac{\eta-1}{2k+A(\eta+1)} - \frac{\eta+1}{2k+A(\eta-1)}\right] - \\ &- \frac{Ae^{-3\theta}\sqrt{1-\eta^2}(1-\eta^2)}{16\eta^2}\times \\ &\times\left[\frac{-2}{2k+A\eta} + \frac{1}{2k+A(\eta+1)} + \frac{1}{2k+A(\eta-1)}\right],\end{aligned} \tag{24}$$

ISSN 2071-0186. Ukr. J. Phys. 2023. Vol. 68, No. 2



a *b*

Fig. 1. Plots of the minus quadrature variance [Eq. (22)] versus η (a) for $\kappa = 0.2$, $\theta = 0.02$, and, for different values of A, (b) for $\kappa = 0.2$, $A = 100$ and for different values of θ

$$\langle\alpha_1\alpha_2\rangle = -\frac{Ae^{-\theta\sqrt{1-\eta^2}}}{16\eta^2}\times$$
$$\times\left[\left(\frac{(\eta^2-1)}{2k+A(\eta+1)}+\frac{(\eta^2-1)}{2k+A(\eta-11)}\right)+\frac{2(\eta^2+1)}{2k+A\eta}\right]+$$
$$+\frac{Ae^{-\theta}\sqrt{1-\eta^2}(1-\eta)}{4\eta^2}\times$$
$$\times\left[\frac{2}{2k+A\eta}+\frac{\eta-1}{2k+A(\eta+1)}+\frac{\eta+1}{2k+A(\eta-1)}\right]-$$
$$-\frac{Ae^{-\theta}\sqrt{1-\eta^2}(1-\eta^2)}{16\eta^2}\times$$
$$\times\left[\frac{-2}{2k+A\eta}+\frac{1}{2k+A(\eta+1)}+\frac{1}{2k+A(\eta-1)}\right]. \quad (25)$$

Figure 1, *a* and *b* shows that the quadrature variance decreases, as η increases, but the degree of squeezing increases with the atomic pumping rate (r_a). The maximum squeezing is 68% for $\eta = 0.8$ in both figures which show that the system is at the perfect squeezing. In addition, the degree of squeezing is independent of the phase fluctuations ($\Delta = e^{-\theta}\sqrt{1-\eta^2}$) for $\eta = 1$, it is the reason why the graph does not excite one. Moreover, comparing Figs. 2, *a* and *b*, one can clearly see that the effect of phase fluctuations is greater than that of the atomic pumping (r_a). Thus, the result agrees with work [33].

4. Entanglement Quantification

Here, we tend to study the degree of entanglement of the two-mode cavity light produced by a nondegenerate three-level cascade laser whose cavity contains a parametric amplifier. A pair of particles is taken to be entangled in quantum theory, if their states cannot be expressed as a product of the states of the individual constituents. The preparation and manipulation of these entangled states that have non-classical and non-local properties lead to a better understanding of the basic quantum principles [34]. If the density operator for the combined state cannot be described as a combination of the product of the density operators of the constituents, $\hat{\rho} \neq \sum_j P_j\hat{\rho}_j^{(1)} \otimes \hat{\rho}_j^{(2)}$, in which $P_j \geq 0$ and $\sum_j P_j = 1$ are set to ensure the normalization of the combined density of state.

A criterion to study the entanglement is the logarithmic negativity which is used for the variables of two-mode continuous based on the negativity of the partial transposition [35, 36]. The negative partial transpose must be parallel with respect to the entanglement monotone in order to obtain the degree of entanglement. The logarithmic negativity is combined with the negative partial transpose in another case where V_S represents the smallest eigenvalue of the symplectic matrix [35]:

$$V_S = \sqrt{\frac{\sigma - \sqrt{(\sigma^2 - 4\det\Gamma)}}{2}} < 1. \quad (26)$$

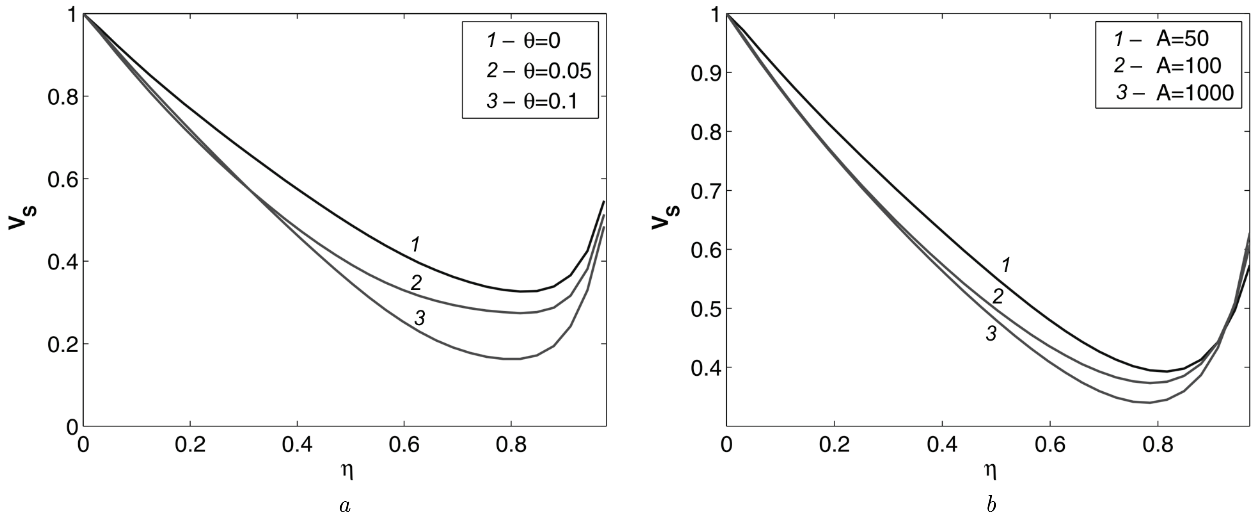


Fig. 2. Plots of the logarithmic negativity [Eq. (26)] versus η (a) for $\kappa = 0.2$, $A = 100$, and for different values of θ, (b) for $\kappa = 0.2$, $A = 25$, $\theta = 0.2$, and for different values of A

Where the invariant and covariance matrices are respectively denoted as:

$$\sigma = \det\Omega_1 + \det\Omega_2 - 2\det\Omega_{12}, \tag{27}$$

$$\Gamma = \begin{pmatrix} \Omega_1\,\Omega_{12} \\ \Omega_{12}^T\,\Omega_2 \end{pmatrix}, \tag{28}$$

in which Ω_1 and Ω_2 are the covariance matrices describing each mode separately, while Ω_{12} are the intermodal correlations. The elements of the matrix in Eq. (28) are given by:

$$\Gamma_{ij} = 1/2\langle\hat{X}_i\hat{X}_j + \hat{X}_j\hat{X}_i\rangle - \langle\hat{X}_i\rangle\langle\hat{X}_j\rangle, \tag{29}$$

in which $i, j = 1, 2, 3, 4$. The quadrature operators are defined as

$$\begin{aligned} &\hat{X}_1 = \hat{a}_1 + \hat{a}_1^\dagger, \\ &\hat{X}_2 = i(\hat{a}_1^\dagger - \hat{a}_1), \\ &\hat{X}_3 = \hat{a}_2 + \hat{a}_2^\dagger, \\ &\hat{X}_4 = i(\hat{a}_2^\dagger - \hat{a}_2). \end{aligned} \tag{30}$$

With this introduction, the extended covariance matrix takes the form

$$\Gamma = \begin{pmatrix} \Sigma_1 & 0 & \Sigma_{12} & 0 \\ 0 & \Sigma_1 & 0 & -\Sigma_{12} \\ \Sigma_{12} & 0 & \Sigma_2 & 0 \\ 0 & -\Sigma_{12} & 0 & \Sigma_2 \end{pmatrix}, \tag{31}$$

where $\Sigma_1 = 2\langle\alpha_1^*\alpha_1\rangle + 1$, $\Sigma_{12} = 2\langle\alpha_1\alpha_2\rangle$, $\Sigma_2 = 2\langle\alpha_2^*\alpha_2\rangle + 1$ are c-number variables associated with the normal ordering. Logarithmic negativity is defined as:

$$E_N = \max\,[0, -\log_2 V_S]. \tag{32}$$

The entanglement is achieved, when E_N is positive within the region of the lowest eigenvalue of the covariance matrix $V_S < 1$ [37].

In view of Eq. (28) along with (31), we can readily show that

$$\begin{aligned} &\det\Omega_1 = 1 + 4\langle\alpha_1^*(t)\,\alpha_1(t)\rangle[\langle\alpha_1^*(t)\alpha_1(t)\rangle + 1], \\ &\det\Omega_2 = 1 + 4\langle\alpha_2^*(t)\,\alpha_2(t)\rangle[\langle\alpha_2^*(t)\alpha_2(t)\rangle + 1], \\ &\det\Omega_{12} = -4\langle\alpha_1(t)\,\alpha_2(t)\rangle^2. \end{aligned} \tag{33}$$

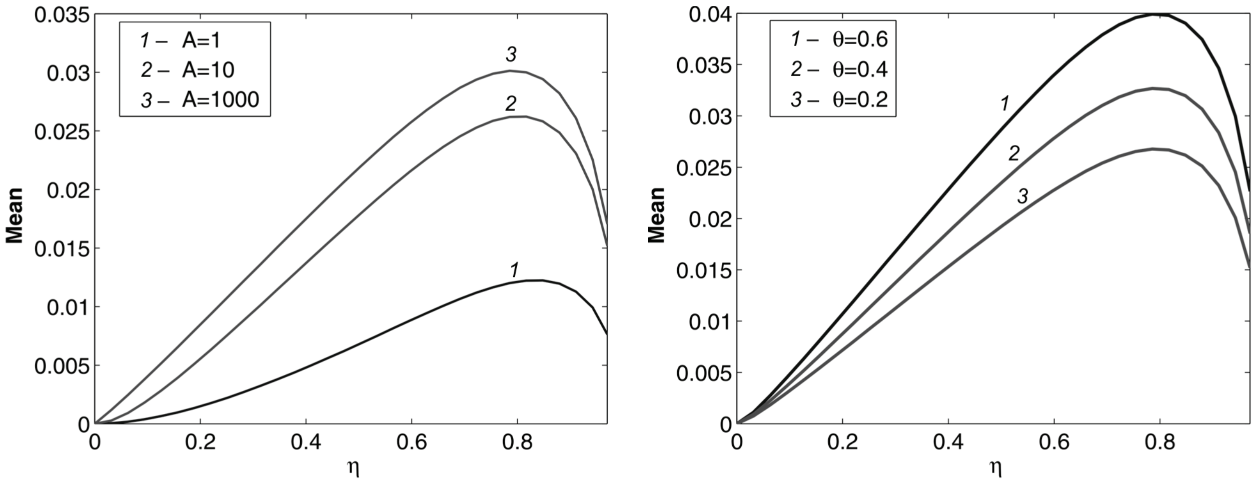
It is also possible to establish that:

$$\det\Gamma = \left[\sqrt{\det\Omega_1\det\Omega_2} - \sqrt{\det\Omega_{12}^T\det\Omega_{12}}\right]^2. \tag{34}$$

It is possible to see from Fig. 3, a and b that the cavity radiation of the nondegenerate three-level laser exhibits the entanglement based on the criteria of Eq. (27). The degree of entanglement is enhanced by the atomic pumping rate (r_a), which agrees with the previous study [38]. The separation of Fig. 3, a is greater than that of b, which indicates that the degree of correlation is greater for Fig. 3, a. From this, we can conclude that the entanglement is highly affected by phase fluctuations, rather than by the atomic pumping rate [39].

5. Photon Statistics

In this section, we study the statistical properties of the cavity radiation of a three-level cascade laser, such



a *b*

Fig. 3. Plots of the mean photon number [Eq. (36)] versus η (a) for $\kappa = 0.6$, $\theta = 0.5$, and for different values of A, (b) for $\kappa = 0.8$, $A = 100$, and for different values of θ

as the mean photon number, Mandel's Q-factor, and the normalized second-order correlation function for the system under consideration.

5.1. Mean photon number

In order to know about the brightness of the generated light [40], it is necessary to study the mean number of photon pairs describing the two-mode cavity radiation that can be defined as [41]

$$\bar{N} = \langle \hat{c}^{\dagger}(t)\hat{c}(t)\rangle. \tag{35}$$

It then follows that

$$\bar{N} = \frac{1}{2}\left[\langle \alpha_1^*(t)\alpha_1(t)\rangle + \langle \alpha_2^*(t)\alpha_2(t)\rangle\right]. \tag{36}$$

Since $\langle \alpha_1^*(t)\alpha_1(t)\rangle$ and $\langle \alpha_2^*(t)\alpha_2(t)\rangle$ represent the mean photon numbers in mode a_1 and mode a_2, respectively, $\bar{N}$ can be interpreted as the mean number of photon pairs. As is seen from Eq. (36), the term wich contains ε represents the contribution from the external driving coherent light of the parametric amplifier to the total mean photon number. Therefore, it is easy to verify that Eq. (36) represents the mean number of photon pairs of the system.

From Fig. 4, a and b, one can observe the mean photon number versus η for different values of κ, r_a and θ. The mean photon number increase with the values of η, r_a and phase fluctuations, since more atoms are expected to participate in the spontaneous emission process. Comparing Fig. 4, a and b, we can state that the mean photon number is highly affected by the atomic pumping rate (r_a), rather than phase fluctuations (Δ), which agrees with work [42].

5.2. Mandel's Q-Factor

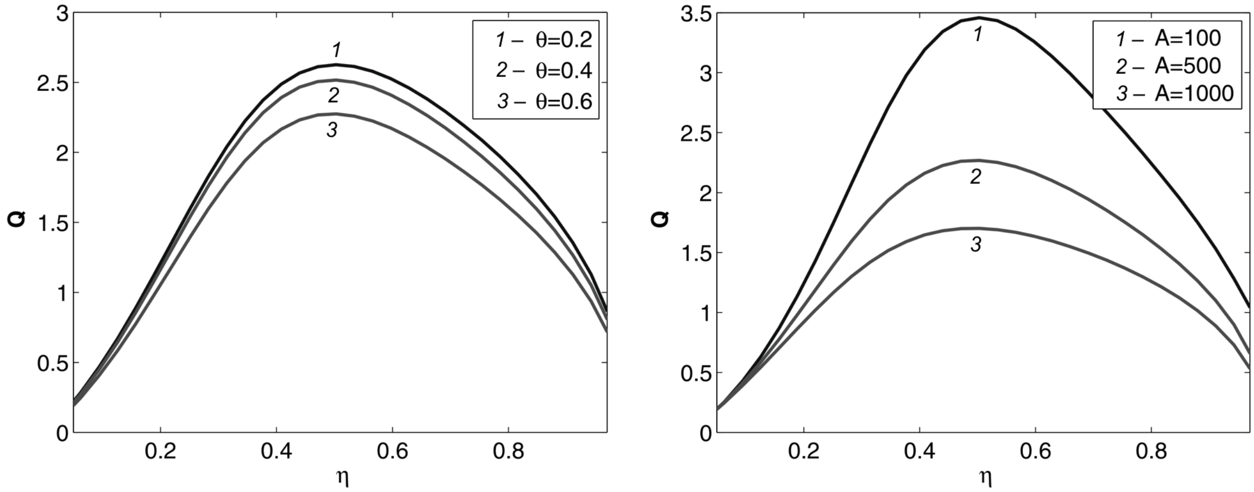
It is a common experience that a nonclassical photon number correlation can be studied by applying the measure of a departure of the photon statistics from the Poisson character [43]. This measure of departure can be represented by Mandel's Q-factor defined as [44]

$$Q = \frac{\langle(\Delta\hat{n})^2\rangle - \langle\hat{n}\rangle}{\langle\hat{n}\rangle}, \tag{37}$$

where $\hat{n} = \hat{c}^{\dagger}\hat{c}$ is the photon number operator of the two-mode cavity radiation. It is not difficult to verify that Eq. (37) can be expressed by putting the operators in the normal ordering as

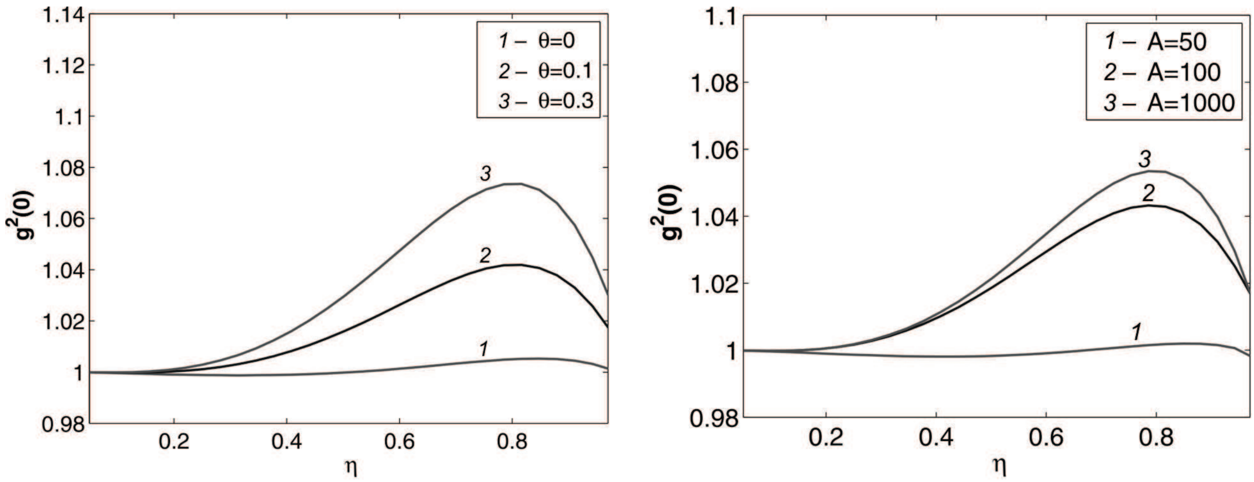
$$Q = \frac{\langle\hat{c}^{\dagger 2}\hat{c}^2\rangle + \langle\hat{c}^{\dagger}\hat{c}\rangle^2}{\langle\hat{c}^{\dagger}\hat{c}\rangle}, \tag{38}$$

where $\hat{c} = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2)$ is the annihilation operator that describes the two-mode cavity radiation. With the help of this, the normal ordering of the operators would not be altered, since $\hat{a}_1$ and $\hat{a}_2$ commute. Hence, it is possible to put the resulting expression in terms of c-number variables associated with



a *b*

Fig. 4. Plots of Mandel's Q-factor [Eq. (40)] versus η (a) for $\kappa = 0.2$, $A = 150$, and for different values of θ, (b) for $\kappa = 0.2$, $\theta = 0.4$, and for different values of A



a *b*

Fig. 5. (a) and b clearly shows that the degree of correlation is greater than that which agrees with the (a) values of θ, (b) for $\kappa = 0$, $\theta = 0.05$, and for different values of A

the normal order as

$$Q = \frac{\langle \gamma^{*2}(t)\gamma^2(t)\rangle + \langle \gamma^*(t)\gamma(t)\rangle^2}{\langle \gamma^*(t)\gamma(t)\rangle}, \tag{39}$$

where $\gamma = \frac{1}{\sqrt{2}}(\alpha_1(t) + \alpha_2(t))$. Hence, employing Eqs. (18) and (19), we get

$$Q = \bar{N} + \frac{\langle \alpha_1(t)\alpha_2(t)\rangle^2}{\bar{N}}. \tag{40}$$

It is clearly seen from Fig. 4, a and b that the Mandel Q-factor for a nondegenerate three-level laser is defiantly positive. This implies the generated radiation has supper-Poissonian photon statistics. Moreover, the result presented in this figure shows that the Mandel Q-factor practically increases ($\eta \leq 5$) with both phase fluctuations and the atomic pumping rate.

It is well known that the negativity of Mandel's parameter refers to a sub-Poissonian character of the photon statistics that essentially refers to a nonclassical property. Since the mean number of photon pairs and $\langle \alpha_1(t)\alpha_2(t)\rangle^2$ are positive, the Mandel's Q-factor in this case is definitely greater than 0. This allows the generated radiation to demonstrate the super-Poissonian photon statistics, while exhibiting nonclassical properties such as the squeezing and entanglement.

5.3. Photon number correlations

The normalized second-order correlation function for the two-mode light can be expressed as[45]

$$g^{(2)}_{(a_1,a_2)}(0) = \frac{\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2 \rangle}{\langle \hat{a}_1^\dagger \hat{a}_1 \rangle \langle \hat{a}_2^\dagger \hat{a}_2 \rangle}. \quad (41)$$

We realize that the operators in Eq. (41) are in the normal order. Therefore, the second-order correlation function can be expressed in terms of the c-number variables associated with the normal ordering as

$$g^{(2)}_{(a_1,a)_2}(0) = 1 + \frac{\langle \alpha_1(t)\alpha_2(t) \rangle^2}{\langle \alpha_1^*(t)\alpha_1(t) \rangle \langle \alpha_2^*(t)\alpha_2(t) \rangle}. \quad (42)$$

Figure 5, *a* and *b* clearly shows that the degree of correlation is greater than one which agrees with the mathematical result. The correlation increase with both the atomic pumping rate and phase fluctuations. Figures 5, *a* and *b* are almost the same, which means that the effects of phase fluctuations and the atomic pumping rate are equal in correlations.

6. Conclusion

In this study, the effects of phase fluctuations and the dephasing on the quantum features and statistical properties of the cavity radiation of a two-photon coherent beat laser are presented by applying the patient master equation. The master equation has additional terms resulting from the change of sign due to the phase fluctuations associated with the partial preparation. This implies that there is a possibility for regaining the quantum properties that have been lost due to the incapability of preparing atoms in a particular atomic coherence by externally driving them with classical radiation. In addition, applying the criterion of density operator (logarithmic negativity) and the sum of the variance of the quadrature operators (Duan–Gieke–Cirac–Zoller criterion) fundamentally lead to similar results, and shows that the squeezing and entanglement increase with both phase fluctuations and the atomic pumping rate.

1. B. Hensen, H. Bernien, A.E. Dr'eau, A. Reiserer, N. Kalb, M.S. Blok, J. Ruitenberg, R.F. Vermeulen, R.N. Schouten, C. Abell'an *et al.* Loophole-free bell inequality violation using electron spins separated by 1.3 kilometres. *Nature* **526**, 682 (2015).
2. H.-A. Bachor, T.C. Ralph. *A Guide to Experiments in Quantum Optics* (John Wiley & Sons, 2019).
3. F. Priolo, T. Gregorkiewicz, M. Galli, T.F. Krauss. Silicon nanostructures for photonics and photovoltaics. *Nature Nanotechnology* **9** (1), 19 (2014).
4. C. Xiong, C. Monat, M.J. Collins, A.S. Clark, C. Grillet, G.D. Marshall, M. Steel, J. Li, L. O'Faolain, T.F. Krauss *et al.* Improved car and noise analysis for photon-pair generation in an ultra-compact silicon slow-light photonic crystal waveguide. In: *2011 2nd International Conference on Photonics* (2011), pp. 1–5.
5. I. Krasnokutska. *Photonic Circuits Engineering in Lithium Niobate on Insulator.* Ph.D. thesis (RMIT University, 2019).
6. C. Gashu, E. Mosisa, T. Abebe. Entanglement quantification of correlated photons generated by three-level laser with parametric amplifier and coupled to a two-mode vacuum reservoir. *Advances in Math. Phys.* **2020** (2020).
7. D. Ayehu, A. Chane. The effect of superposition on the quantum features of the cavity radiation of a three-level laser. *Ukr. J. Phys.* **66** (9), 761 (2021).
8. Y. Guo, L. Yan, W. Pan, B. Luo, K. Wen, Z. Guo, X. Luo. Electromagnetically induced transparency (eit)-like transmission in side-coupled complementary split-ring resonators. *Optics Express* **20** (22), 24348 (2012).
9. A. Mysyrowicz, R. Danylo, A. Houard, V. Tikhonchuk, X. Zhang, Z. Fan, Q. Liang, S. Zhuang, L. Yuan, Y. Liu. Lasing without population inversion in n 2+. *APL Photonics* **4** (11), 110807 (2019).
10. M. Sahrai, S. Asadpour, R. Sadighi-Bonabi. Optical bistability via quantum interference from incoherent pumping and spontaneous emission. *J. Luminescence* **131** (11), 2395 (2011).
11. F. Dell'Anno, S. De Siena, F. Illuminati, Multiphoton quantum optics and quantum state engineering. *Phys. Rep.* **428** (2–3), 53 (2006).
12. A.G. Kumela, A.B. Gemta, A.K. Hordofa, T.A. Desta, M. Dangish, H.D. Mekonnen. Optoplasmonic biosensor for lung cancer telediagnosis: Design and simulation analysis. *Sensors International* **4**, 100232 (2023).
13. S. Tesfa. Role of phase fluctuation and dephasing in the enhancing continuous variable entanglement of a two-photon coherent beat laser. *Chinese Phys. B* **21** (1), 014204 (2012).
14. D. Bruß. Characterizing entanglement. *J. Math. Phys.* **43** (9), 4237 (2002).
15. C. Spengler, M. Huber, S. Brierley, T. Adaktylos, B.C. Hiesmayr. Entanglement detection via mutually unbiased bases. *Phys. Rev. A* **86** (2), 022311 (2012).
16. A.G. Kumela, A.B. Gemta, T.A. Desta, A. Kebede. Noble classical and quantum approach to model the optical prop-

erties of metallic nanoparticles to enhance the sensitivity of optoplasmonic sensors. *RSC advances* **12** (25), 16203 (2022).

17. K. Audenaert, M.B. Plenio, J. Eisert. Entanglement cost under positive-partial-transpose-preserving operations. *Phys. Revi. Lett.* **90** (2), 027901 (2003).
18. F. Martinet, M.K. Olsen. Finite size effects and equilibration in bose-hubbard chains with central well dephasing. *Europ. Phys. J. D* **71**, 1 (2017).
19. S. Tesfa, S. Tesfa. *Quantum features of light. Quantum Optical Processes: From Basics to Applications* (2020), p. 293.
20. F. Sun, D. Mao, Y. Dai, Z. Ficek, Q. He, Q. Gong. Phase control of entanglement and quantum steering in a three-mode optomechanical system. *New J. Phys.* **19** (12), 123039 (2017).
21. S. Qamar, S. Qamar, M.S. Zubairy. Effect of phase fluctuations on entanglement generation in a correlated emission laser with injected coherence. *Optics Commun.* **283** (5), 781 (2010).
22. J.P. Poizat, M. Collett, D. Walls. Nondegenerate two-mode squeezing and quantum-nondemolition measurements using three-level atoms in a cavity. *Physi. Rev. A* **45** (7), 5171 (1992).
23. V. Koverda, V. Skokov. The origin of 1/f fluctuations and scale transformations of time series at nonequilibrium phase transitions. *Phys. A: Stat. Mech. Appl.* **346** (3–4), 203 (2005).
24. M. Alizadeh, G. Shaker, J.C.M. De Almeida, P.P. Morita, S. Safavi-Naeini. Remote monitoring of human vital signs using mm-wave fmcw radar. *IEEE Access* **7**, 54958 (2019).
25. D. Ayehu. Squeezing and entanglement properties of the cavity light with decoherence in a cascade three-level laser. *J. Russian Laser Research* **42**, 136 (2021).
26. M. Anzola, F. Di Maiolo, A. Painelli. Optical spectra of molecular aggregates and crystals: Testing approximation schemes. *Phys. Chem. Chem. Phys.* **21** (36), 19816 (2019).
27. A. Mahmoud, M. Ahmed. Effect of asymmetric intermodal gain suppression on dynamics of multimode semiconductor lasers. *Opt. Commun.* **462**, 125365 (2020).
28. A. Morillo-Candas, C. Drag, J.-P. Booth, T. Dias, V. Guerra, O. Guaitella. Oxygen atom kinetics in co2 plasmas ignited in a dc glow discharge. *Plasma Sources Sci. Techn.* **28** (7), 075010 (2019).
29. S. Tesfa. Role of dephasing in modifying the evolution of the cavity radiation of a coherent beat laser. *Phys. Revi. A* **79** (3), 033810 (2009).
30. S. Tesfa. Dependence of the evolution of the cavity radiation of a coherently pumped correlated emission laser on dephasing and phase fluctuation. *Phys. Rev. A* **83** (2), 023809 (2011).
31. N. Korolkova, G. Leuchs, R. Loudon, T.C. Ralph, C. Silberhorn. Polarization squeezing and continuousvariable polarization entanglement. *Phys. Revi. A* **65** (5), 052306 (2002).
32. M. Servin, J.A. Quiroga, J.L. Marroquin. General n-dimensional quadrature transform and its application to interferogram demodulation. *JOSA A* **20** (5), 925 (2003).
33. Y.-J. Chen, A. Hansen, G.W. Hoth, E. Ivanov, B. Pelle, J. Kitching, E.A. Donley. Single-source multiaxis coldatom interferometer in a centimeter-scale cell. *Phys. Rev. Appl.* **12** (1), 014019 (2019).
34. A. ur Rahman, H. Ali, S. Haddadi, S. Zangi. Generating non-classical correlations in two-level atoms. *Alexandria Engineering J.* **67**, 425 (2023).
35. A.K. Armel, Y.D. Germain, T.A. Giresse, T. Martin. The dynamic of quantum entanglement of two dimensional harmonic oscillator in non-commutative space. *Phys. Scripta* **96** (12), 125731 (2021).
36. A. Neven, J. Carrasco, V. Vitale, C. Kokail, A. Elben, M. Dalmonte, P. Calabrese, P. Zoller, B. Vermersch, R. Kueng *et al.* Symmetry-resolved entanglement detection using partial transpose moments. *npj Quantum Information* **7** (1), 152 (2021).
37. D. Miki, N. Matsumoto, A. Matsumura, T. Shichijo, Y. Sugiyama, K. Yamamoto, N. Yamamoto. Generating quantum entanglement between macroscopic objects with continuous measurement and feedback control. *Phys. Rev. A* **107** (3), 032410 (2023).
38. Y. Yu, F. Ma, X.-Y. Luo, B. Jing, P.-F. Sun, R.-Z. Fang, C.-W. Yang, H. Liu, M.-Y. Zheng, X.-P. Xie *et al.* Entanglement of two quantum memories via fibres over dozens of kilometres. *Nature* **578** (7794), 240 (2020).
39. A. Salmanogli, D. Gokcen, H.S. Gecim. Entanglement sustainability in quantum radar. *IEEE J. Selected Topics in Quantum Electronics* **26** (6), 1 (2020).
40. Y. Ren, S. Duan, W. Xie, Y. Shao, Z. Duan. Antibunched photon-pair source based on photon blockade in a nondegenerate optical parametric oscillator. *Phys. Rev. A* **103** (5), 053710 (2021).
41. T. Abebe, C.G. Feyisa. Dynamics of a nondegenerate three-level laser with parametric amplifier and coupled to a two-mode squeezed vacuum reservoir. *Brazilian J. Phys.* **50** (5), 495 (2020).
42. B. G'abor, D. Nagy, A. Dombi, T. Clark, F. Williams, K. Adwaith, A. Vukics, P. Domokos. Ground-state bistability of cold atoms in a cavity. *Phys. Rev. A* **107** (2), 023713 (2023).
43. J.R. Cuartas, H. Vinck-Posada. Uncover quantumness in the crossover from coherent to quantum-correlated phases via photon statistics and entanglement in the tavis-cummings model. *Optik* **245**, 167672 (2021).
44. T. Abebe, C. Gashu. Generation of entangled light from a nondegenerate three-level laser coupled to a twomode vacuum reservoir. *Ukr. J. Phys.* **66** (7), 551 (2021).
45. D. Safronenkov, N. Borshchevskaya, T. Novikova, K. Katamadze, K. Kuznetsov, G.K. Kitaeva. Measurement of the biphoton second-order correlation function with analog detectors. *Optics Express* **29** (22), 36644 (2021).

Received 05.10.21

А.Г. Кумела, А.Б. Гемта, А.К. Хордофа,
Т.А. Деста, М. Дангиш, Х.Д. Меконнен

КВАНТОВІ ВЛАСТИВОСТІ
КОРЕЛЬОВАНИХ ФОТОНІВ
ІЗ УРАХУВАННЯМ ФЛУКТУАЦІЙ ФАЗИ

Розвинуто теорію впливу флуктуацій фази на корельовані фотони, які випромінюються трирівневими атомами під впливом когерентної хвилі з порожнини лазера. Із застосуванням динамічного рівняння для цієї системи розраховано статистику фотонів, кореляцію чисел фотонів та властивості заплутування. Для субпуассонівської статистики фотонів показано, що ступінь кореляції зростає разом зі швидкістю накачування атомів, та що заплутування залежить від флуктуацій фази, а не від швидкості накачування. Запропонована система може бути використана при обробці квантової інформації.

Ключові слова: квантові властивості, корельовані фотони, лазер з трирівневими атомами, заплутування.