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THE INFLUENCE OF A SPACE-SPACE DEFORMATION ON HIGH- AND LOW-ENERGY SPECTRA OF FERMIONIC PARTICLES AND SPECTRA OF HEAVY QUARKONIA WITH IMPROVED HULTHÉN AND HYPERBOLIC EXPONENTIAL INVERSELY QUADRATIC POTENTIALS

In this work, the modified approximation to the centrifugal barrier term is applied to find approximate bound-state solutions of the deformed Dirac equation for the spin and pseudospin symmetries in a model with the improved hyperbolic Hulthén and hyperbolic exponential inversely quadratic potentials (IHHEIQPs) using the parametric method of Bopp's shift and the standard perturbation theory in the extended relativistic quantum mechanics (ERQM). Our results indicate that the new energy eigenvalues are highly sensitive to the potential parameters (ν_1, A) and to the values of quantum atomic numbers $(j, k, l, m, \widetilde{l}, \widetilde{m}, s, \widetilde{s})$, range of the potential ν, and noncommutativity parameters (Θ, σ, β). We found that the effect of a space-space deformation gives a correction in the energy spectrum, where the main energy term remains due to the effect of the hyperbolic Hulthén and hyperbolic exponential inversely quadratic potentials known in the literature. The new nonrelativistic energies are obtained by applying the nonrelativistic limit to the relativistic spin-energy equation in the extended nonrelativistic quantum mechanics (ENRQM). The proposed potential model reduces to the improved Hulthén and exponential inversely quadratic potentials as special cases in ERQM. The present results are applied for calculating the new mass spectra M_{nc-nl}^{hiqp} of heavy mesons such as $c\overline{c}$, $b\overline{b}$, $b\overline{c}$, $b\overline{s}$, $c\overline{s}$, and $b\overline{q}$, $q = (u, d)$ in ENRQM. It turns out that the values of masses come from the contribution of the mass spectra M_{nl}^{hiqp} in NRQM, while the effect of a space-space deformation δM_{nc-nl}^{hiqp} is an infinitesimal correction as compared with M_{nl}^{hiqp}. Our results seem to be significant and agree perfectly with the ones in the literature.

Keywords: noncommutative space-space, Dirac equation, Schrödinger equation, Hulthén plus hyperbolic exponential inversely quadratic potential, Bopp's shift method, heavy-light mesons.

1. Introduction

Although nearly a century has passed since the emergence of the four fundamental equations aimed at the understanding of physical phenomena at high and low energies, they still attract more attention from researchers in various fields of physics and chemistry. Researchers investigate the low- and high-energy regions by searching for bound states (energy spectrum, wave functions, and two-spinor wave func-

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Цитування: Мереш А. Вплив деформації просторів на високо- та низькоенергетичні спектри ферміонів та спектри важких кварконіїв для модифікованого потенціалу Хюльтена і гіперболічного експоненціально обрізаного обернено квадратичного потенціалу. *Укр. фіз. журн.* **68**, №5, 327 (2023).

tions) and scattering state solutions, at the relativistic and nonrelativistic levels, of the Klein–Gordon, Dirac, and Duffin–Kemmer–Petiau (DKP) equations or the Schrödinger wave equation for low-energy physical systems. Different potentials are used to describe physical systems, and researchers use various methods to solve the equations, including the Nikiforov–Uvarov (NU) approach, the supersymmetric quantum mechanics method, and others. The exact solutions of the various basic equations can only be obtained in exceptional cases, such as a harmonic oscillator and a hydrogen atom. For example, s-wave cases can be considered in addition to the case of $l = 0$, while, in other general cases, which correspond to $(l,k) \neq (0,0)$, it is not possible to obtain exact solutions due to the complexities of the centrifugal terms. In this case, it is necessary to apply some approximations, including the standard Greene–Aldrich approximation. The exponential potentials like the Hulthén potential and the exponential inversely quadratic one play an attractive and extensive role in describing many vital areas such as the molecular and atomic levels and elementary particles. Antia *et al.* found the KGE bound-state solutions for a deformed Hulthén potential with unequal scalar and vector potentials for any l-state and calculated explicitly the energy eigenvalues and the corresponding wave functions expressed in terms of Jacobi polynomials [1]. Using both the NU method and the approximation scheme to deal with the centrifugal (pseudo) term, Ikhdair and Sever found bound-state solutions of the Dirac equation (DE) with the Hulthén potential for all angular momenta under spin and pseudospin (p-spin) symmetries [2]. Hamzavi *et al.* used the NU method to obtain the energy eigenvalue equation and the corresponding eigenfunctions in a closed form for the inversely quadratic Yukawa potential, which included a Coulomb-like tensor potential with arbitrary spin-orbit coupling quantum number k in the framework of the spin and p-spin symmetries [3]. In addition to the thermodynamic properties, Okon *et al.* obtained solutions of DE for spin and pseudospin symmetries in a model with the hyperbolic Hulthén plus hyperbolic exponential inversely quadratic potentials (HHEIQP) using the parametric NU method and the modified approximation to the centrifugal barrier term [4]. The Hulthén potential was proposed by Hulthén in 1942 [5] and is considered as a special case of the Eckart potential[6]. It has received a lot of attention for its effectiveness in describing a variety of phenomena. It is widely applied in many fields of physics, especially at the atomic level and for quarks [7]. Recently, this potential has been studied in many articles such as [8, 9]. Bhaghyesh *et al.* used the Hulthén potential and a linear potential to study $q\bar{q}\,(q = b, c)$ and computed the spin-averaged masses for $c\bar{c}$ and $b\bar{b}$ mesons [10]. The hyperbolic exponential inversely quadratic potential $\left(-\frac{Aq\exp(-\nu r)}{r^2}\right)$ is considered to be worthy of attention because of its important applications in nuclear and high-energy physics [4]. Ikhdair presented an approximate analytic solution of the Klein–Gordon equation (KGE) in the presence of equal scalar and vector generalized deformed hyperbolic potential functions using a parametric generalization of the NU method and calculated the rotational and vibrational energies of diatomic molecules [11]. Through this new research, we aim to restudy the relativistic and nonrelativistic energy bands resulting from this combined potential within the framework of geometric symmetries resulting from a space-space deformation. This space is known as a noncommutative phase-space or extended quantum mechanics in the context of Schrödinger and Dirac theories. We call it the deformed Dirac theory (DDT) and deformed Schrödinger theory (DST) for the purpose of a more profound investigation for new energy values and searching for the possibility of discovering new applications. In addition to the well-known axioms that establish quantum mechanics (QM) known in the literature, we have two additional axioms. The first one was known as the noncommutative (NC) phase-phase $\widehat{p}_{\mu}^{(s,h,i)} * \widehat{p}_{\nu}^{(s,h,i)} \neq$ $\neq \widehat{p}_{\nu}^{(s,h,i)} * \widehat{p}_{\mu}^{(s,h,i)}$; while the second corresponds to $\widehat{x}_{\mu}^{(s,h,i)} * \widehat{x}_{\nu}^{(s,h,i)} \neq \widehat{x}_{\nu}^{(s,h,i)} * \widehat{x}_{\mu}^{(s,h,i)}$, which is known to specialized researchers as the NC space-space (the symbol $*$ denotes the Weyl–Moyal star product). Researchers believe that the NC idea is the best solution to many physical problems that have not found a convincing solution within the framework of QM, such as quantum gravity, string theory, and the divergence problem in the standard model [12–22]. Furthermore, the theory of noncommutativity is a very strong candidate to be the physical tool that unites QM with its three interactions (nuclear strong, electromagnetic, and nuclear weak) with the gravitational interactions represented by Einstein's general relativity and the nonsymmetric gravitational theory of Moffat. The concept of extended quantum mechanics (EQM) is not new; it was proposed by Snyder [23, 24] in 1947, and its geometric analysis was introduced by Connes in 1991 and 1994 [25, 26]. Seiberg and Witten extended earlier ideas about the appearance of a NC geometry in the string theory with a nonzero B-field and obtained a new version of gauge fields in noncommutative gauge theory [27]. On the other hand, the great mathematical progress during the past decades has encouraged researchers to rely on EQM to develop an understanding of atomic and nuclear physical phenomena and to study the interactions between molecules as well. It should be noted that we have had two new contributions to this framework. The first related to a study of the deformed Klein–Gordon equation (DKGE) with generalized modified screened Coulomb plus generalized inversely quadratic Yukawa potential in RNCQM symmetries [28]. The second study concerns the model with a modified unequal mixture of the scalar-vector Hulthén–Yukawa potentials for the quark-antiquark interaction, as well as for neutral atoms [29] and in other cases[30–35]. For this reason, our main goal is to conduct a further investigation in the field of elementary particles. Motivated by previous works in the literature, we will concentrate on a new physical potential, which we call the improved Hulthén plus hyperbolic exponential inversely quadratic potential (IHHEIQP, in short) model ($V^s_{\text{hiq}}(\widehat{r})/V^p_{\text{hiq}}(\widehat{r})$, $S^s_{\text{hiq}}(\widehat{r})/S^p_{\text{hiq}}(\widehat{r})$):

$$\begin{cases} V^s_{\text{hiq}}(\widehat{r}) = V_{\text{hiq}}(r) - \dfrac{\partial V_{\text{hiq}}(r)}{\partial r}\dfrac{\mathbf{L\Theta}}{2r} + O(\Theta^2), \\ S^s_{\text{hiq}}(\widehat{r}) = S_{\text{hiq}}(r) - \dfrac{\partial S_{\text{hiq}}(r)}{\partial r}\dfrac{\mathbf{L\Theta}}{2r} + O(\Theta^2) \end{cases} \tag{1.1}$$

and

$$\begin{cases} V^p_{\text{hiq}}(\widehat{r}) = V_{\text{hiq}}(r) - \dfrac{\partial V_{\text{hiq}}(r)}{\partial r}\dfrac{\widetilde{\mathbf{L}}\mathbf{\Theta}}{2r} + O(\Theta^2), \\ S^p_{\text{hiq}}(\widehat{r}) = S_{\text{hiq}}(r) - \dfrac{\partial S_{\text{hiq}}(r)}{\partial r}\dfrac{\widetilde{\mathbf{L}}\mathbf{\Theta}}{2r} + O(\Theta^2), \end{cases}, \tag{1.2}$$

where $(V_{\text{hiq}}(r), S_{\text{hiq}}(r))$ are the vector and scalar potentials in the commutative quantum mechanics and are known as [4]:

$$\begin{cases} V_{\text{hiq}}(r) = \dfrac{\nu_1 q \exp(-\nu r)}{1 - q\exp(-\nu r)} - \dfrac{Aq\exp(-\nu r)}{r^2}, \\ S_{\text{hiq}}(r) = \dfrac{s_1 q \exp(-\nu r)}{1 - q\exp(-\nu r)} - \dfrac{A_s q\exp(-\nu r)}{r^2}, \end{cases} \tag{2}$$

where $\nu_1\,(s_1)$ is the potential depth, $A\,(A_s)$ is a real constant parameter, $q = \cosh(\omega)$, ω is the optimizing parameter $\nu = \frac{1}{b}$, while b is the screening parameter that represents the strength of the potential, ($\widehat{r}$ and r) are the distances between the two particles in a deformation of Dirac theory symmetries and QM symmetries, respectively. The two couplings ($\mathbf{L\Theta}$ and $\widetilde{\mathbf{L}}\mathbf{\Theta}$) are the scalar product of the usual components of the angular momentum operators $\mathbf{L}(L_x, L_y, L_z)/$ $/\widetilde{\mathbf{L}}(\widetilde{L}_x, \widetilde{L}_y, \widetilde{L}_z)$ and the modified noncommutativity vector $\mathbf{\Theta}(\theta\epsilon_{12}, \theta\epsilon_{23}, \theta\epsilon_{13})/2$ which are related to the noncommutativity elements. The modified algebraic structure of covariant canonical commutation relations (MASCCCRs), canonical structure (CS), Lie structure (LS), and quantum plane (QP) in the DDT in the representations of Schrödinger, Heisenberg, and interaction pictures are as follows (we have used the natural units $\hbar = c = 1$) [36–44]:

$$\left[\widehat{x}^{(s,h,i)}_\mu \overset{*}{,} \widehat{p}^{(s,h,i)}_\nu\right] = i\hbar_{\text{eff}}\delta_{\mu\nu},$$

$$\left[\widehat{x}^{(s,h,i)}_\mu \overset{*}{,} \widehat{x}^{(s,h,i)}_\nu\right] =$$

$$= \begin{cases} i\epsilon_{\mu\nu}\theta & \text{with } \epsilon_{\mu\nu} \in IC, \text{ CS}, \\ if^\alpha_{\mu\nu}\widehat{x}^{(s,h,i)}_\alpha & \text{with } f^\alpha_{\mu\nu} \in IC, \text{ LS}, \\ iC^{\alpha\beta}_{\mu\nu}\widehat{x}^{(s,h,i)}_\alpha \widehat{x}^{(s,h,i)}_\beta & \text{with } C^{\alpha\beta}_{\mu\nu} \in IC, \text{ QP}. \end{cases}$$

$$\left[\widehat{x}^{(s,h,i)}_\mu \overset{*}{,} \epsilon_{\mu\nu}\theta\right] = 0 \Leftrightarrow \left[\widehat{x}^{(s,h,i)}_\mu \overset{*}{,} \left[\widehat{x}^{(s,h,i)}_\nu \overset{*}{,} \widehat{x}^{(s,h,i)}_\xi\right]\right] = 0. \tag{3}$$

These postulates within the framework of QM are known in the literature in the following simplified form:

$$\left[x^{(s,h,i)}_\mu, p^{(s,h,i)}_\nu\right] = i\hbar\delta_{\mu\nu} \text{ and } \left[x^{(s,h,i)}_\mu, x^{(s,h,i)}_\nu\right] = 0. \tag{4}$$

The new NC generalized coordinates in the DDT symmetries ($\widehat{x}^{(s,h,i)}_\mu$ and $\widehat{p}^{(s,h,i)}_\mu$) are equal (($\widehat{x}^s_\mu$, $\widehat{x}^h_\mu$, $\widehat{x}^i_{nc\mu}$) and ($\widehat{p}^s_\mu$, $\widehat{p}^h_\mu$, $\widehat{p}^i_\mu$)), while the corresponding generalizing coordinates ($x^{(s,h,i)}_\mu$ and $p^{(s,h,i)}_\mu$) are equal ((x^s_μ, x^h_μ, $x^i_{nc\mu}$) and (p^s_μ, p^h_μ, p^i_μ)) in QM symmetries, respectively. The symbol IC denotes the complex number field. In EQM symmetries, the above algebraic structures allow us to reformulate the uncertainty relations as

$$\left|\Delta\widehat{x}^{(s,h,i)}_\mu \Delta\widehat{x}^{(s,h,i)}_\nu\right| \geqslant \begin{cases} |\theta_{\mu\nu}|/2 & \text{for CS}, \\ f_{\mu\nu}/2 & \text{for LS}, \\ C_{\mu\nu}/2 & \text{for QP}. \end{cases} \tag{5.1}$$

Here, $f_{\mu\nu}$ and $C_{\mu\nu}$ present the average values

$$\left|\left\langle\sum_{\alpha}^{3}\left(f_{\mu\nu}^{\alpha}\widehat{x}_{\alpha}^{(s,h,i)}\right)\right\rangle\right| \text{ and } \left|\left\langle\sum_{\alpha,\beta}^{3}\left(C_{\mu\nu}^{\alpha\beta}\widehat{x}_{\alpha}^{(s,h,i)}\widehat{x}_{\beta}^{(s,h,i)}\right)\right\rangle\right|,$$

respectively, in addition to the usual uncertainty relation:

$$\left|\Delta\widehat{x}_{\mu}^{(s,h,i)}\Delta\widehat{p}_{\nu}^{(s,h,i)}\right| \geqslant \hbar_{\text{eff}}\delta_{\mu\nu}/2 \tag{5.2}$$

which is obtained by substituting $\hbar$ with new values $\hbar_{\text{eff}}$. It is worth to note that Eqs. (3) and (4) are covariant equations (the same behavior of $\widehat{x}_{\mu}^{(s,h,i)}$) under the Lorentz group transformation $SO(1,3)$, which includes boosts and/or rotations of the observer's inertial frame. We extended the modified equal-time noncommutative canonical commutation relations (METNCCCRs) to include the Heisenberg and interaction pictures in DDT. Here, $\hbar_{\text{eff}} \cong \hbar$ is the effective Planck constant, $\theta_{\mu\nu} = \epsilon_{\mu\nu}\theta$ (θ is the NC parameter, and $\epsilon_{\mu\nu}$ is just an antisymmetric number ($\epsilon_{\mu\nu} = -\epsilon_{\nu\mu} = 1$ with $\mu \neq \nu$) and $\epsilon_{\epsilon\epsilon} = 0$) which is an infinitesimal parameter, if compared to the energy values and elements of antisymmetric (3×3) real matrices, and $\delta_{\mu\nu}$ is the Kronecker symbol. The new deformed product can be expressed with the Weyl–Moyal star product $h(x) * f(x)$ in the symmetries of DDT symmetries as follows: [45–50]:

$$h(x) * f(x) = \\ = \begin{cases} \exp\left(i\epsilon^{\mu\nu}\theta\partial_{\mu}^{x}\partial_{\nu}^{x}\right)(hf)(x), \text{CS}, \\ \exp\left(\frac{i}{2}x_{nc\mu}^{(s,h,i)}g_{k}\left(i\partial_{\mu}^{x}, i\partial_{\nu}^{x}\right)\right)(hf)(x), \text{LS}, \\ iq^{G\left(u,\Lambda,\partial_{\mu}^{u},\partial_{\nu}^{\Lambda}\right)}h(u,\Lambda)f(u',\Lambda')\rfloor_{u'\to u}^{\Lambda'\to\Lambda}, \text{QP}. \end{cases} \tag{6}$$

with

$$f_{\alpha}(u',\Lambda') = -u'_{\mu}\Lambda'_{\nu}f_{k}^{\nu\nu} + \frac{1}{6}u'_{\mu}\Lambda'_{\nu}(\Lambda'_{\alpha}-u'_{\alpha})f_{l}^{\nu\nu}f_{m}^{l\alpha}+\dots.$$

In the current paper, we apply the MASCCCRs to the DDT, which allows us to rewrite it in the following simple form in the first order of the noncommutativity parameter $\epsilon^{\mu\nu}\theta$ as follows [49–51]:

$$(h*f)(x) = (hf)(x) - \\ -\frac{i\epsilon^{\mu\nu}\theta}{2}\partial_{\mu}^{x}h(x)\,\partial_{\nu}^{x}f(x)\rfloor_{x^{\mu}=x^{\nu}} + O\left(\theta^{2}\right). \tag{7}$$

The indices $(\mu,\nu = 1,2,3)$ and $O\left(\theta^{2}\right)$ stand for the second and higher-order terms with the NC parameter. Physically, the second term in the last equation presents the effects of space-space noncommutativity, and, on the other hand, the fields have to be considered as dependent not only on x^{μ}, but also on $\epsilon_{\mu\nu}\theta$. The present study aims at constructing the IHHEIQP model for the application to heavy-light mesons *HLM,* such as $c\bar{c}$, $b\bar{b}$, $b\bar{c}$, $b\bar{s}$, $c\bar{s}$, and $b\bar{q}$, $q = (u,d)$. The present paper is organized as follows: The first section includes the scope and purpose of our investigation, while the remaining parts of the paper are structured as follows. A review of the DE with the Hulthén plus hyperbolic exponential inversely quadratic potential without tensor interaction is presented in Sect. 2. Section 3 is devoted to studying the deformed Dirac equation (DDE) by applying the usual Bopp's shift method and the Greene–Aldrich approximation for the centrifugal terms to obtain the effective potentials of the IHHEIQP model in DDT symmetries. Furthermore, via standard perturbation theory, we find the expectation values of some radial terms to calculate the corrected relativistic energy generated by the effect of the perturbed effective potentials of the IHHEIQP model. We will derive the global corrected energy with the IHHEIQP model in the DDT symmetries. We will also treat some important special cases, including the study of relativistic cases as an NR limit in the next section. The present results are applied to calculate the mass spectra of the previously mentioned HLM systems. In Sect. 6, a brief conclusion of the work is presented.

2. Background and Preparation

2.1. An overview of DE under the HHEIQP model

In this section, in order to achieve complete solutions with the IHHEIQP model in the DDT symmetries, we will mention the most important results of solutions corresponding to this problem within the framework of relativistic mechanics in the literature. A relativistic physical system influenced by the hyperbolic Hulthén plus hyperbolic exponential inversely quadratic potential (HHEIQP) n the context of the following DE:

$$\widehat{H}_{D}^{\text{hiq}}\Psi_{nk}(r,\theta,\varphi) = E_{nk}\Psi_{nk}(r,\theta,\varphi). \tag{8}$$

Here,

$$\widehat{H}_D^{\text{hiq}} = \widehat{\alpha}\mathbf{p}+\widehat{\beta}\left(M + S_{\text{hiq}}\left(r\right)\right)-i\widehat{\beta}\widehat{\mathbf{r}}U\left(r\right)+V_{\text{hiq}}\left(r\right) \quad (9)$$

denotes the Dirac Hamiltonian operator, M is the reduced rest mass, $\mathbf{p} = -\mathbf{i},\hbar\nabla$ is the momentum, and $U\left(r\right)$ is the tensor interaction. The vector potential $V_{\text{hiq}}\left(r\right)$ due to the four-vector linear momentum operator A^{μ} ($V_{\text{hiq}}\left(r\right)$, $\mathbf{A}=\mathbf{0}$), and the space-time scalar potential $S_{\text{hiq}}\left(r\right)$ due to the mass, E_{nk} represents the relativistic eigenvalues, (n, k) represent the principal and spin-orbit coupling terms, respectively, $\widehat{\alpha}_i = \text{anti}-\text{diag}(\sigma_i,\sigma_i)$, $\widehat{\beta} = \text{diag}\left(I_{2\times2},\ -I_{2\times2}\right)$ and σ_i are the usual Pauli matrices. Since the HHEIQP model has spherical symmetry, this allows us to get the solutions of the known form:

$$\Psi_{nk} = \begin{pmatrix}\Psi_{nk}^1\\ \Psi_{nk}^2\end{pmatrix},$$

with Ψ_{nk}^1 and Ψ_{nk}^2 are equal $\frac{F_{nk}(r)}{r}Y_{jm}^{l}\left(\theta,\varphi\right)$ and $i\frac{G_{nk}(r)}{r}Y_{jm}^{\widetilde{l}}\left(\theta,\varphi\right)$, respectively. The two expressions $F_{nk}\left(r\right)$ and $G_{nk}\left(r\right)$ represent the upper and lower components of the Dirac spinors Ψ_{nk}, while $Y_{jm}^{l}\left(\theta,\varphi\right)$ and $Y_{j\widetilde{m}}^{\widetilde{l}}\left(\theta,\varphi\right)$ are the spin and pseudospin spherical harmonics, and m is the projection on the z-axis. The upper and lower components satisfy the two uncoupled differential equations as follows:

$$\left[\frac{d^2}{dr^2} - \frac{k\left(k+1\right)}{r^2} - \left(M + E_{nk} - \Delta_{\text{hiq}}\left(r\right)\right)\times \times\left(M - E_{nk} + \Sigma_{\text{hiq}}\left(r\right)\right) + + \frac{\frac{d\Delta_{\text{hiq}}(r)}{dr}\left(\frac{d}{dr}+\frac{k}{r}\right)}{M + E_{nk} - \Delta_{\text{hiq}}\left(r\right)}\right]F_{nk}^{s}\left(r\right) = 0 \quad (10)$$

and

$$\left[\frac{d^2}{dr^2} - \frac{k\left(k-1\right)}{r^2} - \left(M + E_{nk} - \Delta_{\text{hiq}}\left(r\right)\right)\times \times\left(M - E_{nk} + \Sigma_{\text{hiq}}\left(r\right)\right) + + \frac{\frac{d\Sigma_{\text{hiq}}(r)}{dr}\left(\frac{d}{dr}-\frac{k}{r}\right)}{M + E_{nk} + \Sigma_{\text{hiq}}\left(r\right)}\right]G_{nk}^{ps}\left(r\right) = 0, \quad (11)$$

here $\Sigma_{\text{hiq}}\left(r\right) = V_{\text{hiq}}\left(r\right) + S_{\text{hiq}}\left(r\right)$ and $\Delta_{\text{hiq}}\left(r\right) = V_{\text{hiq}}\left(r\right) - S_{\text{hiq}}\left(r\right)$ are determined by:

$$\begin{cases}\Sigma_{\text{hiq}}\left(r\right) = A\dfrac{\exp\left(-2\alpha r\right)}{r^2} - B\dfrac{\exp\left(-\alpha r\right)}{r},\\ \dfrac{d\Delta_{\text{hiq}}\left(r\right)}{dr} = 0 \text{ For spin } s_y \text{ limit}\end{cases} \quad (12)$$

and

$$\begin{cases}\Delta_{\text{hiq}}\left(r\right) = A\dfrac{\exp\left(-2\alpha r\right)}{r^2} - B\dfrac{\exp\left(-\alpha r\right)}{r},\\ \dfrac{d\Sigma_{\text{hiq}}\left(r\right)}{dr} = 0 \text{ For p-spin } s_y. \text{ limit}\end{cases} \quad (13)$$

The upper and lower components of DE with equal scalar and vector potentials without tensor interaction are given as:

$$\left(\frac{d^2}{dr^2} - \frac{k\left(k+1\right)}{r^2} - - \gamma_s\left(M - E_{nk}^{s} + \Sigma_{\text{hiq}}\left(r\right)\right)\right)F_{nk}^{s}\left(r\right) = 0 \quad (14)$$

and

$$\left(\frac{d^2}{dr^2} - \frac{k\left(k-1\right)}{r^2} - - \gamma_{ps}\left(M + E_{nk}^{ps} - \Delta_{\text{hiq}}\left(r\right)\right)\right)G_{nk}^{ps}\left(r\right) = 0, \quad (15)$$

with $k\left(k-1\right)$ and $k\left(k+1\right)$ to be equal to $\widetilde{l}(\widetilde{l}-1)$ and $l\left(l+1\right)$, respectively. The authors of Ref. [4] used the NU method and Greene–Aldrich approximation for the centrifugal term to obtain the expressions for the wave function as hypergeometric polynomials $P_n^{\left(\sqrt{-\beta^{s2}b^2},2\beta_{nk}^{s}\right)}\left(1-2s\cosh(\omega)\right)$ and $P_n^{\left(\sqrt{-\beta^{ps2}b^2},2\beta_{nk}^{ps}\right)}\left(1-2s\cosh(\omega)\right)$ in RQM symmetries as

$$F_{nk}\left(s\right) = B_{nk}^{s}\left(s\cosh(\omega)\right)^{\sqrt{-\beta^{s2}b^2}}\left(1 - s\cosh(\omega)\right)^{\eta_{nk}^{sp}}, P_n^{\left(\sqrt{-\beta^{s2}b^2},2\beta_{nk}^{s}\right)}\left(1-2s\cosh(\omega)\right), \quad (16)$$

$$G_{nk}\left(s\right) = B_{nk}^{ps}\left(s\cosh(\omega)\right)^{\sqrt{-\beta^{ps2}b^2}}\left(1 - s\cosh(\omega)\right)^{\eta_{nk}^{ps}}, P_n^{\left(\sqrt{-\beta^{ps2}b^2},2\beta_{nk}^{ps}\right)}\left(1-2s\cosh(\omega)\right), \quad (17)$$

with $s = \exp\left(-\upsilon r\right)$, while β_{nk}^{s}, β_{nk}^{ps}, γ_s, γ_{ps}, β^{s}, β^{ps}, η_{nk}^{sp} and η_{nk}^{ps} are given by

$$\left.\begin{array}{l}\beta_{nk}^{s} = \cosh(\omega) + \\ + \sqrt{\begin{array}{l}4k\left(k+1\right)\left(\cosh(\omega)\right)^2 - \\ -8A\gamma_s\left(\cosh(\omega)\right)^2 + \left(\cosh(\omega)\right)^2\end{array}} + \\ + 2\cosh(\omega)\sqrt{-\beta^{s2}b^2} - 2\sqrt{-\beta^{s2}b^2} - 2,\\ \beta_{nk}^{ps} = \cosh(\omega) + 4k\left(k-1\right)\left(\cosh(\omega)\right)^2 - \\ - 8A\gamma_{ps}\left(\cosh(\omega)\right)^2 + \left(\cosh(\omega)\right)^2 + 2\cosh(\omega)\times \\ \times\sqrt{-\beta^{ps2}b^2} - 2\sqrt{-\beta^{ps2}b^2} - 2,\\ \gamma_s = M + E_{nk}^{s} - C_s, \gamma_{ps} = M - E_{nk}^{ps} + C_{ps},\\ \beta^{s2} = E_{nk}^{s2} - M^2\ ,\ \beta^{ps2} = E_{nk}^{ps2} - M^2,\end{array}\right\}$$

and

$$\left.\begin{aligned}&\eta_{nk}^{sp}=-\sqrt{-\beta^{s2}b^2}-2\cosh(\omega)\sqrt{-\beta^{s2}b^2}-\\&-\left[2\cosh(\omega)+2\sqrt{\begin{aligned}&4k(k+1)(\cosh(\omega))^2-\\&-8A\gamma_s(\cosh(\omega))^2+(\cosh(\omega))^2\end{aligned}}\right],\\&\eta_{nk}^{ps}=-\sqrt{-\beta^{ps2}b^2}-2\cosh(\omega)\sqrt{-\beta^{ps2}b^2}-\\&-\left[2\cosh(\omega)+2\sqrt{\begin{aligned}&4k(k-1)(\cosh(\omega))^2-\\&-8A\gamma_{ps}(\cosh(\omega))^2+(\cosh(\omega))^2\end{aligned}}\right],\end{aligned}\right\}$$

while B_{nk}^{s} and B_{nk}^{ps} are the normalization constants. For the spin symmetry and the p-spin symmetry, the equations of energy are given by:

$$\begin{aligned}&(E_{nk}^{s}-M)(E_{nk}^{s}+M)=\\&=-\frac{1}{b^2}\left(\frac{qk(k+1)-2q(E_{nk}^{s}+M)\nu_1 b^2}{(2n+1)^2+q\sqrt{(2k+1)^2-8A(E_{nk}^{s}+M)}}+\right.\\&\left.+\frac{\left(n^2+n+\frac{1}{2}\right)q+q(2n+1)\times\sqrt{(2k+1)^2-8A(E_{nk}^{s}+M)}}{(2n+1)^2+q\sqrt{(2k+1)^2-8A(E_{nk}^{s}+M)}}\right)^2\end{aligned}\tag{18}$$

and

$$\begin{aligned}&(M+E_{nk}^{ps})(M-E_{nk}^{ps})=-\frac{1}{b^2}\times\\&\times\left(\frac{qk(k-1)-2q(E_{nk}^{ps}-M)\nu_1 b^2}{(2n+1)^2+q\sqrt{(2k-1)^2-8A(E_{nk}^{ps}-M)}}+\right.\\&\left.+\frac{\left(n^2+n+\frac{1}{2}\right)q+q(2n+1)\sqrt{\begin{aligned}&(2k-1)^2-\\&-8A(E_{nk}^{ps}-M)\end{aligned}}}{(2n+1)^2+q\sqrt{(2k-1)^2-8A(E_{nk}^{ps}-M)}}\right)^2.\end{aligned}\tag{19}$$

From the definition of Jacobi polynomials, we have

$$\begin{aligned}&P_n^{(c_n,d_n)}(1-2s)=\frac{\Gamma(n+c_n+1)}{n!\Gamma(c_n+1)},\\&{}_2F_1(-n,n+c_n+d_n+1;1+c_n,s).\end{aligned}\tag{20}$$

We obtain that $F_{nk}^{s}(r)$ and $G_{nk}^{ps}(r)$ represent the upper and lower components of the Dirac spinors $\Psi_{nk}(r,\theta,\varphi)$ in terms of the wave function as hypergeometric polynomials ${}_2F_1(-n,D_{nk}^{sp};K^s,qs)$ and ${}_2F_1(-n,D_{nk}^{ps};K^{ps},qs)$ as follows:

$$\begin{aligned}&F_{nk}^{s}(s)=B_{nk}^{sn}(qs)^{\sqrt{-\beta^{s2}b^2}}(1-qs)^{\eta_{nk}^{sp}},\\&{}_2F_1(-n,D_{nk}^{sp};K^s,qs)\end{aligned}\tag{21.1}$$

and

$$\begin{aligned}&G_{nk}^{ps}(s)=B_{nk}^{psn}(qs)^{\sqrt{-\beta^{ps2}b^2}}(1-qs)^{\eta_{nk}^{ps}},\\&{}_2F_1(-n,D_{nk}^{ps};K^{ps},qs),\end{aligned}\tag{21.2}$$

with

$$\left.\begin{aligned}&D_{nk}^{(s,ps)}=n+\sqrt{-\beta^{(s,ps)2}b^2}+2\beta_{nk}^{(s,ps)}+1,\\&K^{(s,ps)}=1+\sqrt{-\beta^{(s,ps)2}b^2},\\&B_{nk}^{psn}=\frac{\Gamma\left(n+\sqrt{-\beta^{ps2}b^2}+1\right)}{n!\Gamma\left(\sqrt{-\beta^{ps2}b^2}+1\right)}B_{nk}^{ps}.\end{aligned}\right\}$$

The lower component $G_{nk}^{sp}(s)$ of spin symmetry and the upper component $F_{nk}^{ps}(s)$ of p-spin symmetry are obtained as

$$G_{nk}^{sp}(s)=\frac{1}{M+E_{nk}^{s}-C_s}\left(\frac{d}{dr}+\frac{k}{r}\right)F_{nk}^{s}(s)\tag{22}$$

and

$$F_{nk}^{ps}(s)=\frac{1}{M-E_{nk}^{ps}+C_p}\left(\frac{d}{dr}-\frac{k}{r}\right)G_{nk}^{ps}(s).\tag{23}$$

3. New Solutions of *DDE* under the IHHEIQP in DDT symmetries

3.1. Review of Bopp's shift method

In this subsection, we are going to solve the DDE with the IHHEIQP model by using the BS method. We can find the *DDE* expression using the concepts of the Weyl–Moyal star product mentioned in the introduction. These data allow us to rewrite the upper and lower components $F_{nk}^{s}(r)$ and $G_{nk}^{ps}(r)$ without tensor interaction in Eqs. (14) and (15) in the DDT symmetries as follows:

$$\left(\begin{aligned}&\frac{d^2}{dr^2}-\frac{k(k+1)}{r^2}-\\&-\gamma_s(M-E_{nk}^{s}+\Sigma_{hiq}(r))\end{aligned}\right)*F_{nk}^{s}(r)=0\tag{24}$$

and

$$\left(\begin{aligned}&\frac{d^2}{dr^2}-\frac{k(k-1)}{r^2}-\\&-\gamma_{ps}(M+E_{nk}^{ps}-\Delta_{hiq}(r))\end{aligned}\right)*G_{nk}^{ps}(r)=0.\tag{25}$$

Among the possible paths to find solutions of Eqs. (24) and (25), we indicate, according to the application of the Connes method [25, 26], the Seiberg–Witten map [27]. The star product construction that we saw in the above two equations can be converted to the normal product process using what is known as Bopp's shift (BS) method. F. Bopp was the first who considered pseudo-differential operators obtained from a symbol by the quantization rules $(x, p_x) \rightarrow (\widehat{x} = x - \frac{i}{2}\frac{\partial}{\partial p_x}, \widehat{p}_x = p_x + \frac{i}{2}\frac{\partial}{\partial x})$ instead of the ordinary correspondence $(x, p_x) \rightarrow (\widehat{x} = x, \widehat{p}_x = p_x + \frac{i}{2}\frac{\partial}{\partial x})$, respectively. This procedure is known as the BS method for the researchers, and this quantization procedure is known as Bopp quantization [52–55]. This method has achieved a considerable success in recent years. In the search for solutions of the NR deformed Schrödinger equation DSE under the influence of several different potentials [56–64]. The success of this method was not limited to the DSE, but was extended to the study of various relativistic physics problems, for example, for the DKGE [65–71], for the DDE [72–80] and the deformed DKP equation [81, 82]. Thus, the BS method allows us to simplify second-order linear differential equations of the DSE, DKG, DDE, and DDKPE with the Weyl–Moyal star product to the second-order linear differential SE, KGE, DE, and DKP equations with the ordinary product with simultaneous translation in the space-space. Therefore, the two differential equations (24) and (25) will take the following form:

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k+1)\widehat{r}^{-2} - \\ -\gamma_s\left(M - E_{nk}^s + \Sigma_{hiq}(\widehat{r})\right) \end{bmatrix} F_{nk}^s(r) = 0 \tag{26}$$

and

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k-1)\widehat{r}^{-2} - \\ -\left(M + E_{nk}^{ps} - \Delta_{hiq}(\widehat{r})\right)\gamma_{ps} \end{bmatrix} G_{nk}^{ps}(r) = 0. \tag{27}$$

In Eqs. (3), the METNCCCRs with the notion of a Weyl–Moyal star product become new METNCCCRs with ordinary products in the literature, as follows: (see, e.g.,[52–55]):

$$\begin{cases} \left[\widehat{x}_\mu^{(s,h,i)}, \widehat{p}_\nu^{(s,h,i)}\right] = i\hbar_{\text{eff}}\delta_{\mu\nu}, \\ \left[\widehat{x}_\mu^{(s,h,i)}, \widehat{x}_\nu^{(s,h,i)}\right] = i\theta_{\mu\nu}. \end{cases} \tag{28}$$

The BS method enables us to express the generalized Hermitian operators ($\widehat{x}_\mu^{(s,h,i)}$ and $\widehat{p}_\mu^{(s,h,i)}$) in the deformation Dirac theory symmetries on the corresponding parameters ($x_\mu^{(s,h,i)}$ and $p_\mu^{(s,h,i)}$) in ordinary QM as [52–55]:

$$\begin{cases} \widehat{x}_\mu^{(s,h,i)} = x_\mu^{(s,h,i)} - \theta \sum_{\nu=1}^{3} \frac{i\epsilon_{\mu\nu}}{2} p_\nu^{(s,h,i)}, \\ \widehat{p}_\mu^{(s,h,i)} = p_\mu^{(s,h,i)}. \end{cases} \tag{29}$$

This allows us to find $\widehat{r}^2$ equal $\left(r^2 - \mathbf{L\Theta}\right)$ and $(r^2 - \widetilde{\mathbf{L}}\mathbf{\Theta})$ for the spin and p-spin symmetries, respectively [72–80] which we will use in the next subsection.

3.2. Constructing the IHHEIQP Model in DDT symmetries

In this subsection, we aim to find the expressions of $\Sigma_{\text{hiq}}(\widehat{r})$ and $\Delta_{\text{hiq}}(\widehat{r})$ in the DDT symmetries, which will allow us to find the values of new energies in the framework of extended symmetries, and, accordingly, this allows us to compare them with their known counterparts in the relativistic framework, which, of course, results from the effects of both $\Sigma_{\text{hiq}}(r)$ and $\Delta_{\text{hiq}}(r)$. To achieve this goal, we begin to search for the new operators $V_{\text{hiq}}(\widehat{r})$, $V_{\text{hiq}}^{ps}(\widehat{r})$, $k(k+1)\widehat{r}^{-2}$ and $k(k-1)\widehat{r}^{-2}$ which we can obtain through the following operations:

$$\begin{cases} V_{\text{hiq}}^s(\widehat{r}) = V_{\text{hiq}}(r) - \frac{\partial V_{\text{hiq}}(r)}{\partial r}\frac{\mathbf{L\Theta}}{2r} + O\left(\Theta^2\right), \\ V_{\text{hiq}}^{ps}(\widehat{r}) = V_{\text{hiq}}(r) - \frac{\partial V_{\text{hiq}}(r)}{\partial r}\frac{\widetilde{\mathbf{L}}\mathbf{\Theta}}{2r} + O\left(\Theta^2\right), \\ k(k+1)\widehat{r}^{-2} = k(k+1)r^{-2} + \\ + k(k+1)r^{-4}\mathbf{L\Theta} + O\left(\Theta^2\right), \\ k(k-1)\widehat{r}^{-2} = k(k-1)r^{-2} + \\ + k(k-1)r^{-4}\widetilde{\mathbf{L}}\mathbf{\Theta} + O\left(\Theta^2\right). \end{cases} \tag{30}$$

Substituting Eqs. (30) into Eqs. (26) and (27), we obtain the following two Schrödinger-like equations:

$$\begin{bmatrix} \frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - \Sigma_{\text{hiq}}^{\text{pert}}(r) - \\ -\gamma_s\left(M - E_{nk}^s + \Sigma_{\text{hiq}}(r)\right) \end{bmatrix} F_{nk}^s(r) = 0 \tag{31}$$

and

$$\begin{bmatrix} \frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - \Delta_{\text{hiq}}^{\text{pert}}(r) - \\ -\left(M + E_{nk}^{ps} - \Delta_{\text{hiq}}(r)\right)\gamma_{ps} \end{bmatrix} G_{nk}^{ps}(r) = 0, \tag{32}$$

with

$$\Sigma_{\text{hiq}}^{\text{pert}}(r) = \left(\frac{k(k+1)}{r^4} - \frac{\gamma_s}{2r}\frac{\partial V_{\text{hiq}}(r)}{\partial r}\right)\mathbf{L\Theta} \tag{33}$$

and

$$\Delta_{\rm hiq}^{\rm pert}(r)=\left(\frac{k(k-1)}{r^4}-\frac{\gamma_{ps}}{2r}\frac{\partial V_{\rm hiq}(r)}{\partial r}\right)\widetilde{\mathbf{L}}\mathbf{\Theta}. \qquad (34)$$

By comparing Eqs. (14) and (15) with (Eqs. (31) and (32)), we observe two additive potentials $\Sigma_{\rm hiq}^{\rm pert}(r)$ and $\Delta_{\rm hiq}^{\rm pert}(r)$. Moreover, these terms are proportional to the infinitesimal noncommutativity vector $\mathbf{\Theta}$. From the physical point of view, this means that these two spontaneously generated terms $\Sigma_{\rm hiq}^{\rm pert}(r)$ and $\Delta_{\rm hiq}^{\rm pert}(r)$, as a result of the topological properties of the space-space deformation, can be considered very small compared to the fundamental terms $\Sigma_{\rm hiq}(r)$ and $\Delta_{\rm hiq}(r)$, respectively. A direct calculation gives $\frac{\partial V_{\rm hiq}(r)}{\partial r}$ as follows:

$$\frac{\partial V_{\rm hiq}(r)}{\partial r}=-\frac{\nu\nu_1 q\exp(-\nu r)}{(1-q\exp(-\nu r))^2}+$$
$$+\frac{A\nu q}{r^2}\exp(-\nu r)+\frac{2Aq}{r^3}\exp(-\nu r). \qquad (35)$$

Substituting Eq. (35) into Eqs. (33) and (34), we obtain the spontaneously generated terms $\Sigma_{\rm hiq}^{\rm pert}(r)$ as follows:

$$\Sigma_{\rm hiq}^{\rm pert}(r)=\left(\frac{k(k+1)}{r^4}+\frac{\nu\nu_1 q\gamma_s\exp(-\nu r)}{2r(1-q\exp(-\nu r))^2}-\right.$$
$$\left.-\frac{A\nu q\gamma_s\exp(-\nu r)}{2r^3}-\frac{Aq\gamma_s\exp(-\nu r)}{r^4}\right)\mathbf{L}\mathbf{\Theta}. \qquad (36)$$

Whereas, the generated potential $\Delta_{\rm hiq}^{\rm pert}(r)$ can be obtained by applying the two simultaneous transformations from Eq. (36):

$$\Delta_{\rm hiq}^{\rm pert}(r)=\Sigma_{\rm hiq}^{\rm pert}(r)\begin{bmatrix}\gamma_s\to\gamma_{ps},\ \mathbf{L}\to\widetilde{\mathbf{L}}\\ \text{and } k(k+1)\to k(k-1)\end{bmatrix}. \qquad (37)$$

Furthermore, using the unit step function (also known as the Heaviside step function $\theta(x)$ or simply the theta function), we can rewrite the globally induced two potentials $\Sigma_{t\_{\rm hiq}}^{\rm pert}(r)$ and $\Delta_{t-{\rm hiq}}^{\rm pert}(r)$ for spin and p-spin symmetries corresponding upper and lower components ($F_{nk}^{s}(s)$ and $G_{nk}^{sp}(s)$) and ($F_{nk}^{ps}(s)$ and $G_{nk}^{ps}(s)$), respectively as:

$$\Sigma_{t\_{\rm hiq}}^{\rm pert}(r)=\Sigma_{\rm hiq}^{\rm pert}(r)\,\theta\left(E_{nc}^{{\rm hiq}-s}\right)-$$
$$-\Sigma_{\rm hiq}^{\rm pert}(r)\,\theta\left(-E_{nc}^{{\rm hiq}-s}\right)=$$
$$=\begin{cases}\Sigma_{\rm hiq}^{\rm pert}(r) & \text{for upper component of spin } s_y,\\ -\Sigma_{\rm hiq}^{\rm pert}(r) & \text{for lower component of spin } s_y.\end{cases} \qquad (38.1)$$

and

$$\Delta_{t\_{\rm hiq}}^{\rm pert}(r)=\Delta_{\rm hiq}^{\rm pert}(r)\,\theta\left(E_{nc}^{{\rm hiq}-p}\right)-$$
$$-\Delta_{\rm hiq}^{\rm pert}(r)\,\theta\left(-E_{nc}^{{\rm hiq}-p}\right)=$$
$$=\begin{cases}\Delta_{\rm hiq}^{\rm pert}(r) & \text{for upper component of p-spin } s_y,\\ -\Delta_{\rm hiq}^{\rm pert}(r) & \text{for lower component of p-spin sy.}\end{cases} \qquad (38.2)$$

Here, the step function $\theta(x)$ is given by:

$$\theta(x)=\begin{cases}1 \text{ for } x\geq 0,\\ 0 \text{ for } x<0.\end{cases} \qquad (38.3)$$

For the spin symmetry, we first consider Eq. (31), which contains the improved Hulthén plus hyperbolic exponential inversely quadratic potential in the deformation of Dirac theory symmetries. It can be solved exactly only for $k=0$ and $k=-1$ without tensor interaction, since the two centrifugal terms (proportional to $k(k+1)r^{-2}$ and $k(k+1)r^{-4}$) vanish. In the case of arbitrary k, an appropriate approximation needs to be employed on the centrifugal terms. We apply the following improved approximation which was applied by Greene and Aldrich [83–86, 90]:

$$\frac{1}{r^2}\approx\frac{\nu^2 q e^{-\nu r}}{(1-qe^{-\nu r})^2}=\frac{\nu^2 qs}{(1-qs)^2}\Leftrightarrow$$
$$\Leftrightarrow\frac{1}{r}\approx\frac{\nu q^{1/2}e^{-\frac{\nu}{2}r}}{1-qe^{-\nu r}}=\frac{\nu q^{1/2}s^{\frac{1}{2}}}{1-qs}. \qquad (39)$$

For the p-spin symmetry, we now consider Eq. (32) and will follow similar steps with the spin symmetry case in the deformation of Dirac theory symmetries. As above, Eq. (31) cannot be solved exactly for $k=0$ and $k=1$ without tensor interaction, since the two centrifugal terms (proportional to $k(k-1)r^{-2}$ and $k(k-1)r^{-4}$). Applying approximations (38) to the centrifugal terms of Eqs. (36) and (37), the general form of the additive potentials $\Sigma_{\rm hiq}^{\rm pert}(s)$ and $\Delta_{\rm hiq}^{\rm pert}(s)$ will be as follows:

$$\Sigma_{\rm hiq}^{\rm pert}(s)=\left(\frac{\delta_{nk}^{1s}s^2}{(1-qs)^4}+\frac{\delta_{nk}^{2s}s^{\frac{3}{2}}}{(1-qs)^3}+\right.$$
$$\left.+\frac{\delta_{nk}^{3s}s^{5/2}}{(1-qs)^3}+\frac{\delta_{nk}^{4s}s^3}{(1-qs)^4}\right)\mathbf{L}\mathbf{\Theta} \qquad (40.1)$$

and

$$\Delta_{\rm hiq}^{\rm pert}(s)=\left(\frac{\delta_{nk}^{1ps}s^2}{(1-qs)^4}+\frac{\delta_{nk}^{2ps}s^{\frac{3}{2}}}{(1-qs)^3}+\right.$$

$$+\frac{\delta_{nk}^{3ps}s^{5/2}}{(1-qs)^3}+\frac{\delta_{nk}^{4ps}s^3}{(1-qs)^4}\Bigg)\widetilde{\mathbf{L}}\mathbf{\Theta}, \quad (40.2)$$

with

$$\begin{cases}\delta_{nk}^{1s}=\nu^4k(k+1)q^2, & \delta_{nk}^{2s}=\frac{\nu^2\nu_1q^{3/2}\gamma_s}{2},\\ \delta_{nk}^{3s}=-\frac{A\nu^4q^3\gamma_s}{2}, & \delta_{nk}^{4s}=-Aq^3\gamma_s\nu^4,\\ \delta_{nk}^{1ps}=\nu^4k(k-1)q^2, & \delta_{nk}^{2ps}=\frac{\nu^2\nu_1q^{3/2}\gamma_{ps}}{2},\\ \delta_{nk}^{3ps}=-\frac{A\nu^4q^3\gamma_{ps}}{2}, & \delta_{nk}^{4ps}=-Aq^3\gamma_{ps}\nu^4.\end{cases} \quad (41)$$

It is important to remember that the approximation used in our study is valid in the case $\nu r \ll 1$, which corresponds to the short-hand distance, and it is very suitable for us. We have replaced the terms $(k(k+1)r^{-4}$ and $k(k-1)r^{-4})$ with the approximation in Eq. (38). The Hulthén plus hyperbolic exponential inversely quadratic potential under spin(pseudo) symmetries without tensor interaction is extended by including two new additive potentials $(\Sigma_{\text{hiq}}^{\text{pert}}(r)$ and $\Delta_{\text{hiq}}^{\text{pert}}(r))$ expressed in terms proportional to the radial terms:

$$R(s)=\left\{\frac{s^2}{(1-qs)^4},\frac{s^{\frac{3}{2}}}{(1-qs)^3},\frac{s^{5/2}}{(1-qs)^3},\frac{s^3}{(1-qs)^4}\right\}$$

to become the improved Hulthén plus hyperbolic exponential inversely quadratic potential under spin(pseudo) symmetries without tensor interaction in deformation Dirac theory symmetries. The generated new two effective potentials $\Sigma_{\text{hiq}}^{\text{pert}}(s)$ and $\Delta_{\text{hiq}}^{\text{pert}}(s)$ are also proportional to the infinitesimal vector $\mathbf{\Theta}$. This allows us to consider the new additive parts of the effective potential $\Sigma_{\text{hiq}}^{\text{pert}}(s)$ and $\Delta_{\text{hiq}}^{\text{pert}}(s)$ as perturbation potentials compared with the main potentials $\Sigma_{\text{hiq}}(s)$ and $\Delta_{\text{hiq}}(s)$, the parent potential operator in the symmetries of the deformation Dirac theory, i.e., the two inequalities $\left(\Sigma_{\text{hiq}}^{\text{pert}}(s),\Delta_{\text{hiq}}^{\text{pert}}(s)\right) \ll (\Sigma_{\text{hiq}}(s),\Delta_{\text{hiq}}(s))$ are achieved to calculate the expectation values of the previous radial terms. In other words, all physical justifications for applying the time-independent perturbation theory become satisfied. This allows us to give a complete prescription for determining the energy level of the generalized $(n,l,m,s,\widetilde{l},\widetilde{m},\widetilde{s})^{\text{th}}$ excited states.

3.3. The expectation values under IHHEIQP in the DDT for spin symmetry

For the purpose of finding the energetic corrections resulting from the topological deformations of space-space, we devote this subsection to calculating the expectation values. In the case of deformation Dirac theory symmetries, we find $R_{1(nlms)}^{s-\text{hiq}}$, $R_{2(nlms)}^{s-\text{hiq}}$, $R_{3(nlms)}^{s-\text{hiq}}$ and $R_{4(nlms)}^{s-\text{hiq}}$ which are equal to $\left\langle\frac{s^2}{(1-qs)^4}\right\rangle_{(nlms)}^{s-\text{hiq}}$, $\left\langle\frac{s^{\frac{3}{2}}}{(1-qs)^3}\right\rangle_{(nlms)}^{s-\text{hiq}}$, $\left\langle\frac{s^{5/2}}{(1-qs)^3}\right\rangle_{(nlms)}^{s-\text{hiq}}$ and $\left\langle\frac{s^3}{(1-qs)^4}\right\rangle_{(nlms)}^{s-\text{hiq}}$, respectively, for the spin symmetry, by applying the perturbative theory accounting for the unperturbed wave function of the HHEIQP model seen previously in Eq. (21.1). Thus, after straightforward calculations, we obtain the following results:

$$R_{1(nlms)}^{s-\text{hiq}}=\frac{B_{nk}^{sn2}}{q^2}\int_0^{+\infty}(qs)^{\rho^{sp}+2}(1-qs)^{2\eta_{nk}^{sp}-4}Ddr, \quad (42.1)$$

$$R_{2(nlms)}^{s-\text{hiq}}=\frac{B_{nk}^{sn2}}{q^{3/2}}\int_0^{+\infty}(qs)^{\rho^{sp}+3/2}(1-qs)^{2\eta_{nk}^{sp}-3}Ddr, \quad (42.2)$$

$$R_{3(nlms)}^{s-\text{hiq}}=\frac{B_{nk}^{sn2}}{q^{5/2}}\int_0^{+\infty}(qs)^{\rho^{sp}+5/2}(1-qs)^{2\eta_{nk}^{sp}-3}Ddr, \quad (42.3)$$

$$R_{4(nlms)}^{s-\text{hiq}}=\frac{B_{nk}^{sn2}}{q^3}\int_0^{+\infty}(qs)^{\rho^{sp}+3}(1-qs)^{2\eta_{nk}^{sp}-4}Ddr, \quad (42.4)$$

with $\rho^{sp}=2\sqrt{-\beta^s b^2}$ and $D=[{}_2F_1(-n,D_{nk}^{sp};K^s,qs)]^2$. We have used the useful abbreviations $\langle R(s)\rangle_{(nlms)}^{s-\text{hiq}}=\langle n,l,m.\,|R(s)|.\,n,l,m\rangle$ to avoid the extra burden to write equations. Furthermore, we have applied the property of the spherical harmonics, which has the form:

$$\int Y_l^m(\theta',\varphi')Y_{l'}^{m'}(\theta,\varphi)d\Omega=\delta_{ll'}\delta_{mm'}, \quad (43)$$

with $d\Omega=\sin(\theta)\,d\theta d\varphi$. As we saw in the second section, performing a change of variables via $s = \exp(-\nu r)$ in Eqs. (42), we map the region $0 \leqslant r < \infty$ into $0 \leqslant qs \le q$. This allows us to obtain

$dr = -\frac{ds}{\nu s}$. Now, we transform Eqs. $(42, i = \overline{1,4})$ into the following form:

$$R^{s-\text{hiq}}_{1(nlms)} = = \frac{B^{sn2}_{nk}}{\nu q^2} \int_0^{+q} (qs)^{\rho^{sp}+2-2} (1-qs)^{2\eta^{sp}_{nk}-4} D d(qs), \quad (44.1)$$

$$R^{s-\text{hiq}}_{2(nlms)} = = \frac{B^{sn2}_{nk}}{\nu q^{3/2}} \int_0^{+q} (qs)^{\rho^{sp}+3/2-1} (1-qs)^{2\eta^{sp}_{nk}-3} D d(qs), \quad (44.2)$$

$$R^{s-\text{hiq}}_{3(nlms)} = = \frac{B^{sn2}_{nk}}{\nu q^{5/2}} \int_0^{+q} (qs)^{\rho^{sp}+5/2-1} (1-qs)^{2\eta^{sp}_{nk}-3} D d(qs), \quad (44.3)$$

$$R^{s-\text{hiq}}_{4(nlms)} = = \frac{B^{sn2}_{nk}}{\nu q^3} \int_0^{+q} (qs)^{\rho^{sp}+3-1} (1-qs)^{2\eta^{sp}_{nk}-4} D d(qs). \quad (44.4)$$

For $q = 1$, we can evaluate the above integrals either in a recurrence way through the physical values of the principal quantum number $(n = = 0, 1, ...)$ and then generalize the result to the general $(n, l, m, s, \widetilde{l}, \widetilde{m}, \widetilde{s})^{\text{th}}$ excited state, or we use the method, which was proposed by Dong *et al.* [89] and applied by Zhang [89], to obtain the general excited state directly. We calculate the integrals in Eqs. $(44, i = \overline{1,4})$ with the help of the special integral formula:

$$\int_0^{+1} s^{\alpha-1} (1-s)^{\beta-1} \; {}_2F_1(c_1, c_2; c_3; s)\, ds = = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \; {}_3F_2(c_1, c_2, \beta; c_3, \beta+\alpha; 1). \quad (45)$$

Here, ${}_3F_2$ $(c_1, c_2, \beta; c_3, \hat{\beta}+\alpha; 1)$ equal to $\sum_{n=0}^{+\infty} \frac{(c_1)_n (c_2)_n (\sigma)_n}{(c_3)_n (\sigma+\xi) n!}$, $(c_1)_n$ is the rising factorial or Pochhammer symbol, while $\Gamma(\alpha)$ denotes the usual Gamma function. By identifying Eq. (45) with the integrals, we obtain the following results:

$$R^{s-\text{hiq}}_{1(nlms)} = \gamma^{1s}_{nk} \times \times \sum_{q=0}^{n} \frac{(-1)^q (n + \lambda^{sp}_{nk} - 2)_q}{(q + \rho^{sp} + 2)(n-q)! q! \Gamma(q + \lambda^{sp}_{nk} - 1)} \times \times {}_3F_2 \begin{pmatrix} -n, q + \rho^{sp} + 2, n + \lambda^{sp}_{nk} - 2, \\ \rho^{sp} + 3; q + \lambda^{sp}_{nk} - 1; 1 \end{pmatrix} \quad (46.1)$$

$$R^{s-\text{hiq}}_{2(nlms)} = \gamma^{2s}_{nk} \times \sum_{q=0}^{n} \frac{(-1)^q (n + \lambda^{sp}_{nk} - 3/2)_q}{(q + \rho^{sp} + 3/2)(n-q)! q! \Gamma(q + \lambda^{sp}_{nk} - 1/2)} \times \times {}_3F_2 \begin{pmatrix} -n, q + \rho^{sp} + 3/2, n + \lambda^{sp}_{nk} - 3/2, \\ \rho^{sp} + 5/2; q + \lambda^{sp}_{nk} - 1/2; 1 \end{pmatrix} \quad (46.2)$$

$$R^{s-\text{hiq}}_{3(nlms)} = \gamma^{3s}_{nk} \times \times \sum_{q=0}^{n} \frac{(-1)^q (n + \lambda^{sp}_{nk} - 1/2)_q}{(q + \rho^{sp} + 5/2)(n-q)! q! \Gamma(q + \lambda^{sp}_{nk} + 1/2)} \times \times {}_3F_2 \begin{pmatrix} -n, q + \rho^{sp} + 5/2, n + \lambda^{sp}_{nk} - 1/2, \\ \rho^{sp} + 7/2; q + \lambda^{sp}_{nk} + 1/2; 1 \end{pmatrix} \quad (46.3)$$

$$R^{s-\text{hiq}}_{4(nlms)} = \gamma^{4s}_{nk} \times \times \sum_{q=0}^{n} \frac{(-1)^q (n + \lambda^{sp}_{nk} - 1)_q}{(q + \rho^{sp} + 3)(n-q)! q! \Gamma(q + \lambda^{sp}_{nk})} \times {}_3F_2 \begin{pmatrix} -n, q + \rho^{sp} + 3, n + \lambda^{sp}_{nk} - 1, \\ \rho^{sp} + 4; q + \lambda^{sp}_{nk}; 1 \end{pmatrix}, \quad (46.4)$$

with

$$\lambda^{sp}_{nk} = \rho^{sp} + 2\eta^{sp}_{nk},$$

$$\gamma^{1s}_{nk} = \frac{B^{sn2}_{nk}}{\nu} n! \Gamma(\rho^{sp} + 3) \Gamma(2\eta^{sp}_{nk} - 3),$$

$$\gamma^{2s}_{nk} = \frac{B^{sn2}_{nk}}{\nu} n! \Gamma(\rho^{sp} + 5/2) \Gamma(2\eta^{sp}_{nk} - 2),$$

$$\gamma^{3s}_{nk} = \frac{B^{sn2}_{nk}}{\nu} n! \Gamma(\rho^{sp} + 7/2) \Gamma(2\eta^{sp}_{nk} - 2),$$

$$\gamma^{4s}_{nk} = \frac{B^{sn2}_{nk}}{\nu} n! \Gamma(\rho^{sp} + 4) \Gamma(2\eta^{sp}_{nk} - 3)$$

and

$$(n + \lambda^{sp}_{nk} - 2)_q = \frac{\Gamma(n + \lambda^{sp}_{nk} - 2 + q)}{\Gamma(n + \lambda^{sp}_{nk} - 2)},$$

$$(n + \lambda^{sp}_{nk} - 3/2)_q = \frac{\Gamma(n + \lambda^{sp}_{nk} - 3/2 + q)}{\Gamma(n + \lambda^{sp}_{nk} - 3/2)},$$

$$(n + \lambda^{sp}_{nk} - 1/2)_q = \frac{\Gamma(n + \lambda^{sp}_{nk} - 1/2 + q)}{\Gamma(n + \lambda^{sp}_{nk} - 1/2)},$$

$$(n + \lambda^{sp}_{nk} - 1)_q = \frac{\Gamma(n + \lambda^{sp}_{nk} - 1 + q)}{\Gamma(n + \lambda^{sp}_{nk} - 1)}.$$

3.4. The expectation values under the IHHEIQP in the DDT for the p-spin symmetry

Now, we apply the perturbative theory to find the following expectation values: $R_{1(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-\text{hiq}}$, $R_{2(nlms)}^{ps-\text{hiq}}$, $R_{3(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-\text{hiq}}$ and $R_{4(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-\text{hiq}}$ which are equal to $\left\langle\frac{s^2}{(1-qs)^4}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-\text{hiq}}$, $\left\langle\frac{s^{\frac{3}{2}}}{(1-qs)^3}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-\text{hiq}}$, $\left\langle\frac{s^{5/2}}{(1-qs)^3}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-\text{hiq}}$ and $\left\langle\frac{s^3}{(1-qs)^4}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-\text{hiq}}$, respectively, for the p-spin symmetry under the IHHEIQP model in the DDT accounting for the unperturbed wave function of the HHEIQP model which we have seen previously in Eq. (21.2). It will be used to determine the appropriate energetic corrections resulting from the topological properties of the space. By examining the two expressions of the two unperturbed wave functions of the *HHEIQP* model in Eqs. (21.1) and (21.2), we note that there is a possibility of passing from the upper wave function $F_{nk}(r)$ to the lower wave function $G_{nk}(r)$ by making the following substitutions:

$$B_{nk}^{sn} \Longleftrightarrow B_{nk}^{psn},\ \beta^{sp} \Leftrightarrow \beta^{ps} \text{and } \eta_{nk}^{sp} \Leftrightarrow \eta_{nk}^{ps}\ , \tag{47}$$

which allows us to obtain the expectation values for the p-spin symmetry from Eqs. $(46, i=\overline{1,4})$ without recalculation, as follows:

$$R_{1(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-\text{hiq}} = \gamma_{nk}^{1s} \times \\ \times \sum_{q=0}^{n} \frac{(-1)^q \left(n+\lambda_{nk}^{ps}-2\right)_q}{(q+\rho^{ps}+2)(n-q)!q!\Gamma\left(q+\lambda_{nk}^{ps}-1\right)} \times \\ \times {}_3F_2\begin{pmatrix} -n, q+\rho^{ps}+2, n+\lambda_{nk}^{ps}-2, \\ \rho^{ps}+3; q+\lambda_{nk}^{ps}-1; 1 \end{pmatrix}, \tag{48.1}$$

$$R_{2(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-\text{hiq}} = \gamma_{nk}^{2s} \times \\ \times \sum_{q=0}^{n} \frac{(-1)^q \left(n+\lambda_{nk}^{ps}-3/2\right)_q}{(q+\rho^{ps}+3/2)(n-q)!q!\Gamma\left(q+\lambda_{nk}^{ps}-1/2\right)} \times \\ \times {}_3F_2\begin{pmatrix} -n, q+\rho^{ps}+3/2, n+\lambda_{nk}^{ps}-3/2, \\ \rho^{ps}+5/2; q+\lambda_{nk}^{ps}-1/2; 1 \end{pmatrix}, \tag{48.2}$$

$$R_{3(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-\text{hiq}} = \gamma_{nk}^{3s} \times \\ \times \sum_{q=0}^{n} \frac{(-1)^q \left(n+\lambda_{nk}^{ps}-1/2\right)_q}{(q+\rho^{ps}+5/2)(n-q)!q!\Gamma\left(q+\lambda_{nk}^{ps}+1/2\right)} \times \\ \times {}_3F_2\begin{pmatrix} -n, q+\rho^{ps}+5/2, n+\lambda_{nk}^{ps}-1/2, \\ \rho^{ps}+7/2; q+\lambda_{nk}^{ps}+1/2; 1 \end{pmatrix}, \tag{48.3}$$

$$R_{4(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-\text{hiq}} = \gamma_{nk}^{4s} \times \\ \times \sum_{q=0}^{n} \frac{(-1)^q \left(n+\lambda_{nk}^{ps}-1\right)_q}{(q+\rho^{sp}+3)(n-q)!q!\Gamma\left(q+\lambda_{nk}^{ps}\right)} \times \\ \times {}_3F_2\begin{pmatrix} -n, q+\rho^{ps}+3, n+\lambda_{nk}^{ps}-1, \\ \rho^{ps}+4; q+\lambda_{nk}^{ps}; 1 \end{pmatrix}, \tag{48.4}$$

with

$$\lambda_{nk}^{ps} = \rho^{ps} + 2\eta_{nk}^{ps},$$
$$\rho^{ps} = 2\sqrt{-\beta^{ps} b^2},$$
$$\gamma_{nk}^{1s} = \frac{B_{nk}^{psn2}}{\nu} n!\Gamma\left(\rho^{ps}+3\right)\Gamma\left(2\eta_{nk}^{ps}-3\right),$$
$$\gamma_{nk}^{2s} = \frac{B_{nk}^{psn2}}{\nu} n!\Gamma\left(\rho^{ps}+5/2\right)\Gamma\left(2\eta_{nk}^{ps}-2\right),$$
$$\gamma_{nk}^{3s} = \frac{B_{nk}^{psn2}}{\nu} n!\Gamma\left(\rho^{ps}+7/2\right)\Gamma\left(2\eta_{nk}^{ps}-2\right),$$
$$\gamma_{nk}^{4s} = \frac{B_{nk}^{psn2}}{\nu} n!\Gamma\left(\rho^{ps}+4\right)\Gamma\left(2\eta_{nk}^{ps}-3\right)$$

and

$$\left(n+\lambda_{nk}^{ps}-2\right)_q = \frac{\Gamma\left(n+\lambda_{nk}^{ps}-2+q\right)}{\Gamma\left(n+\lambda_{nk}^{ps}-2\right)},$$
$$\left(n+\lambda_{nk}^{ps}-3/2\right)_q = \frac{\Gamma\left(n+\lambda_{nk}^{ps}-3/2+q\right)}{\Gamma\left(n+\lambda_{nk}^{sp}-3/2\right)},$$
$$\left(n+\lambda_{nk}^{sp}-1/2\right)_q = \frac{\Gamma\left(n+\lambda_{nk}^{ps}-1/2+q\right)}{\Gamma\left(n+\lambda_{nk}^{ps}-1/2\right)},$$
$$\left(n+\lambda_{nk}^{ps}-1\right)_q = \frac{\Gamma\left(n+\lambda_{nk}^{ps}-1+q\right)}{\Gamma\left(n+\lambda_{nk}^{ps}-1\right)}.$$

3.5. The corrected energy for the IHHEIQP model in DDT symmetries

In this subsection, we will focus on the physical implications and will obtain the relative corrections resulting from the topological properties of a physical system that is affected by the IHHEIQP. The total value of the relative energy resulting from the effect of $(\Sigma_{\text{hiq}}(\widehat{r}), \Delta_{\text{hiq}}(\widehat{r}))$ gives us the main contribution, which we saw in Sect. 2 through Eqs. (18) and (19) for the spin and p-spin symmetries, whereas the secondary contributions are caused by topological defects of $\Sigma_{\text{hiq}}^{\text{pert}}(s)$ and $\Delta_{\text{hiq}}^{\text{pert}}(s)$. These additional effects have an efficient action on the spontaneous generation of several intrinsic physical phenomena. The first one is generated from the effect of the perturbed spin-orbit and pseudospin-orbit effective potentials $\Sigma_{\text{hiq}}^{\text{pert}}(s)$ and $\Delta_{\text{hiq}}^{\text{pert}}(s)$ and corresponds to the spin and pseudospin symmetries. These perturbed effective potentials are obtained by replacing the coupling of the angular momentums ($\mathbf{L}$ and $\widetilde{\mathbf{L}}$) operators and the NC vector η with the new equivalent couplings $\Theta\mathbf{LS}$ and $\Theta\widetilde{\mathbf{L}}\widetilde{\mathbf{S}}$ for the spin and p-spin symmetries, respectively (with $\Theta^2 = \Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2$). This degree of freedom comes considering that the infinitesimal NC vector $\boldsymbol{\Theta}$ is arbitrary. We have oriented the spins ($\mathbf{S}$,$\widetilde{\mathbf{S}}$) of the fermionic particles to be in parallel to the vector $\boldsymbol{\Theta}$ which interact with the improved Hulthén plus hyperbolic exponential inversely quadratic potential. Moreover, we replace the new spin-orbit and pseudospin-orbit couplings $\eta\mathbf{LS}$ and $\eta\widetilde{\mathbf{L}}\widetilde{\mathbf{S}}$ with the corresponding new physical forms $(\eta/2)\mathbf{G}^2$ and $(\eta/2)\widetilde{\mathbf{G}}^2$, with $\mathbf{G}^2 = \mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2$ and $\widetilde{\mathbf{G}}^2 = \mathbf{J}^2 - \widetilde{\mathbf{L}}^2 - \widetilde{\mathbf{S}}^2$ for a spin/(p-spin) symmetry, respectively. Furthermore, in RQM, the operators ($\widehat{\mathbf{H}}_{rnc}^{\text{hiq}}$, $\mathbf{J}^2$, $\mathbf{L}^2$, $\mathbf{S}^2$ and $\mathbf{J}_z$) form a complete set of conserved physical quantities. The eigenvalues of the operators $\mathbf{G}^2$ and $\widetilde{\mathbf{G}}^2$ are equal to the values:

$$\begin{cases}\Xi\,(j,l,s) = \left[j(j+1) - k(k+1) - 3/4)\right]/2 \\ \text{with } |l-1/2| \leq j \leq \ |l+1/2|, \\ \Xi(j,\widetilde{l},\widetilde{s}) = \left[j(j+1) - k(k-1) - 3/4)\right]/2 \\ \text{with } \left|\widetilde{l} - 1/2\right| \leq j \leq \ \left|\widetilde{l} + 1/2\right| \end{cases}$$

for the spin and p-spin symmetries, respectively. As a direct consequence, the partially corrected energies $\Delta E_{\text{hiq}}^{so-s}(n,\nu,\nu_1,A,\Theta,j,l,s) \equiv \Delta E_{\text{hiq}}^{so-s}$ and $\Delta E_{\text{hiq}}^{so-ps}(n,\nu,\nu_1,A,\Theta,j,\widetilde{l},\widetilde{s}) \equiv \Delta E_{\text{hiq}}^{so-ps}$ due to the perturbed effective potentials $\Sigma_{\text{hiq}}^{\text{pert}}(r)$ and $\Delta_{\text{hiq}}^{\text{pert}}(r)$ produced for the $(n,l,m,s,\widetilde{l},\widetilde{m},\widetilde{s})^{\text{th}}$ excited state, in DDT symmetries are as follows:

$$\begin{cases}\Delta E_{\text{hiq}}^{so-s} = \Theta\left(j(j+1) - k(k+1) - \dfrac{3}{4}\right), \\ \langle\Lambda\rangle_{(nlms)}^{\text{hiq}}(n,\nu,\nu_1,A), \\ \Delta E_{\text{hiq}}^{so-ps} = \Theta\left(j(j+1) - k(k-1) - \dfrac{3}{4}\right), \\ \langle\widetilde{\Lambda}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{\text{hiq}}(n,\nu,\nu_1,A). \end{cases} \tag{49}$$

The global two expectations values $\langle\Lambda\rangle_{(nlms)}^{\text{hiq}}(n,\nu,\nu_1,A)$ and $\langle\widetilde{\Lambda}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{\text{hiq}}(n,\ \nu,\nu_1,A)$ for a spin(p-spin) -symmetry, respectively are determined from the following expressions:

$$\begin{cases}\langle\Lambda\rangle_{(nlms)}^{\text{hiq}}(n,\nu,\nu_1,A) = \sum\limits_{4}^{\mu=1}\delta_{nk}^{\mu s}R_{\mu(nlms)}^{s-\text{hiq}}, \\ \langle\widetilde{\Lambda}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{\text{hiq}}(n,\nu,\nu_1,A) = \sum\limits_{4}^{\mu=1}\delta_{nk}^{\mu ps}R_{\mu(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-\text{hiq}}. \end{cases} \tag{50}$$

We get the coefficients $\delta_{nk}^{\mu s}$ and $\delta_{nk}^{\mu ps}$ from Eq. (41), while $R_{\mu(nlms)}^{s-\text{hiq}}$ and $R_{\mu(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-\text{hiq}}$ from Eqs. (46,i) and Eqs. (48,i), $\left(i = \overline{1,4}\right)$, respectively. The second main part is obtained from the magnetic effect of the perturbative effective potentials $\Sigma_{\text{hiq}}^{\text{pert}}(r)$ and $\Delta_{\text{hiq}}^{\text{pert}}(r)$ under the IHHEIQP model in the deformed Dirac theory symmetries. These effective potentials are achieved, when we replace both ($\mathbf{L}\boldsymbol{\Theta}$ and $\widetilde{\mathbf{L}}\boldsymbol{\Theta}$) by ($\sigma\aleph L_z$ and $\sigma\aleph\widetilde{L}_z$), respectively, and Θ_{12} by $\sigma\aleph$, here ($\aleph$ and σ) are present the intensity of the magnetic field induced by the effect of the space-space geometry deformation. So that the physical unit of the original noncommutativity parameter $\Theta_{12}(\text{length})^2$ is the same unit of $\sigma\aleph$, we have also need to apply

$$\begin{cases}\langle n',l',m'\ L_z\ n,l,m\rangle = m\delta_{m'm}\delta_{l'l}\delta_{n'n} \\ \text{with } -l \leq m \leq \ l \end{cases}$$

and

$$\begin{cases}\langle n',\widetilde{l'},\widetilde{m'}\widetilde{L}_z n,\widetilde{l},\widetilde{m}\rangle = \widetilde{m}\delta_{\widetilde{m'}\widetilde{m}}\delta_{\widetilde{l'}\widetilde{l}}\delta_{n'n} \\ \text{with } -\widetilde{l'} \leq \widetilde{m'} \leq \ \widetilde{l} \end{cases}$$

for a spin (p-spin) symmetry, respectively. All of these data allow the discovery of new energy shifts

$\Delta E_{\text{hiq}}^{mg-s}(n,\nu,\nu_1,A,\sigma,m)$ and

$\Delta E_{\text{hiq}}^{mg-ps}(n,\nu,\nu_1,A,\sigma,\widetilde{m})$

due to the perturbed Zeeman effect created by the influence of the IHHEIQP for the $(n,l,m,s,\widetilde{l},\widetilde{m},\widetilde{s})^{\text{th}}$ excited state in the deformed Dirac theory symmetries as follows:

$$\begin{cases}\dfrac{\Delta E_{\text{hiq}}^{mg-s}(n,\nu,\nu_1,A,\sigma,m)}{\sigma\aleph} = \\ = \langle\Lambda\rangle_{(nlms)}^{\text{hiq}}(n,\nu,\nu_1,A)\,m, \\ \dfrac{\Delta E_{\text{hiq}}^{mg-ps}(n,\nu,\nu_1,A,\sigma,\widetilde{m})}{\sigma\aleph} = \\ = \langle\widetilde{\Lambda}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{\text{hiq}}(n,\nu,\nu_1,A)\,\widetilde{m}. \end{cases} \tag{51}$$

We will now refer to the generation of another phenomenon as a result of the influence of topological properties in the IHHEIQP model in DDT symmetries. This physical phenomenon is produced automatically from the influence of the perturbed effective potentials $\Sigma_{\text{hiq}}^{\text{pert}}(r)$ and $\Delta_{\text{hiq}}^{\text{pert}}(r)$ which we have seen in Eqs. (39) and (40). We consider the fermionic particles undergoing the rotation with angular velocity $\mathbf{\Omega}$. The features of this subjective phenomenon are determined by replacing an arbitrary vector $\mathbf{\Theta}$ with $\beta\mathbf{\Omega}$. We now replace two couplings ($\mathbf{L\Theta}$ and $\widetilde{\mathbf{L}}\mathbf{\Theta}$) with ($\beta\mathbf{L\Omega}$ and $\beta\widetilde{\mathbf{L}}\mathbf{\Omega}$), respectively:

$$\begin{pmatrix}\mathbf{L}\\ \widetilde{\mathbf{L}}\end{pmatrix}\mathbf{\Theta}\to\beta\begin{pmatrix}\mathbf{L}\\ \widetilde{\mathbf{L}}\end{pmatrix}\mathbf{\Omega}\begin{cases}\text{for spin-sy,}\\ \text{for p-spin-sy.}\end{cases} \tag{52}$$

Here β is just an infinitesimal real constant. We can express the effective potential $\Sigma_{\text{pert}}^{\text{hiq-rot}}(s)$ and $\Delta_{\text{pert}}^{\text{hiq-rot}}(s)$ which induce the rotational movements of fermionic particles as follows:

$$\Sigma_{\text{pert}}^{\text{hiq-rot}}(s)=\beta\left(\frac{k(k+1)}{r^4}+\frac{\nu\nu_1 q\gamma_s\exp(-\nu r)}{2r(1-q\exp(-\nu r))^2}-\right.$$
$$\left.-\frac{A\nu q\gamma_s\exp(-\nu r)}{2r^3}-\frac{Aq\gamma_s\exp(-\nu r)}{r^4}\right)\mathbf{L\Omega}+O(\Omega^2) \tag{53.1}$$

and

$$\Delta_{\text{pert}}^{\text{hiq-rot}}(s)=\beta\left(\frac{k(k-1)}{r^4}+\frac{\nu\nu_1 q\gamma_s\exp(-\nu r)}{2r(1-q\exp(-\nu r))^2}-\right.$$
$$\left.-\frac{A\nu q\gamma_s\exp(-\nu r)}{2r^3}-\frac{Aq\gamma_s\exp(-\nu r)}{r^4}\right)\widetilde{\mathbf{L}}\mathbf{\Omega}+O(\Omega^2). \tag{53.2}$$

To simplify the calculations, we choose the rotational velocity $\mathbf{\Omega}$ to be parallel to the (Oz) axis ($\mathbf{\Omega}=\Omega\mathbf{e}_z$). Of course, this does not change the physical nature of the studied problem and simplifies the calculations. Then we transform the spin-orbit couplings into new physical phenomena as follows:

$$\begin{pmatrix}\mathbf{L}\\ \widetilde{\mathbf{L}}\end{pmatrix}\mathbf{\Omega}\to\beta\Omega\begin{pmatrix}L_z\\ \widetilde{L}_z\end{pmatrix}. \tag{54}$$

All of these data allow us to find new corrected energies $\Delta E_{\text{hiq}}^{\text{rot}-s}$ $(n,\nu,\nu_1,A,\ \beta,\ m)$ and $\Delta E_{\text{hiq}}^{\text{rot}-ps}$ $(n,\nu,\nu_1,A,\beta\,\widetilde{m})$ due to the perturbed effective potentials $\Sigma_{\text{pert}}^{\text{hiq-rot}}(s)$ and $\Delta_{\text{pert}}^{\text{hiq-rot}}(s)$ which are generated at once by the influence of the improved Hulthén plus hyperbolic exponential inversely quadratic potential for the $(n,l,m,s,\widetilde{l},\widetilde{m},\widetilde{s})^{\text{th}}$ excited state in DDT symmetries as follows:

$$\begin{pmatrix}\Delta E_{\text{hiq}}^{\text{rot}-s}\\ \Delta E_{\text{hiq}}^{\text{rot}-ps}\end{pmatrix}=\beta\Omega\begin{pmatrix}\langle\Lambda\rangle_{(nlms)}^{\text{hiq}}\,m\\ \left\langle\widetilde{\Lambda}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{\text{hiq}}\,\widetilde{m}\end{pmatrix}. \tag{55}$$

It is worth mentioning that the authors of Ref. [90] studied a rotating isotropic or anisotropic harmonically confined ultra-cold Fermi gas in the two- and three-dimensional spaces at the zero temperature. But, in this study, the rotational term was added to the Hamiltonian operator in contrast to our recent study, where two rotation operators $\Sigma_{\text{pert}}^{\text{hiq-rot}}(s)\,\mathbf{L\Omega}$ and $\Delta_{\text{pert}}^{\text{hiq-rot}}(s)\,\widetilde{\mathbf{L}}\mathbf{\Omega}$ appear due to the augmented symmetries resulting from a space-space deformation under the improved Hulthén plus hyperbolic exponential inversely quadratic potential.

3.5.1. Global relativistic correction of energies

Having obtained the physical form of three main parts of energies through the effect of the IHHEIQP, the next task is to obtain the formula for bound-state energies for a system moving in this potential. We have seen that the eigenvalues $\Xi(j,l,s)$ and $\Xi(j,\widetilde{l},\widetilde{s})$ of the operators $\mathbf{G}^2$ and $\widetilde{\mathbf{G}}^2$are, respectively, are equal to

$$\begin{cases}\Xi(j,l,s)=[j(j+1)-l(l+1)-3/4]/2,\\ \Xi(j,\widetilde{l},\widetilde{s})=[j(j+1)-\widetilde{l}(\widetilde{l}-1)-3/4]/2.\end{cases} \tag{56}$$

Thus, for the case of spin-1/2 fields corresponding to the up polarity and the down polarity, the possible values of j are $l\pm1/2$ and $\widetilde{l}\pm1/2$ for the spin symmetry $\Xi(j,l,s)$ and the pseudospin symmetry $\Xi(j,\widetilde{l},\widetilde{s})$, are as follows:

$$\Xi(j=l\pm1/2,s=1/2)=$$
$$\frac{1}{2}\begin{cases}l & \text{for Up-p: } j=l+1/2,\\ -(l+1) & \text{for down-p: } j=l-1/2\end{cases} \tag{57.1}$$

and

$$\Xi(j=\widetilde{l}\pm1/2,\widetilde{s}=1/2)=$$
$$\frac{1}{2}\begin{cases}\widetilde{l} & \text{for Up-p: } j=\widetilde{l}+1/2,\\ -(\widetilde{l}+1) & \text{for Down-p: } j=\ \widetilde{l}-1/2.\end{cases} \tag{57.2}$$

The global relativistic energy

$E_{nc}^{s-\text{hiq}}(n,\nu,\nu_1,A,\eta,\sigma,\beta,j,l,s,m)$ and $E_{nc}^{ps-\text{hiq}}(n,\,\nu,\,\nu_1,\,A,\,\eta,\,\sigma,\,\beta,\,j,\,\widetilde{l},\,\widetilde{s},\,\widetilde{m})$

for the case of spin-1/2 with the improved Hulthén plus hyperbolic exponential inversely quadratic potential without tensor interaction, in the DDT symmetries corresponding to the generalized $(n,l,m,s,\widetilde{l},\widetilde{m},\widetilde{s})^{\text{th}}$ excited states:

$$E_{nc}^{s-\text{hiq}}=E_{nk}^{s}+\langle\Lambda\rangle_{(nlms)}^{\text{hiq}}(n,\nu,\nu_1,A)\Bigg[(\sigma\aleph+\beta\Omega)\,m+ + \frac{\Theta}{2}\begin{cases} l & \text{for Up-p: } j=l+1/2,\\ -(l+1) & \text{for Down-p: } j=l-1/2\end{cases}\Bigg] \tag{58.1}$$

and

$$E_{nc}^{ps-\text{hiq}}=E_{nk}^{ps}+\langle\widetilde{\Lambda}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{\text{hiq}}(n,\nu,\nu_1,A)\Bigg[(\sigma\aleph+\beta\Omega)\,\widetilde{m}+ + \frac{\Theta}{2}\begin{cases} \widetilde{l} & \text{for Up-p: } j=\widetilde{l}+1/2,\\ -(\widetilde{l}+1) & \text{for Down-p: } j=\widetilde{l}-1/2\end{cases}\Bigg]. \tag{58.2}$$

Here, E_{nk}^{s} and E_{nk}^{ps} are usual relativistic energies under the hyperbolic Hulthén plus the hyperbolic exponential inversely quadratic potential which can be obtained from Eqs. (18) and (19) in the second section. We can now generalize our obtained energies $E_{g-nc}^{\text{hiq}-s}$ and $E_{g-nc}^{\text{hiq}-ps}$ which are produced with two globally induced potentials $\Sigma_{t\_\text{hiq}}^{\text{pert}}(r)$ and $\Delta_{t-\text{hiq}}^{\text{pert}}(r)$ for the spin and p-spin symmetries corresponding to upper and lower components ($F_{nk}^{s}(s)$ and $G_{nk}^{sp}(s)$) and ($F_{nk}^{ps}(s)$ and $G_{nk}^{ps}(s)$), respectively, as:

$$E_{g-nc}^{\text{hiq}-s}=E_{nc}^{\text{hiq}-s}\theta\left(\left|E_{nc}^{\text{hiq}-s}\right|\right)-E_{nc}^{\text{hiq}-s}\theta\left(-\left|E_{nc}^{\text{hiq}-s}\right|\right)= =\begin{cases}E_{nc}^{\text{hiq}-s} & \text{Upper component of spin } s_y,\\ -E_{nc}^{\text{hiq}-s} & \text{Lower component of spin } s_y.\end{cases} \tag{59.1}$$

and

$$E_{g-nc}^{\text{hiq}-ps}=E_{nc}^{\text{hiq}-ps}\theta\left(\left|E_{nc}^{\text{hiq}-ps}\right|\right)- -E_{nc}^{\text{hiq}-ps}\theta\left(-\left|E_{nc}^{\text{hiq}-ps}\right|\right)= =\begin{cases}E_{nc}^{\text{hiq}-ps} & \text{Upper component of p-spin } s_y,\\ -E_{nc}^{\text{hiq}-ps} & \text{Lower component of p-spin } s_y.\end{cases} \tag{59.2}$$

3.6. Study of important relativistic particular cases in the context of DDT

Here, we will examine some particular cases regarding the new bound-state energy eigenvalues in Eqs. (58.1) and (58.2). We could derive some particular potentials useful for other physical systems, by adjusting relevant parameters of the IHHEIQP model in DDT symmetries such as the improved Hulthén potential and the improved Dirac exponential inversely quadratic potential.

3.6.1. Deformed Dirac equation with the improved Hulthén potential without tensor interaction

The improved Hulthén potential is obtained from the improved Hulthén plus hyperbolic exponential inversely quadratic potential, in DDT symmetries, as follows:

$$V_{hp}(\widehat{r})=\lim_{(A,q)\to(0,1)}V_{\text{hiq}}(\widehat{r})= =\frac{\nu_1\exp(-\nu r)}{1-\exp(-\nu r)}+\frac{\nu\nu_1}{2r}\frac{\exp(-\nu r)}{(1-\exp(-\nu r))^2}\,\mathbf{L\Theta}, \tag{60}$$

where we have used Eqs. (1), (2), and (35). The first part is the Hulthén potential in usual relativistic quantum mechanics [5, 91], while the second part is related to the effect of topological properties on the Hulthén potential. The global energies for the improved Hulthén potential under the spin and p-spin symmetries are obtained from Eqs. (58.1) and (58.2) as

$$E_{nc-nk}^{s-hp}=E_{nk}^{s-hp}+\langle\Lambda\rangle_{(nlms)}^{hp}(n,\nu,\nu_1)\Bigg[(\sigma\aleph+\beta\Omega)\,m+ +\frac{\Theta}{2}\begin{cases} l & \text{for Up-p: } j=l+1/2,\\ -(l+1) & \text{for Down-p: } j=l-1/2\end{cases}\Bigg] \tag{61}$$

and

$$E_{nc-nk}^{ps-hp}=E_{nk}^{ps-hp}+ +\langle\widetilde{\Lambda}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{hp}(n,\nu,\nu_1,A)\Bigg[(\sigma\aleph+\beta\Omega)\,\widetilde{m}+ +\frac{\Theta}{2}\begin{cases} \widetilde{l} & \text{for Up-p: } j=\widetilde{l}+1/2,\\ -(\widetilde{l}+1) & \text{for Down-p: } j=\widetilde{l}-1/2\end{cases}\Bigg]. \tag{62}$$

Here, E_{nk}^{s-hp} and E_{nk}^{ps-hp} are the Hulthén-potential energies under the spin and pseudospin symmetries obtained from the energy equations:

$$\left(E_{nk}^{s-hp}-M\right)\left(E_{nk}^{s-hp}+M\right)=-\frac{1}{b^2}\times$$

$$\times \frac{\begin{bmatrix} n^2+n+\frac{1}{2}+(2n+1)(2k+1)+ \\ +k(k+1)-2\left(E_{nk}^{s-hp}+M\right)\nu_1 b^2 \end{bmatrix}}{(2n+1)^2+(2k+1)} \tag{63}$$

and

$$\left(M+E_{nk}^{ps-hp}\right)\left(M-E_{nk}^{ps-hp}\right)=-\frac{1}{b^2}\times$$
$$\times \frac{\begin{bmatrix} n^2+n+\frac{1}{2}+(2n+1)(2k-1)+ \\ +k(k-1)-2\left(E_{nk}^{ps-hp}-M\right)\nu_1 b^2 \end{bmatrix}}{(2n+1)^2+(2k-1)}. \tag{64}$$

The new expectation values $\langle\Lambda\rangle_{(nlms)}^{hp}$ and $\langle\widetilde{\Lambda}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{hp}$ are determined from Eq. (50), by applying the compensation referred to above at the beginning of the current subsection as follows:

$$\begin{pmatrix} \langle\Lambda\rangle_{(nlms)}^{mt} \\ \langle\widetilde{\Lambda}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{mt} \end{pmatrix} = \begin{pmatrix} \gamma_{nk}^{1s}R_{1(nlms)}^{s-hp}+\gamma_{nk}^{1s}R_{1(nlms)}^{s-hp} \\ \gamma_{nk}^{1ps}R_{1(nlms)}^{ps-hp}+\gamma_{nk}^{1ps}R_{1(nlms)}^{ps-hp} \end{pmatrix} \tag{65}$$

with

$$\left.\begin{aligned} \gamma_{nk}^{1s} &= \nu^4 k(k+1), \\ \delta_{nk}^{2s} &= \frac{\nu^2\nu_1\gamma_s}{2}, \\ \gamma_{nk}^{1ps} &= \nu^4 k(k-1), \\ \gamma_{nk}^{2ps} &= \frac{\nu^2\nu_1\gamma_{ps}}{2}. \end{aligned}\right\}$$

3.6.2. Deformed Dirac improved exponential inversely quadratic potential

When $\nu_1 = 0$ and $q = 1$, the improved Hulthén plus hyperbolic exponential inversely quadratic potential reduces the improved exponential inversely quadratic potential:

$$V_{iq}(\widehat{r}) = \lim_{(\nu_1,q)\to(0,1)} V_{\text{hiq}}(\widehat{r}), \tag{66.1}$$

Which gives

$$V_{iq}(\widehat{r}) = -\frac{A\exp(-\nu r)}{r^2} - $$
$$-\left[\frac{A\nu}{2r^3}\exp(-\nu r)+\frac{A}{r^4}\exp(-\nu r)\right]\mathbf{L\Theta}. \tag{66.2}$$

We have used Eqs. (1), (2), and (35). The first part is the exponential inversely quadratic potential (inverse quadratic Yukawa potential) [92], while the second part is the effect of topological properties on the inverse quadratic Yukawa potential. The global energy for this potential under the spin and p-spin symmetries is obtained from Eqs. (58.1) and (58.2) as

$$E_{nc-nk}^{s-iq} = E_{nk}^{s-iq} + \langle\Lambda\rangle_{(nlms)}^{iq}(n,\nu,A)\left[(\sigma\aleph+\beta\Omega)m + \right.$$
$$\left. +\frac{\Theta}{2}\begin{cases} l & \text{for Up polarity: } j=l+1/2, \\ -(l+1) & \text{for Down polarity: } j=l-1/2 \end{cases}\right] \tag{67}$$

and

$$E_{nc-nk}^{ps-iq} = E_{nk}^{ps-iq} +$$
$$+\langle\widetilde{\Lambda}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{iq}(n,\nu,A)\left[(\sigma\aleph+\beta\Omega)\widetilde{m} + \right.$$
$$\left. +\frac{\Theta}{2}\begin{cases} \widetilde{l} & \text{for Up polarity: } j=\widetilde{l}+1/2, \\ -(\widetilde{l}+1) & \text{for Down polarity: } j=\widetilde{l}-1/2 \end{cases}\right]. \tag{68}$$

Here, E_{nk}^{s-iq} and E_{nk}^{ps-iq} are the energy for the exponential inversely quadratic Yukawa potential under the spin and pseudospin symmetries obtained from the energy equations:

$$\left(E_{nk}^{s-iq}-M\right)\left(E_{nk}^{s-iq}+M\right) = -b^{-2}\times$$
$$\left(\frac{\begin{matrix}\left(n^2+n+\frac{1}{2}\right)+(2n+1)\times \\ \times\sqrt{(2k+1)^2-8A\left(E_{nk}^{s-iq}+M\right)}+k(k+1)\end{matrix}}{(2n+1)^2+\sqrt{(2k+1)^2-8A\left(E_{nk}^{s}+M\right)}}\right)^2 \tag{69}$$

and

$$\left(M+E_{nk}^{ps-iq}\right)\left(M-E_{nk}^{ps-iq}\right) = -b^{-2}\times$$
$$\left(\frac{\begin{matrix}n^2+n+\frac{1}{2}+(2n+1)\times \\ \times\sqrt{(2k-1)^2-8A\left(E_{nk}^{ps-iq}-M\right)}+k(k-1)\end{matrix}}{(2n+1)^2+\sqrt{(2k-1)^2-8A\left(E_{nk}^{ps-iq}-M\right)}}\right)^2, \tag{70}$$

while the new expectation values $\langle\Lambda\rangle_{(nlms)}^{iq}$ and $\langle\widetilde{\Lambda}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{iq}$ are determined from Eq. (50), by applying the compensation referred to above at the beginning of the current subsection as follows:

$$\begin{cases} \langle\Lambda\rangle^{iq}_{(nlms)} = \varepsilon^{1s}_{nk} R^{s-iq}_{1(nlms)} + \\ + \varepsilon^{3s}_{nk} R^{s-iq}_{3(nlms)} + \varepsilon^{4s}_{nk} R^{s-iq}_{4(nlms)}, \\ \langle\widetilde{\Lambda}\rangle^{iq}_{(n\widetilde{l}\widetilde{m}\widetilde{s})} = \varepsilon^{1qs}_{nk} R^{qs-iq}_{1(nlms)} + \\ + \varepsilon^{3s}_{nk} R^{qs-iq}_{3(nlms)} + \varepsilon^{4s}_{nk} R^{qs-iq}_{4(nlms)} \end{cases} \quad (71)$$

with

$$\left.\begin{array}{l} \epsilon^{1s}_{nk} = \nu^4 k\,(k+1)\,, \\ \epsilon^{3s}_{nk} = -\dfrac{A\nu^4\gamma_s}{2}, \\ \epsilon^{4s}_{nk} = -Aq^3\gamma_s\nu^4, \\ \epsilon^{1ps}_{nk} = \nu^4 k\,(k-1)\,, \\ \epsilon^{3ps}_{nk} = -\dfrac{A\nu^4\gamma_{ps}}{2}, \\ \epsilon^{4ps}_{nk} = -A\gamma_{ps}\nu^4. \end{array}\right\} \quad (72)$$

3.6.3. Deformed IHHEIQP model problems in ENRQM symmetries

To realize a study of the nonrelativistic limit, in the extended nonrelativistic quantum mechanics (ENRQM) symmetries of the IHHEIQP model, two steps must be applied; the first step corresponds to the nonrelativistic limit, in usual nonrelativistic quantum mechanics. This is done, by applying the following steps. We replace C_s, $E^s_{nk}+M$, $E^s_{nk}-M$, $k(k+1)$, $F_{nk}(r)$ and ϵ^2_s by 0, 2μ, E^{nr}_{nl}, $l(l+1)$ and $R_{nk}(r)$, respectively, which allows us to obtain the nonrelativistic energy levels as:

$$E^{nr}_{nl} = -\frac{1}{2\mu b^2}\times \times\left[\frac{\left(n^2+n+\frac{1}{2}\right)q + q\,(2n+1)\times \times\sqrt{(2l+1)^2-8A\mu} + ql\,(l+1) - 2q\mu\nu_1 b^2}{(2n+1)\,q + \sqrt{(2l+1)^2-8A\mu}}\right]^2. \quad (73)$$

Now, the second step corresponds to the new coefficients:

$$\left.\begin{array}{l} \delta^{1nr}_{nl} = \nu^4 l\,(l+1)\,q^2, \\ \delta^{2nr}_{nl} = \nu^2\nu_1 q^{3/2}\mu, \\ \delta^{3nr}_{nl} = -A\nu^4 q^3\mu, \\ \delta^{4nr}_{nl} = -A2q^3\mu\nu^4, \end{array}\right\}$$

which are obtained, by applying the previous limits to Eq. (41). This allows us to reexport the relativistic expectation values $\langle\Lambda\rangle^{\text{hiq}}_{(nlms)}(n,\nu,\nu_1,A)$ of the spin symmetry in Eq. (50) from the corresponding nonrelativistic expectation values $\langle\Lambda\rangle^{nr-\text{hiq}}_{(nlms)}(n,\nu,\nu_1,A)$ as:

$$\langle\Lambda\rangle^{nr-\text{hiq}}_{(nlms)}(n,\nu,\nu_1,A) = \sum_{\mu=1}^{4}\delta^{4nr}_{nl} R^{s-\text{hiq}}_{\mu(nlms)} \quad (74)$$

This allows us to express the nonrelativistic correction energy $\Delta E^{\text{hiq}}_{nc-nr}$ (n, ν, ν_1, A, Θ, σ, β, j, l, s, m) $\equiv \Delta E^{\text{hiq}}_{nc-nr}$ produced by the new modified potential problems as

$$\Delta E^{\text{hiq}}_{nc-nl} = \langle\Lambda\rangle^{nr-\text{hiq}}_{(nlms)}\left[(\sigma\aleph+\beta\Omega)\,m + +\frac{\Theta}{2}\begin{cases} l & \text{for } j=l+1/2, \\ -(l+1) & \text{for } j=l-1/2 \end{cases}\right]. \quad (75)$$

The global NR energy $E^{\text{hiq}}_{nc-nl}(n,\alpha,A,B,\Theta,\sigma,\beta,j,l,s,m) \equiv \Delta E^{\text{hiq}}_{nc-nr}$ produced with the new hyperbolic Hulthén plus hyperbolic exponential inversely quadratic potential in ENRQM symmetries. A result of the topological properties of a space-space deformation is the sum of usual energy E^{hiq}_{nl} in Eq. (73) under the Hulthén plus hyperbolic exponential inversely quadratic potential in NRQM symmetries and the obtained correction $\Delta E^{\text{hiq}}_{nc-nl}$ in Eq. (75) as follows:

$$E^{\text{hiq}}_{nc-nl} = E^{nr}_{nl} + \langle\Lambda\rangle^{nr-\text{hiq}}_{(nlms)}\left[(\sigma\aleph+\beta\Omega)\,m + +\frac{\Theta}{2}\begin{cases} l & \text{for } j=l+1/2, \\ -(l+1) & \text{for } j=\ l-1/2 \end{cases}\right]. \quad (76)$$

It should be noted that the corrected energy $\Delta E^{\text{hiq}}_{nc-nl}$ expressed in Eq. (76) is due to the effect of the perturbed potential $\Lambda^{\text{hiq}}_{nr-\text{pert}}(r)$:

$$\Lambda^{\text{hiq}}_{nr-\text{pert}}(r) = = \left(l\,(l+1)\,r^{-4} - \frac{1}{2r}\frac{\partial V_{\text{hiq}}(r)}{\partial r}\right)\mathbf{L\Theta} + O\left(\Theta^2\right). \quad (77)$$

The first term in Eq. (68) is due to the centrifugal term $l\,(l+1)\,\widehat{r}^{-2}$ in ENRQM symmetries which equals the usual centrifugal term $l\,(l+1)\,r^{-2}$ plus the perturbative centrifugal term $l\,(l+1)\,r^{-4}\mathbf{L\Theta}$, while the second term is produced with the effect of the IHHEIQP model. This is one of the most important new results of this research.

4. Spin-Averaged New Mass Spectra of HLM under the IHHEIQP Model in ENRQM symmetries

Our new theoretical model, in ENRQM symmetries, developed so far is applicable to several QM physical systems and to obtain the values of related quantum physical quantities. In this section, we use the NR energies that represent the binding energy between a quark and an anti-quark to determine the modified spin-averaged mass spectra of heavy and heavy-light mesons (HLM) such as $c\bar{c}$, $b\bar{b}$, $b\bar{c}$, $b\bar{s}$, $c\bar{s}$, and $b\bar{q}$, $q = (u, d)$ under IHHEIQP by using the following formula:

$$M_{nl}^{\text{hiq}p} = m_Q + m_{\bar{q}} + E_{nc-nr}^{\text{hiq}} \rightarrow$$

$$\rightarrow M_{nc-nl}^{\text{hiq}p} = m_Q + m_{\bar{q}} +$$

$$+ \begin{cases} \frac{1}{3}\left(E_{nl}^{nc-u} + E_{nl}^{nc-m} + E_{nl}^{nc-l}\right) & \text{for spin 1,} \\ E_{nl}^{nc} & \text{for spin 0.} \end{cases} \quad (78)$$

The LHS of Eq. (78) $M_{nl}^{\text{hiq}p}$ describes the spin-averaged mass spectra of HLM such as $c\bar{c}$, $b\bar{b}$, $b\bar{c}$, $b\bar{s}$, $c\bar{s}$, and $b\bar{q}$, $q = (u, d)$ under the HHEIQP model in usual NR QM symmetries, [93–96], while the RHS is our generalization to this equation in ENRQM symmetries, m_Q and $m_{\bar{q}}$ are the quark mass and the anti-quark mass, E_{nl}^{nr} is the nonrelativistic energy under HHEIQP model which is determined in Eq. (73), while (E_{nl}^{nc-u}, E_{nl}^{nc-m}, E_{nl}^{nc-l}) are the modified energies of HLM which have spin 1, while E_{nl}^{nc} is the modified energies of HLM which have spin 0. We need to replace the factor $\Xi\,(j = l \pm 1/2, s = 1/2)$ with new generalized values $\Xi^n\,(j, l, s = 0, 1)$ as follows:

$$2\Xi^n\,(j, l, s = 0, 1) = j(j+1) - l(l+1) - s(s+1) =$$

$$= \begin{cases} l & \text{for: } j = l + 1 \text{ and } s = 1, \\ -2 & \text{for: } j = l \text{ and } s = 1, \\ -2\,(l+1) & \text{for: } j = l - 1 \text{ and } s = 1, \\ 0 & \text{for: } j = l \text{and} s = 0. \end{cases} \quad (79)$$

Now, we can obtain (E_{nl}^{nc-u}, E_{nl}^{nc-m}, E_{nl}^{nc-l}) and E_{nl}^{nc} of the HLM such as $c\bar{c}$, $b\bar{b}$, $b\bar{c}$, $b\bar{s}$, $c\bar{s}$, and $b\bar{q}$, $q = (u, d)$ as:

$$E_{nl}^{nc-u} = E_{nc-nl}^{\text{hiq}} + \langle\Lambda\rangle_{(nlms)}^{nr-\text{hiq}} \left((\sigma\aleph + \beta\Omega)\, m + \Theta\frac{l}{2}\right)$$

For: $j = l + 1$ and $s = 1$, (80.1)

$$E_{nl}^{nc-m} = E_{nc-nl}^{\text{hiq}} + \langle\Lambda\rangle_{(nlms)}^{nr-\text{hiq}} \left((\sigma\aleph + \beta\Omega)\, m - \Theta\right)$$

For: $j = l$ and $s = 1$, (80.2)

$$E_{nl}^{nc-l} = E_{nc-nl}^{\text{hiq}} + \langle\Lambda\rangle_{(nlms)}^{nr-\text{hiq}} \left((\sigma\aleph + \beta\Omega)\, m - \Theta\,(l+1)\right)$$

For: $j = l - 1$ and $s = 1$, (80.3)

$$E_{nl}^{nc} = E_{nc-nl}^{\text{hiq}} + \langle\Lambda\rangle_{(nlms)}^{nr-\text{hiq}} (\sigma\aleph + \beta\Omega)\, m$$

For: $j = l$ and $s = 0$. (80.4)

By substituting Eqs. (80, i =1,2,3) and (76) into Eq. (78), the new mass spectrum $M_{nc-nl}^{\text{hiq}\,p}$ of the meson systems in the ENRQM symmetries under the IHHEIQP model for any arbitrary radial and angular momentum quantum numbers (nl) become:

$$M_{nc-nl}^{\text{hiq}\,p} = M_{nl}^{\text{hiq}\,p} + \langle\Lambda\rangle_{(nlms)}^{nr-\text{hiq}} \times$$

$$\times \begin{cases} (\sigma\aleph + \beta\Omega)\, m - \frac{\Theta}{3} & \text{for spin-1,} \\ (\sigma\aleph + \beta\Omega)\, m & \text{for spin-0.} \end{cases} \quad (81)$$

Thus, the spin-averaged mass spectra $M_{nl}^{\text{hiq}\,p}$ of HLM such as $c\bar{c}$, $b\bar{b}$, $b\bar{c}$, $b\bar{s}$, $c\bar{s}$, and $b\bar{q}$, $q = (u, d)$ in the HHEIQP model in usual NRQM symmetries are as follows:

$$M_{nc-nl}^{\text{hiq}\,p} = m_Q + m_{\bar{q}} - \frac{1}{2\mu b^2} \times \left[\frac{\left(n^2 + n + \frac{1}{2}\right) q + q\,(2n+1) \times \sqrt{(2l+1)^2 - 8A\mu} + ql\,(l+1) - 2q\mu\nu_1 b^2}{(2n+1)\, q + \sqrt{(2l+1)^2 - 8A\mu}}\right]^2. \quad (82)$$

Thus, we have $\delta M_{nc-nl}^{\text{hiq}\,p}$ in ENRQM symmetries:

$$\delta M_{nc-nl}^{\text{hiq}\,p} = M_{nc-nl}^{\text{hiq}\,p} - M_{nl}^{\text{hiq}\,p} =$$

$$= \langle Z\rangle_{(nlms)}^{qy-nr} \begin{cases} (\sigma\aleph + \beta\Omega)\, m - \frac{\Theta}{3} & \text{for spin 1,} \\ (\sigma\aleph + \beta\Omega)\, m & \text{for spin 0.} \end{cases} \quad (83)$$

This allows us to obtain the physical limit:

$$\lim_{(\Theta,\sigma,\beta)\rightarrow(0,0,0)} M_{nc-nl}^{\text{hiq}\,p} = M_{nl}^{\text{hiq}\,p}$$

to be achieved. It is worth to mention that, for the three simultaneous limits $(\Theta, \sigma, \beta) \rightarrow (0, 0, 0)$, we recover the energy equations for the spin and p-spin symmetries under the hyperbolic Hulthén plus hyperbolic exponential inversely quadratic potential.

5. Conclusions

In summary, this work presents an approximate analytic solution of the three-dimensional deformed Dirac equation with a new hyperbolic Hulthén plus hyperbolic exponential inversely quadratic potential without tensor interaction under pseudospin and spin symmetries with arbitrary spin-orbit coupling quantum numbers k. To do so, we have dealt with the centrifugal potential term using the Greene–Aldrich approximation. We have obtained new approximate bound-state energies that appear sensitive to the quantum numbers $(j, k, l, m, \widetilde{l}, \widetilde{m}, s, \widetilde{s})$, potential depths (n, ν_1, A) of the studied potentials, potential range, and noncommutativity parameters (Θ, σ, β) under the condition of spin and pseudospin symmetries. As we know, the improved Hulthén plus hyperbolic exponential inversely quadratic potential reduces the improved Hulthén potential and the improved Dirac exponential inversely quadratic potential. In this research, we also studied the nonrelativistic limit of the IHHEIQP model in ENRQM symmetries. The present results are applied for calculating the new mass spectra $M_{nc-nl}^{\text{hiq}\,p}$ of heavy-light mesons such as $c\overline{c}$, $b\overline{b}$, $b\overline{c}$, $b\overline{s}$, $c\overline{s}$, and $b\overline{q}$, $q = (u, d)$ under the effect of the IHHEIQP in ENRQM symmetries. The new mass spectra $M_{nc-nl}^{\text{hiq}\,p}$ of heavy-light mesons in ENCQM symmetries equal the corresponding values $M_{nl}^{\text{hiq}\,p}$ in ENRQM symmetries plus the contribution of noncommutativity $\delta M_{nc-nl}^{\text{hiq}\,p}$ which is an infinitesimal correction as compared with the main part $M_{nl}^{\text{hiq}\,p}$. It is worth mentioning that, for all cases, to make the three simultaneous limits $(\Theta, \sigma, \beta) \to \to (0, 0, 0)$, the ordinary physical quantities are covered in Ref. [4]. Finally, a feature of the noncommutative geometry on the 3D deformed Dirac equation with the improved Hulthén plus hyperbolic exponential inversely quadratic potential would be the presence of many physical phonemes which usually appear automatically, such as the spin-orbit and pseudospin-orbit, modified Zeeman effect, and others, and cause the behavior of topological properties of a space-space deformation. Our studied physical model in the current paper may be useful in investigating many interesting physical systems such as heavy-light mesons and can also include other applications.

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ВПЛИВ ДЕФОРМАЦІЇ ПРОСТОРІВ НА ВИСОКО- ТА НИЗЬКОЕНЕРГЕТИЧНІ СПЕКТРИ ФЕРМІОНІВ ТА СПЕКТРИ ВАЖКИХ КВАРКОНІЇВ ДЛЯ МОДИФІКОВАНОГО ПОТЕНЦІАЛУ ХЮЛЬТЕНА І ГІПЕРБОЛІЧНОГО ЕКСПОНЕНЦІАЛЬНО ОБРІЗАНОГО ОБЕРНЕНО КВАДРАТИЧНОГО ПОТЕНЦІАЛУ

Знайдено наближені розв'язки модифікованого рівняння Дірака для зв'язаних станів з урахуванням спінової або псевдоспінової симетрії. При цьому використано наближений вираз для відцентрового бар'єра у вигляді потенціала Хюльтена та гіперболічного експоненціально обрізаного обернено квадратичного потенціалу, метод зсуву Боппа та теорію збурень в узагальненій релятивістичній квантовій механіці. Розраховано спектри мас $M_{nc-nl}^{\mathrm{hiq}\,p}$ важких мезонів $c\bar{c}$, $b\bar{b}$, $b\bar{c}$, $b\bar{s}$, $c\bar{s}$ та $b\bar{q}$, $q = (u, d)$. Результати добре узгоджуються з даними інших робіт.

Ключові слова: некомутативні простори, рівняння Дірака, рівняння Шрьодінгера, потенціал Хюльтена з гіперболічним експоненціально обрізаним обернено квадратичним потенціалом, метод зсуву Боппа, мезони.