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DEVELOPMENT AND ANALYSIS OF NOVEL INTEGRABLE NONLINEAR DYNAMICAL SYSTEMS ON QUASI-ONE-DIMENSIONAL LATTICES. TWO-COMPONENT NONLINEAR SYSTEM WITH THE ON-SITE AND SPATIALLY DISTRIBUTED INERTIAL MASS PARAMETERS

The main principles of developing the evolutionary nonlinear integrable systems on quasi-one-dimensional lattices are formulated in clear mathematical and physical terms discarding the whimsical mathematical formulations and computer-addicted presentations. These basic principles are substantiated by the actual development of novel semi-discrete integrable nonlinear system, whose auxiliary spectral and evolutionary operators are given by 4×4 *square matrices. The procedure of reduction from the prototype nonlinear integrable system with twelve field functions to the physically meaningful nonlinear integrable system with four field functions is described in details prompted by our previous cumulative experience. The obtained ultimate semi-discrete nonlinear integrable system comprises the two subsystems of essentially distinct physical origins. Thus, the first subsystem is the subsystem of the Toda type. It is characterized by the on-site (spatially local) mass parameter and the positively defined elasticity coefficient. In contrast, the second subsystem is characterized by the spatially distributed mass parameters and the negatively defined elasticity coefficient responsible for the low-amplitude instability. We believe our scrupulous consideration of all main steps in developing the semi-discrete nonlinear integrable systems will be useful for the researchers unfamiliar with the numerous stumbling blocks inevitable in such an interesting and prospective scientific field as the theory of semi-discrete nonlinear integrable systems.*

Keywords: nonlinear dynamics, integrable system, quasi-one-dimensional lattice, stable displacements, unstable displacements.

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1. Introduction

The contribution of Krylov–Bogolyubov–Mytropolskyy school to the theoretical investigation of nonlinear dynamical problems concerning the diverse types of oscillations in a variety of mechanical, physical, and technical systems by the asymptotic analytic methods is widely acknowledged [1–4]. In turn, the numerical study of dynamical problems in the chains of nonlinearly coupled oscillators appears to be initiated by Fermi, Pasta, Ulam, and Tsingou [5]. Evidently, both these approximate approaches have been extremely valuable for the subsequent development of exactly integrable analytic dynamical models of nonlinear oscillators on quasi-one-dimensional lattices inspired by Toda [6–8].

In the present paper, we suggest the novel integrable two-component nonlinear system of stable and unstable displacement fields on a quasi-one-dimensional lattice. One of its subsystems is characterized by the spatially distributed inertial mass parameters shared between each pair of neighboring lattice sites. This rather unusual property gives rise to the unexpectedly curious set of semi-discrete (*i.e.* continuous in time variable τ and discrete in space variable n) nonlinear equations for the whole system. The preliminary analysis of the obtained essentially nondissipative semi-discrete equations shows that they seem to be treated beyond the standard dynamical procedure of Lagrangian or Hamiltonian formalism.

2. Basic Principles for Developing the Quasi-One-Dimensional Semi-Discrete Nonlinear Systems Integrable in the Lax Sense

Though the basic principles for developing the quasi-one-dimensional integrable semi-discrete nonlinear systems are widely exposed in the scientific literature [9–17], however, their concise formulation, supplemented by some important but usually undeservedly ignored observations, will be appropriate for scientists disliking the intricacies of chimerical mathematical notations.

The first step consists in inventing some square spectral matrix $L(n|z)$, whose matrix elements $L_{jk}(n|z)$ have to be chosen as some Laurent or Taylor polynomials with respect to time-independent spectral parameter z. The spectral-independent coefficients of such polynomials should be treated as the prototype field functions of continuous time variable τ and discrete space variable n in a future set of nonlinear evolutionary equations. For the sake of definiteness we assume the discrete space variable n to span over all integers from minus infinity to plus infinity.

The second step is to construct the proper evolution square matrix $A(n|z)$, whose matrix elements $A_{jk}(n|z)$, sought as Laurent or Taylor polynomials with respect to time-independent spectral parameter z, would be recoverable from the semi-discrete matrix-valued equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}L(n|z) = A(n+1|z)L(n|z) - L(n|z)A(n|z). \quad (2.1)$$

This equation is claimed to be the semi-discrete zero-curvature representation for the future successfully developed semi-discrete nonlinear system.

Provided both these two steps have been successfully overcome, and the spectrally independent parts of evolutionary matrix have been isolated, the same matrix-valued condition (2.1) yields additionally the set of nonlinear semi-discrete equations for the prototype field functions. Here, we would like to stress that the above described procedure is not so simple in practice, and it can be performed only in some lucky cases among numerous unsuccessful attempts, followed sometimes by the unexpected insight during a sleepless night.

The third step is to simplify or reduce the obtained prototype set of nonlinear semi-discrete equations to the nonlinear semi-discrete equations written in terms of physically meaningful field variables. Though this step is dictated by the scrupulous analysis of obtained prototype evolutionary equations, but, as a rule, it can be substantially facilitated by means of the universal local conservation law

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln[\det L(n|z)] = \mathrm{Sp}A(n+1|z) - \mathrm{Sp}A(n|z) \quad (2.2)$$

appearing as a simple contraction of system's zero-curvature representation (2.1).

Once the explicit uncontradictory forms of the spectral $L(n|z)$ and evolution $A(n|z)$ matrices have been found, the matrix-valued equation (2.1) automatically acquires the status of compatibility condition (referred to as the semi-discrete zero-curvature condition [10, 18]) between two auxiliary matrix-valued linear equations

$$X(n+1|z) = L(n|z)X(n|z), \quad (2.3)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}X(n|z) = A(n|z)X(n|z) \quad (2.4)$$

for the auxiliary matrix-function $X(n|z)$. The ranks of all three involved square matrices $L(n|z)$, $A(n|z)$ and $X(n|z)$ are assumed to be the same. The set of matrix-valued linear equations (2.3)–(2.4) itself is referred to as auxiliary linear problem for the developed semi-discrete nonlinear system. As a consequence, early claimed notion of semi-discrete matrix-valued equation (2.1) as the semi-discrete zero-curvature representation for the developed semi-discrete nonlinear system becomes truly justified.

The existence of system's zero-curvature representation (2.1) in combination with the respective auxiliary linear problem (2.3)–(2.4) establishes the integrability of a semi-discrete nonlinear system in the Lax sense. Moreover, as a rule, it gives the green light for the integrability of a semi-discrete nonlinear system in the Liouville sense [18].

In conclusion of this Section we would like to provide the detailed logical scheme leading from the auxiliary linear problem (2.3)–(2.4) to the zero-curvature condition (2.1). First of all, the set of auxiliary linear equations (2.3)–(2.4) is overdetermined. To achieve the compatibility of this overdetermined set, the operation of differentiation with respect to the time variable τ and the operation of shifting along the spatial variable n as applied to the auxiliary matrix-function $X(n|z)$ within the auxiliary linear set (2.3)–(2.4) must commutate, *i.e.* [9]

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau}X(m|z)\right]_{m=n+1} = \frac{\mathrm{d}}{\mathrm{d}\tau}X(n+1|z). \tag{2.5}$$

Then the zero-curvature condition (2.1) arises from the auxiliary linear set of linear equations (2.3)–(2.4) as the immediate consequence of such a commutative procedure.

3. Semi-Discrete Nonlinear Integrable System in Terms of Prototype Field Functions

As a particular successful realization of the roadmap, sketched in the previous second Section, we adopt the ansätze for the spectral $L(n|z)$ and evolution $A(n|z)$ operators to be in the forms of following 4×4 square matrices

$$L(n|z) = \begin{pmatrix} 0 & t_{12}(n) & s_{13}(n)z & 0 \\ t_{21}(n) & t_{22}(n) & u_{23}(n)z^{-1} & s_{24}(n)z \\ u_{31}(n)z^{-1} & s_{32}(n)z & t_{33}(n) & t_{34}(n) \\ 0 & u_{42}(n)z^{-1} & t_{43}(n) & 0 \end{pmatrix} \tag{3.1}$$

and

$$A(n|z) = \begin{pmatrix} 0 & b_{12}(n) & a_{13}(n)z & 0 \\ b_{21}(n) & b_{22}(n) & c_{23}(n)z^{-1} & a_{24}(n)z \\ c_{31}(n)z^{-1} & a_{32}(n)z & b_{33}(n) & b_{34}(n) \\ 0 & c_{42}(n)z^{-1} & b_{43}(n) & 0 \end{pmatrix}. \tag{3.2}$$

The direct substitution of these ansätze (3.1)–(3.2) into the zero-curvature equation (2.1) establishes the following ten relations:

$$a_{13}(n+1)\, s_{32}(n) = s_{13}(n)\, a_{32}(n), \tag{3.3}$$

$$s_{32}(n)\, a_{24}(n) = a_{32}(n+1)\, s_{24}(n), \tag{3.4}$$

$$b_{12}(n+1)\, u_{23}(n) = t_{12}(n)\, c_{23}(n), \tag{3.5}$$

$$s_{32}(n)\, b_{21}(n) = a_{32}(n+1)\, t_{21}(n), \tag{3.6}$$

$$u_{23}(n)\, b_{34}(n) = c_{23}(n+1)\, t_{34}(n), \tag{3.7}$$

$$b_{43}(n+1)\, s_{32}(n) = t_{43}(n)\, a_{32}(n), \tag{3.8}$$

$$u_{23}(n)\, c_{31}(n) = c_{23}(n+1)\, u_{31}(n), \tag{3.9}$$

$$c_{42}(n+1)\, u_{23}(n) = u_{42}(n)\, c_{23}(n), \tag{3.10}$$

$$a_{32}(n+1)\, u_{23}(n) = s_{32}(n)\, c_{23}(n), \tag{3.11}$$

$$c_{23}(n+1)\, s_{32}(n) = u_{23}(n)\, a_{32}(n) \tag{3.12}$$

between the ten spectrally independent ingredients $a_{13}(n)$, $a_{24}(n)$, $a_{32}(n)$, $b_{12}(n)$, $b_{21}(n)$, $b_{34}(n)$, $b_{43}(n)$, $c_{23}(n)$, $c_{31}(n)$, $c_{42}(n)$ of evolution matrix $A(n|z)$ and the ten prototype field functions $s_{13}(n)$, $s_{24}(n)$, $s_{32}(n)$, $t_{12}(n)$, $t_{21}(n)$, $t_{34}(n)$, $t_{43}(n)$, $u_{23}(n)$, $u_{31}(n)$, $u_{42}(n)$ of spectral matrix $L(n|z)$. The rest of martix elements $b_{22}(n)$ and $b_{33}(n)$, referred to as the sampling functions, remains to be free for the time being.

As for the promised semi-discrete nonlinear system, generated by the semi-discrete matrix-valued zero-curvature equation (2.1) on the suggested spectral and evolution matrices (3.1)–(3.2), its the most general (prototype) form is given by the set of twelve semi-discrete nonlinear equations

$$\dot{s}_{13}(n) = a_{13}(n+1)\, t_{33}(n) - s_{13}(n)\, b_{33}(n), \tag{3.13}$$

$$\dot{u}_{31}(n) = b_{33}(n+1)\, u_{31}(n) - t_{33}(n)\, c_{31}(n), \tag{3.14}$$

$$\dot{s}_{24}(n) = b_{22}(n+1)\, s_{24}(n) - t_{22}(n)\, a_{24}(n), \tag{3.15}$$

$$\dot{u}_{42}(n) = c_{42}(n+1)\, t_{22}(n) - u_{42}(n)\, b_{22}(n), \tag{3.16}$$

$$\dot{t}_{12}(n) = b_{12}(n+1)\, t_{22}(n) - t_{12}(n)\, b_{22}(n), \tag{3.17}$$

$$\dot{t}_{21}(n) = b_{22}(n+1)\,t_{21}(n) - t_{22}(n)\,b_{21}(n), \tag{3.18}$$

$$\dot{t}_{34}(n) = b_{33}(n+1)\,t_{34}(n) - t_{33}(n)\,b_{34}(n), \tag{3.19}$$

$$\dot{t}_{43}(n) = b_{43}(n+1)\,t_{33}(n) - t_{43}(n)\,b_{33}(n), \tag{3.20}$$

$$\dot{s}_{32}(n) = a_{32}(n+1)t_{22}(n) + b_{33}(n+1)\,s_{32}(n) - \\ - s_{32}(n)\,b_{22}(n) - t_{33}(n)\,a_{32}(n), \tag{3.21}$$

$$\dot{u}_{23}(n) = b_{22}(n+1)u_{23}(n) + c_{23}(n+1)\,t_{33}(n) - \\ - t_{22}(n)\,c_{23}(n) - u_{23}(n)\,b_{33}(n), \tag{3.22}$$

$$\dot{t}_{22}(n) = b_{21}(n+1)\,t_{12}(n) + b_{22}(n+1)\,t_{22}(n) + \\ + c_{23}(n+1)\,s_{32}(n) + a_{24}(n+1)\,u_{42}(n) - \\ - t_{21}(n)\,b_{12}(n) - t_{22}(n)\,b_{22}(n) - \\ - u_{23}(n)\,a_{32}(n) - s_{24}(n)\,c_{42}(n), \tag{3.23}$$

$$\dot{t}_{33}(n) = c_{31}(n+1)\,s_{13}(n) + a_{32}(n+1)\,u_{23}(n) + \\ + b_{33}(n+1)\,t_{33}(n) + b_{34}(n+1)\,t_{43}(n) - \\ - u_{31}(n)\,a_{13}(n) - s_{32}(n)\,c_{23}(n) - \\ - t_{33}(n)\,b_{33}(n) - t_{34}(n)\,b_{43}(n). \tag{3.24}$$

The overdot in each of above-written formulas (3.13)–(3.24) stands for the differentiation with respect to the time variable τ.

4. Amazing Metamorphoses of Prototype Field Functions

Having analyzed the subset of ten prototype semi-discrete nonlinear equations (3.13)–(3.22) with the use of two somewhat curious relations (3.11)–(3.12) between two ingredients $a_{32}(n)$, $c_{23}(n)$ of evolution matrix and two prototype functions $s_{32}(n)$, $u_{23}(n)$ we come to the following five shrunken equations:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln[s_{24}(n)\,u_{42}(n)] = b_{22}(n+1) - b_{22}(n), \tag{4.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln[u_{31}(n)\,s_{13}(n)] = b_{33}(n+1) - b_{33}(n), \tag{4.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln[t_{21}(n)\,t_{12}(n)] = b_{22}(n+1) - b_{22}(n), \tag{4.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln[t_{34}(n)\,t_{43}(n)] = b_{33}(n+1) - b_{33}(n), \tag{4.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln[s_{32}(n)\,u_{23}(n)] = \\ = b_{22}(n+1) + b_{33}(n+1) - b_{22}(n) - b_{33}(n). \tag{4.5}$$

In addition, the universal local conservation law (2.2) yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln[s_{13}(n)\,u_{42}(n) - t_{12}(n)\,t_{43}(n)] + \\ + \frac{\mathrm{d}}{\mathrm{d}\tau}\ln[s_{24}(n)\,u_{31}(n) - t_{34}(n)\,t_{21}(n)] + \\ = b_{22}(n+1) + b_{33}(n+1) - b_{22}(n) - b_{33}(n). \tag{4.6}$$

The analytic structures of six above-written equations (4.1)–(4.6) prompt us to fix the sampling functions $b_{22}(n)$ and $b_{33}(n)$ by eliminating a presumable spatial dependence in each of them. Thus, we have

$$b_{22}(n) = b_{22}, \tag{4.7}$$

$$b_{33}(n) = b_{33}. \tag{4.8}$$

As a consequence, the just mentioned shrunken semi-discrete equations (4.1)–(4.6) are transformed into six differential constraints upon the ten involved prototype functions. Precisely, the list of these constraints acquires the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[s_{24}(n)\,u_{42}(n)] = 0, \tag{4.9}$$

$$\frac{d}{d\tau}[u_{31}(n)\,s_{13}(n)] = 0, \tag{4.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[t_{21}(n)\,t_{12}(n)] = 0, \tag{4.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[t_{34}(n)\,t_{43}(n)] = 0, \tag{4.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[s_{32}(n)\,u_{23}(n)] = 0, \tag{4.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{s_{13}(n)\,t_{34}(n)\,u_{42}(n)\,t_{21}(n)}{t_{43}(n)\,u_{31}(n)\,t_{12}(n)\,s_{24}(n)}\right\} + \\ + \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{t_{43}(n)\,u_{31}(n)\,t_{12}(n)\,s_{24}(n)}{s_{13}(n)\,t_{34}(n)\,u_{42}(n)\,t_{21}(n)}\right\} = 0. \tag{4.14}$$

These constraints (4.9)–(4.14) are convertible into the sheer identities by means of the following parametrization formulas

$$s_{24}(n) = s_{24}\exp[+q_-(n)], \tag{4.15}$$

$$u_{42}(n) = u_{42}\exp[-q_-(n)], \tag{4.16}$$

$$u_{31}(n) = u_{31}\exp[-q_+(n)], \tag{4.17}$$

$$s_{13}(n) = s_{13}\exp[+q_+(n)] \tag{4.18}$$

$$t_{21}(n) = t_{21}\exp[-q_+(n) - r(n)], \tag{4.19}$$

$$t_{12}(n) = t_{12}\exp[+q_+(n) + r(n)], \tag{4.20}$$

$$t_{34}(n) = t_{34}\exp[+q_-(n) + r(n)], \tag{4.21}$$

$$t_{43}(n) = t_{43}\exp[-q_-(n) - r(n)], \tag{4.22}$$

$$s_{32}(n) = s_{32}\exp[+q_0(n)], \tag{4.23}$$

$$u_{23}(n) = u_{23}\exp[-q_0(n)]. \tag{4.24}$$

Here, each of the spatially independent parameters s_{24}, u_{42}, u_{31}, s_{13}, t_{21}, t_{12}, t_{34}, t_{43}, s_{32}, u_{23} are assumed to be time-independent without any loss of generality.

In order to reconcile the field function $q_0(n)$ with the two relationships (3.11)–(3.12) between the field functions $s_{32}(n)$, $u_{23}(n)$ and the spectrally independent ingredients $a_{32}(n)$, $c_{23}(n)$ of the evolution matrix, we are obliged to introduce the auxiliary field function $w(n|n-1)$ by means of the equality

$$q_0(n) = w(n+1|n) + w(n|n-1). \tag{4.25}$$

The explicit result of the announced reconcilement reads:

$$s_{32}(n) = s_{32}\exp[+w(n+1|n) + w(n|n-1)], \tag{4.26}$$

$$u_{23}(n) = u_{23}\exp[-w(n+1|n) - w(n|n-1)], \tag{4.27}$$

$$a_{32}(n) = \beta s_{32}\exp[+2w(n|n-1)], \tag{4.28}$$

$$c_{23}(n) = \beta u_{23}\exp[-2w(n|n-1)], \tag{4.29}$$

where β is a spatially independent parameter, which can be taken as an arbitrary function of time.

The quantities $q_-(n)$, $q_+(n)$, $r(n)$, $w(n+1|n)$, $t_{22}(n)$, $t_{33}(n)$ can be treated as the intermediately reduced field functions, inasmuch as some of them will be shown to undergo an unexpected additional reduction.

Now, relying upon the explicit formulas (4.28) and (4.29) for the spectrally independent ingredient functions $a_{32}(n)$ and $c_{23}(n)$ having been applied to the equations (3.3)–(3.10) for determining the rest of ingredient functions, we obtain

$$a_{13}(n+1) = \\ = \beta s_{13}\exp[-w(n+1|n) + w(n|n-1) + q_+(n)], \tag{4.30}$$

$$a_{24}(n) = \\ = \beta s_{24}\exp[+w(n+1|n) - w(n|n-1) + q_-(n)], \tag{4.31}$$

$$b_{12}(n+1) = \beta t_{12}\times \\ \times\exp[+w(n+1|n) - w(n|n-1) + q_+(n) + r(n)], \tag{4.32}$$

$$b_{21}(n) = \beta t_{21}\times \\ \times\exp[+w(n+1|n) - w(n|n-1) - q_+(n) - r(n)], \tag{4.33}$$

$$b_{34}(n) = \beta t_{34}\times \\ \times\exp[-w(n+1|n) + w(n|n-1) + q_-(n) + r(n)], \tag{4.34}$$

$$b_{43}(n+1) = \beta t_{43}\times \\ \times\exp[-w(n+1|n) + w(n|n-1) - q_-(n) - r(n)], \tag{4.35}$$

$$c_{31}(n) = \\ = \beta u_{31}\exp[-w(n+1|n) + w(n|n-1) - q_+(n)], \tag{4.36}$$

$$c_{42}(n+1) = \\ = \beta u_{42}\exp[+w(n+1|n) - w(n|n-1) - q_-(n)]. \tag{4.37}$$

5. Semi-Discrete Nonlinear Integrable System in Terms of Intermediately Reduced Field Functions

The elementary manipulations with the set of prototype semi-discrete nonlinear equations (3.13)–(3.24) supported by the parametrization formulas (4.15)–(4.22), (4.26)–(4.27), and (4.28)–(4.37) for the spectrally independent parts of the spectral and evolution matrices give rise to the following set of semi-discrete nonlinear integrable equations in terms of intermediately reduced field functions:

$$\dot{q}_-(n) = b_{22} - \beta t_{22}(n)\,\exp\left[+w(n+1|n) - w(n|n-1)\right], \tag{5.1}$$

$$\dot{q}_+(n) = \beta t_{33}(n)\,\exp\left[-w(n+1|n) + w(n|n-1)\right] - b_{33}, \tag{5.2}$$

$$\dot{r}(n) = b_{33} - b_{22} + \\ + \beta t_{22}(n)\exp[+w(n+1|n) - w(n|n-1)] - \\ - \beta t_{33}(n)\exp[-w(n+1|n) + w(n|n-1)], \tag{5.3}$$

$$\dot{w}(n+1|n) + \dot{w}(n|n-1) = b_{33} - b_{22} + \\ + \beta t_{22}(n)\exp[+w(n+1|n) - w(n|n-1)] - \\ - \beta t_{33}(n)\exp[-w(n+1|n) + w(n|n-1)], \tag{5.4}$$

$$\dot{t}_{22}(n) = \beta t_{21} t_{12} \exp[+w(n+2|n+1) - w(n+1|n)] \times$$
$$\times \exp[-q_+(n+1) + q_+(n) - r(n+1) + r(n)] +$$
$$+ \beta s_{24} u_{42} \exp[+w(n+2|n+1) - w(n+1|n)] \times$$
$$\times \exp[+q_-(n+1) - q_-(n)] -$$
$$- \beta t_{21} t_{12} \exp[-w(n-1|n-2) + w(n|n-1)] \times$$
$$\times \exp[+q_+(n-1) - q_+(n) + r(n-1) - r(n)] -$$
$$- \beta s_{24} u_{42} \exp[-w(n-1|n-2) + w(n|n-1)] \times$$
$$\times \exp[-q_-(n-1) + q_-(n)], \tag{5.5}$$
$$\dot{t}_{33}(n) = \beta t_{34} t_{43} \exp[-w(n+2|n+1) + w(n+1|n)] \times$$
$$\times \exp[+q_-(n+1) - q_-(n) + r(n+1) - r(n)] +$$
$$+ \beta u_{31} s_{13} \exp[-w(n+2|n+1) + w(n+1|n)] \times$$
$$\times \exp[-q_+(n+1) + q_+(n)] -$$
$$- \beta t_{34} t_{43} \exp[+w(n-1|n-2) - w(n|n-1)] \times$$
$$\times \exp[-q_-(n-1) + q_-(n) - r(n-1) + r(n)] -$$
$$- \beta u_{31} s_{13} \exp[+w(n-1|n-2) - w(n|n-1)] \times$$
$$\times \exp[+q_+(n-1) - q_+(n)]. \tag{5.6}$$

This set of equations admits the shifted set of some field functions given by the expressions

$$Q_-(n) = q_-(n) - B_{22}, \tag{5.7}$$
$$Q_+(n) = q_+(n) + B_{33}, \tag{5.8}$$
$$R(n) = r(n) + B_{22} - B_{33}, \tag{5.9}$$
$$2W(n+1|n) = 2w(n+1|n) + B_{22} - B_{33}, \tag{5.10}$$
$$2W(n|n-1) = 2w(n|n-1) + B_{22} - B_{33}, \tag{5.11}$$

where

$$\dot{B}_{22} = b_{22}, \tag{5.12}$$
$$\dot{B}_{33} = b_{33}. \tag{5.13}$$

The above shift procedure prompts us to eliminate the parameters b_{22} and b_{33} without the loss of generality. Thus, we safely assume

$$b_{22} = 0 = b_{33} \tag{5.14}$$

in the first four equations (5.1)–(5.4) of our semi-discrete nonlinear system (5.1)–(5.6).

On the other hand, the time-independent preexponential parameters s_{32} and u_{23} are seen to be canceled from the intermediate set of semi-discrete nonlinear equations (5.1)–(5.6). Thus, without the loss of generality, we equalize each of them to unity

$$s_{32} = 1 = u_{23}. \tag{5.15}$$

Moreover, it is possible to introduce the rescaled time variable $\mathcal{T}$ by means of differential equality

$$\mathrm{d}\mathcal{T} = \beta \mathrm{d}\tau. \tag{5.16}$$

This observation allows us to specify the parameter β by the simple equality

$$\beta = 1 \tag{5.17}$$

in each of equations related to the obtained semi-discrete nonlinear system (5.1)–(5.6).

6. Ultimate Reduction of Field Functions

The first four equations (5.1)–(5.4) of the intermediate nonlinear system (5.1)–(5.6) are seen to be mutually dependent, inasmuch as they maintain the following two differential constraints

$$\dot{q}_+(n) + \dot{q}_-(n) = -\dot{w}(n+1|n) - \dot{w}(n|n-1), \tag{6.1}$$
$$\dot{r}(n) = \dot{w}(n+1|n) + \dot{w}(n|n-1). \tag{6.2}$$

As a consequence, we have the strong reasons to reduce the number of field functions by means of substitutions

$$q_-(n) = +q(n) -$$
$$- w(n+1|n)\sin^2(\alpha_+) - w(n|n-1)\cos^2(\alpha_-), \tag{6.3}$$
$$q_+(n) = -q(n) -$$
$$- w(n+1|n)\cos^2(\alpha_+) - w(n|n-1)\sin^2(\alpha_-), \tag{6.4}$$
$$r(n) = w(n+1|n) + w(n|n-1), \tag{6.5}$$

where the parameters α_+ and α_- are assumed to be time-independent.

Thus, instead of twelve original prototype field functions, announced in the expression (3.1) for the spectral matrix $L(n|z)$, we come only to the four ultimate field functions $q(n)$, $w(n+1|n)$ and $t_{22}(n)$,

$t_{33}(n)$. For the stylistic reasons it is appropriate to rename the last two functions by formulas

$$t_{22}(n) = -p_-(n), \tag{6.6}$$

$$t_{33}(n) = -p_+(n). \tag{6.7}$$

Having combined the final set of semi-discrete nonlinear integrable equations in terms of four ultimately reduced field functions $q(n)$, $w(n+1|n)$ and $p_-(n)$, $p_+(n)$ we surprisingly revealed that time-independent parameters s_{24}, u_{42}, u_{31}, s_{13}, t_{21}, t_{12}, t_{34}, t_{43} manifest themselves only in the form of two combinations

$$\Omega_-^2 = -t_{21}t_{12} - s_{24}u_{42}, \tag{6.8}$$

$$\Omega_+^2 = -t_{34}t_{43} - u_{31}s_{13}. \tag{6.9}$$

In view of these observations, it is reasonable to introduce the following parametrization formulas

$$t_{21} = +\Omega_- \sin(\varphi), \tag{6.10}$$

$$t_{12} = -\Omega_- \sin(\varphi), \tag{6.11}$$

$$s_{24} = +\Omega_- \cos(\varphi), \tag{6.12}$$

$$u_{42} = -\Omega_- \cos(\varphi), \tag{6.13}$$

$$t_{34} = +\Omega_+ \sin(\varphi), \tag{6.14}$$

$$t_{43} = -\Omega_+ \sin(\varphi), \tag{6.15}$$

$$u_{31} = -\Omega_+ \cos(\varphi), \tag{6.16}$$

$$s_{13} = +\Omega_+ \cos(\varphi). \tag{6.17}$$

Due to the above parametrization formulas (6.10)–(6.17) and the early written parametrization formulas (4.15)–(4.22), the expression for the determinant $\det L(n|z)$ of the spectral matrix (3.1) acquires the very simple form

$$\det L(n|z) = \Omega_+^2 \Omega_-^2. \tag{6.18}$$

7. Semi-Discrete Nonlinear Integrable System in Terms of Ultimately Reduced Field Functions

Now we are ready to write down our semi-discrete nonlinear integrable system in terms of ultimately reduced field functions

$$\dot{q}(n) - \dot{w}(n+1|n)\sin^2(\alpha_+) - \dot{w}(n|n-1)\cos^2(\alpha_-) =$$
$$= p_-(n)\exp[+w(n+1|n) - w(n|n-1)], \tag{7.1}$$

$$\dot{q}(n) + \dot{w}(n+1|n)\cos^2(\alpha_+) + \dot{w}(n|n-1)\sin^2(\alpha_-) =$$
$$= p_+(n)\exp[-w(n+1|n) + w(n|n-1)], \tag{7.2}$$

$$\dot{p}_-(n)\exp[+w(n+1|n) - w(n|n-1|n)] =$$
$$= \Omega_-^2 \exp[+q(n+1) - q(n)] \times$$
$$\times \exp\{+[w(n+2|n+1) - w(n+1|n)]\cos^2(\alpha_+)\} \times$$
$$\times \exp\{+[w(n+1|n) - w(n|n-1)]\sin^2(\alpha_-)\} -$$
$$- \Omega_-^2 \exp[+q(n) - q(n-1)] \times$$
$$\times \exp\{+[w(n+1|n) - w(n|n-1)]\cos^2(\alpha_+)\} \times$$
$$\times \exp\{+[w(n|n-1) - w(n-1|n-2)]\sin^2(\alpha_-)\}, \tag{7.3}$$

$$\dot{p}_+(n)\exp[-w(n+1|n) + w(n|n-1|n)] =$$
$$= \Omega_+^2 \exp[+q(n+1) - q(n)] \times$$
$$\times \exp\{-[w(n+2|n+1) - w(n+1|n)]\sin^2(\alpha_+)\} \times$$
$$\times \exp\{-[w(n+1|n) + w(n|n-1)]\cos^2(\alpha_-)\} -$$
$$- \Omega_+^2 \exp[+q(n) - q(n-1)] \times$$
$$\times \exp\{-[w(n+1|n) - w(n|n-1)]\sin^2(\alpha_+)\} \times$$
$$\times \exp\{-[w(n|n-1) - w(n-1|n-2)]\cos^2(\alpha_-)\}. \tag{7.4}$$

Alternatively, we are able to rewrite this first order in time set of four equations (7.1)–(7.4) as the second order in time set of just two equations and thus to exclude the field functions $p_-(n)$ and $p_+(n)$ from further consideration. Then, the alternative form of our semi-discrete nonlinear system is formalized by the equations

$$\ddot{q}(n) - \ddot{w}(n+1|n)\sin^2(\alpha_+) - \ddot{w}(n|n-1)\cos^2(\alpha_-) -$$
$$- [\dot{q}(n) - \dot{w}(n+1|n)\sin^2(\alpha_+) - \dot{w}(n|n-1)\cos^2(\alpha_-)] \times$$
$$\times [+\dot{w}(n+1|n) - \dot{w}(n|n-1)] =$$
$$= \Omega_-^2 \exp[+q(n+1) - q(n)] \times$$
$$\times \exp\{+[w(n+2|n+1) - w(n+1|n)]\cos^2(\alpha_+)\} \times$$
$$\times \exp\{+[w(n+1|n) - w(n|n-1)]\sin^2(\alpha_-)\} -$$
$$- \Omega_-^2 \exp[+q(n) - q(n-1)] \times$$
$$\times \exp\{+[w(n+1|n) - w(n|n-1)]\cos^2(\alpha_+)\} \times$$
$$\times \exp\{+[w(n|n-1) - w(n-1|n-2)]\sin^2(\alpha_-)\}, \tag{7.5}$$

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$$\ddot{q}(n) + \ddot{w}(n+1|n)\cos^2(\alpha_+) + \ddot{w}(n|n-1)\sin^2(\alpha_-) +$$
$$+ [\dot{q}(n)+\dot{w}(n+1|n)\cos^2(\alpha_+)+\dot{w}(n|n-1)\sin^2(\alpha_-)] \times$$
$$\times [+\dot{w}(n+1|n) - \dot{w}(n|n-1)] =$$
$$= \Omega_+^2 \exp[+q(n+1) - q(n)] \times$$
$$\times \exp\bigl\{-[w(n+2|n+1) - w(n+1|n)]\sin^2(\alpha_+)\bigr\} \times$$
$$\times \exp\bigl\{-[w(n+1|n) - w(n|n-1)]\cos^2(\alpha_-)\bigr\} -$$
$$- \Omega_+^2 \exp[+q(n) - q(n-1)] \times$$
$$\times \exp\bigl\{-[w(n+1|n) - w(n|n-1)]\sin^2(\alpha_+)\bigr\} \times$$
$$\times \exp\bigl\{-[w(n|n-1) - w(n-1|n-2)]\cos^2(\alpha_-)\bigr\}. \quad (7.6)$$

Of course, it would be interesting to press these rather bulky equations (7.5)–(7.6) into a Procrustean bed of concise Lagrangian or Hamiltonian formulation. However, the solution of this problem is not so simple. Maybe it can be facilitated by the explicit extraction of some physically important local conservation laws within the recurrence technique well developed for the semi-discrete nonlinear integrable systems [11, 19]. Nevertheless, sometimes even the knowledge of basic local conservation laws does not guarantee the routes to the system's exact Hamiltonian presentation in physically meaningful terms [20, 21].

The unbiased estimation of the proposed semi-discrete nonlinear system (7.5)–(7.6) shows that it consists of two nonlinearly coupled subsystems. The subsystem associated with the field function $q(n)$ is the vibrational subsystem of the Toda type [6–8] inasmuch as it is characterized by the positively defined elasticity coefficients Ω_-^2 and Ω_+^2. As for the subsystem associated with the field function $w(n)$, it is characterized by the negatively defined elasticity coefficients $-\Omega_-^2$ and $-\Omega_+^2$. Therefore, the latter subsystem can not be treated as the vibrational one. Moreover, this anti-vibrational subsystem strictly manifests itself as the subsystem with the spatially distributed inertial mass parameters shared between each pair of neighboring lattice sites.

This preliminary naïve consideration finds its partially modified confirmation in the standard low-amplitude (linear) analysis of the system under study (7.5)–(7.6). Indeed, the respective dispersion relation between the eigenfrequency ω and the wave vector $\varkappa$

$$\omega^4 - 16\,\Omega_+^2\Omega_-^2 \sin^4(\varkappa/2) = 0 \quad (7.7)$$

clarifies, that only the part

$$\omega^2 - 4|\Omega_+\Omega_-|\sin^2(\varkappa/2) = 0 \quad (7.8)$$

is responsible for the true vibrations with the real valued eigenfrequency $\omega(\varkappa) = 2\sqrt{|\Omega_+\Omega_-|}\,|\sin(\varkappa/2)|$. Conversely, the part

$$\omega^2 + 4|\Omega_+\Omega_-|\sin^2(\varkappa/2) = 0 \quad (7.9)$$

indicates on a sort of instability detectable at least in the system's low-amplitude (linear) dynamical regime.

8. Complete List of Ultimate Reduction Formulas for the Functional Components of Spectral and Evolution Matrices

For the convenience of future applications we present the complete list of ultimate reduction formulas for the functional components of spectral and evolution matrices.

Thus, the list of reduction formulas specifying the spectral matrix $L(n|z)$ (3.1) reads as follows

$$s_{24}(n) = +\Omega_- \cos(\varphi)\exp[+q(n)] \times$$
$$\times \exp\left[-w(n+1|n)\sin^2(\alpha_+)\right] \times$$
$$\times \exp\left[-w(n|n-1)\cos^2(\alpha_-)\right], \quad (8.1)$$
$$u_{42}(n) = -\Omega_- \cos(\varphi)\exp[-q(n)] \times$$
$$\times \exp\left[+w(n+1|n)\sin^2(\alpha_+)\right] \times$$
$$\times \exp\left[+w(n|n-1)\cos^2(\alpha_-)\right], \quad (8.2)$$
$$u_{31}(n) = -\Omega_+ \cos(\varphi)\exp[+q(n)] \times$$
$$\times \exp\left[+w(n+1|n)\cos^2(\alpha_+)\right] \times$$
$$\times \exp\left[+w(n|n-1)\sin^2(\alpha_-)\right], \quad (8.3)$$
$$s_{13}(n) = +\Omega_+ \cos(\varphi)\exp[-q(n)] \times$$
$$\times \exp\left[-w(n+1|n)\cos^2(\alpha_+)\right] \times$$
$$\times \exp\left[-w(n|n-1)\sin^2(\alpha_-)\right], \quad (8.4)$$
$$t_{21}(n) = +\Omega_- \sin(\varphi)\exp[+q(n)] \times$$
$$\times \exp\left[-w(n+1|n)\sin^2(\alpha_+)\right] \times$$
$$\times \exp\left[-w(n|n-1)\cos^2(\alpha_-)\right], \quad (8.5)$$
$$t_{12}(n) = -\Omega_- \sin(\varphi)\exp[-q(n)] \times$$
$$\times \exp\left[+w(n+1|n)\sin^2(\alpha_+)\right] \times$$
$$\times \exp\left[+w(n|n-1)\cos^2(\alpha_-)\right], \quad (8.6)$$

$$t_{34}(n) = +\Omega_+ \sin(\varphi)\exp[+q(n)] \times$$
$$\times \exp\left[+w(n+1|n)\cos^2(\alpha_+)\right] \times$$
$$\times \exp\left[+w(n|n-1)\sin^2(\alpha_-)\right], \qquad (8.7)$$

$$t_{43}(n) = -\Omega_+ \sin(\varphi)\exp[-q(n)] \times$$
$$\times \exp\left[-w(n+1|n)\cos^2(\alpha_+)\right] \times$$
$$\times \exp\left[-w(n|n-1)\sin^2(\alpha_-)\right], \qquad (8.8)$$

$$s_{32}(n) = \exp[+w(n+1|n) + w(n|n-1)], \qquad (8.9)$$

$$u_{23}(n) = \exp[-w(n+1|n) - w(n|n-1)], \qquad (8.10)$$

$$t_{22}(n) = -p_-(n) \qquad (8.11)$$

$$t_{33}(n) = -p_+(n). \qquad (8.12)$$

The list of reduction formulas specifying the evolution matrix $A(n|z)$ (3.2) is given below

$$a_{24}(n) = +\Omega_- \cos(\varphi)\exp[+q(n)] \times$$
$$\times \exp\left[+w(n+1|n)\cos^2(\alpha_+)\right] \times$$
$$\times \exp\left\{-w(n|n-1)\left[1+\cos^2(\alpha_-)\right]\right\}, \qquad (8.13)$$

$$c_{42}(n+1) = -\Omega_- \cos(\varphi)\exp[-q(n)] \times$$
$$\times \exp\left\{+w(n+1|n)\left[1+\sin^2(\alpha_+)\right]\right\} \times$$
$$\times \exp\left[-w(n|n-1)\sin^2(\alpha_-)\right], \qquad (8.14)$$

$$c_{31}(n) = -\Omega_+ \cos(\varphi)\exp[+q(n)] \times$$
$$\times \exp\left[-w(n+1|n)\sin^2(\alpha_+)\right] \times$$
$$\times \exp\left\{+w(n|n-1)\left[1+\sin^2(\alpha_-)\right]\right\}, \qquad (8.15)$$

$$a_{13}(n+1 = +\Omega_+ \cos(\varphi)\exp[-q(n)] \times$$
$$\times \exp\left\{-w(n+1|n)\left[1+\cos^2(\alpha_+)\right]\right\} \times$$
$$\times \exp\left[+w(n|n-1)\cos^2(\alpha_-)\right], \qquad (8.16)$$

$$b_{21}(n) = +\Omega_- \sin(\varphi)\exp[+q(n)] \times$$
$$\times \exp\left[+w(n+1|n)\cos^2(\alpha_+)\right] \times$$
$$\times \exp\left\{-w(n|n-1)\left[1+\cos^2(\alpha_-)\right]\right\}, \qquad (8.17)$$

$$b_{12}(n+1) = -\Omega_- \sin(\varphi)\exp[-q(n)] \times$$
$$\times \exp\left\{+w(n+1|n)\left[1+\sin^2(\alpha_+)\right]\right\} \times$$
$$\times \exp\left[-w(n|n-1)\sin^2(\alpha_-)\right], \qquad (8.18)$$

$$b_{34}(n) = +\Omega_+ \sin(\varphi)\exp[+q(n)] \times$$
$$\times \exp\left[-w(n+1|n)\sin^2(\alpha_+)\right] \times$$
$$\times \exp\left\{+w(n|n-1)\left[1+\sin^2(\alpha_-)\right]\right\}, \qquad (8.19)$$

$$b_{43}(n+1) = -\Omega_+ \sin(\varphi)\exp[-q(n)] \times$$
$$\times \exp\left\{-w(n+1|n)\left[1+\cos^2(\alpha_+)\right]\right\} \times$$
$$\times \exp\left[+w(n|n-1)\cos^2(\alpha_-)\right], \qquad (8.20)$$

$$a_{32}(n) = \exp[+2w(n|n-1)], \qquad (8.21)$$

$$c_{23}(n) = \exp[-2w(n|n-1)], \qquad (8.22)$$

$$b_{22} = 0, \qquad (8.23)$$

$$b_{33} = 0. \qquad (8.24)$$

Although the arbitrary constant parameter φ is presented almost in each of above written reduction formulas (8.1)–(8.24), however it is absent in each of the two permissible incarnations (7.1)–(7.4) and (7.5)–(7.6) of developed semi-discrete nonlinear integrable system. This remarkable fact allows us to simplify drastically the ultimate expressions for the spectral $L(n|z)$ and evolution $A(n|z)$ matrices by either of the two appropriate choices

$$\sin(\varphi) = 0, \qquad (8.25)$$

or

$$\cos(\varphi) = 0 \qquad (8.26)$$

for the most suitable fixation of adjusting parameter φ.

9. Conclusion

The main objective of our present research was to formulate the basic principles for the development of semi-discrete nonlinear integrable systems and to comprehensively illustrate them starting with a particular properly invented spectral and evolution matrices. Following these basic principles, we have shown how the prototype semi-discrete nonlinear integrable system, characterized by the twelve field functions, is reduced to the offspring semi-discrete nonlinear integrable system, characterized only by the four physically meaningful field functions.

The obtained system comprises the two coupled nonlinear quasi-one-dimensional evolutionary subsystems of vibrational and anti-vibrational origins. The vibrational subsystem is the subsystem of the pronounced Toda type, and it is characterized by the usual on-site (spatially local) mass parameter and positively defined elasticity coefficient. In contrast, the anti-vibrational subsystem is characterized by the spatially distributed mass parameters and negatively defined elasticity coefficient.

Due to its Lax integrability, the suggested semi-discrete nonlinear system permits the exact analytic solutions, which can be obtained in the framework of modern mathematical methods such as the method of inverse scattering transform [9, 11–14] and the method of Darboux–Bäcklund transformation [15–17]. On the other hand, the Lax integrability opens the door for the selection of physically motivated local conservation laws among the infinite hierarchy of local conservation laws typical of the integrable nonlinear evolutionary systems on quasi-one-dimensional lattices [11, 19].

The forthcoming investigation of our nonlinear dynamical system appears to give some new insight onto the very important and interesting permanent problems of the applied nonlinear dynamics [22, 23].

We think, that a class of low-dimensional semi-discrete nonlinear models dealing with the semi-analytic description of traffic control problems [24–26] deserves to be enriched by the strict analytic approaches too.

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1. N.M. Krylov, N.N. Bogoliubov. *Introduction to Non-Linear Mechanics* (Princeton University Press, 1950) [ISBN: 9780691079851].
2. N.N. Bogoliubov, Y.A. Mitropolsky. *Asymptotic Methods in the Theory of Non-Linear Oscillations* (Gordon and Breach Science Publishers, 1961) [ISBN-10: 067720051X].
3. Yu.A. Mitropolskii, Nguen Van Dao. *Applied Asymptotic Methods in Nonlinear Oscillations* (Springer Science + Business Media, 1997).
4. A.R.E. Oliveira. History of Krylov–Bogoliubov–Mitropolsky methods of nonlinear oscillations. *Adv. Histor. Stud.* **6**, 40 (2017).
5. E. Fermi, P. Pasta, S. Ulam, M. Tsingou. Studies of the nonlinear problems. I, *Los Alamos Report* **LA-1940**, 1 (1955).
6. M. Toda. Vibration of a chain with nonlinear interaction. *J. Phys. Soc. Japan* **22** (2), 431 (1967).
7. M. Toda. Wave propagation in anharmonic lattices. *J. Phys. Soc. Japan* **23** (3), 501 (1967).
8. M. Toda. Studies of a non-linear lattice. *Phys. Rep.* **18** (1), 1 (1975).
9. M.J. Ablowitz. Lectures on the inverse scattering transform. *Stud. Appl. Math.* **58** (1), 17 (1978).
10. L.D. Faddeev, L.A. Takhtajan. *Hamiltonian Methods in the Theory of Solitons* (Springer-Verlag, 1987).
11. T. Tsuchida, H. Ujino, M. Wadati. Integrable semi-discretization of the coupled nonlinear Schrödinger equations. *J. Phys. A: Math. Gen.* **32** (11), 2239 (1999).
12. O.O. Vakhnenko. Nonlinear beating excitations on ladder lattice. *J. Phys. A: Math. Gen.* **32** (30), 5735 (1999).
13. O.O. Vakhnenko, M.J. Velgakis. Transverse and longitudinal dynamics of nonlinear intramolecular excitations on multileg ladder lattices. *Phys. Rev. E* **61** (6), 7110 (2000).
14. M.J. Ablowitz, B. Prinari, A.D. Trubatch. *Discrete and Continuous Nonlinear Schrödinger Systems* (Cambridge University Press, 2004).
15. O.O. Vakhnenko. Integrable nonlinear Schrödinger system on a lattice with three structural elements in the unit cell. *J. Math. Phys.* **59** (5), 053504 (2018).
16. O.O. Vakhnenko, A.P. Verchenko. Nonlinear system of $\mathcal{PT}$-symmetric excitations and Toda vibrations integrable by the Darboux–Bäcklund dressing method. *Proc. R. Soc. A* **477** (2256), 20210562 (2021).
17. O.O. Vakhnenko, A.P. Verchenko. Dipole–monopole alternative in nonlinear dynamics of an integrable gauge-coupled exciton-phonon system on a one-dimensional lattice. *Eur. Phys. J. Plus* **137** (10), 1176 (2022).
18. G.-Z. Tu. A trace identity and its applications to the theory of discrete integrable systems. *J. Phys. A: Math. Gen.* **23** (17), 3903 (1990).
19. O.O. Vakhnenko. Semidiscrete integrable nonlinear systems generated by the new fourth-order spectral operator. Local conservation laws. *J. Nonlin. Math. Phys.* **18** (3), 401 (2011).
20. O.O. Vakhnenko, M.J. Velgakis. Multimode soliton dynamics in pertrubed ladder lattices. *Phys. Rev. E* **63** (1), 016612 (2001).
21. O.O. Vakhnenko. Enigma of probability amplitudes in Hamiltonian formulation of integrable semidiscrete nonlinear Schrödinger systems. *Phys. Rev. E* **77**(2), 026604 (2008).
22. S. Timoshenko. *Vibration Problems in Engineering* (D. Van Nostrand Company, Inc., 1937).
23. V.F. Nesterenko. *Dynamics of Heterogeneous Materials* (Springer-Verlag, 2001).

24. Yu.B. Gaididei, C. Gorria, R. Berkemer, A. Kawamoto, T. Shiga, P.L. Christiansen, M.P. Sørensen, J. Starke. Controlling traffic jams by time modulating the safety distance. *Phys. Rev. E* **88** (4), 042803 (2013).
25. Yu.B. Gaididei, P.L. Christiansen, M.P. Sørensen, J.J. Rasmussen. Analytical solutions of pattern formation for a class of discrete Aw–Rascle–Zhang traffic models. *Commun. Nonlin. Sci. Numer. Simul.* **73**, 391 (2019).
26. Yu.B. Gaididei, C. Marschler, M.P. Sørensen, P.L. Christiansen, J.J. Rasmussen. Pattern formation in flows of asymmetrically interacting particles: Peristaltic pedestrian dynamics as a case study. *Evolution Equations and Control Theory* **8** (1), 73 (2019).

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ПОБУДОВА ТА АНАЛІЗ НОВИХ ІНТЕҐРОВНИХ НЕЛІНІЙНИХ ДИНАМІЧНИХ СИСТЕМ НА КВАЗИОДНОВИМІРНИХ ҐРАТКАХ. ДВОКОМПОНЕНТНА НЕЛІНІЙНА СИСТЕМА З ПРОСТОРОВО ЛОКАЛЬНИМИ ТА ПРОСТОРОВО РОЗПОДІЛЕНИМИ ІНЕРЦІЙНИМИ МАСОВИМИ ПАРАМЕТРАМИ

Основні принципи побудови інтеґровних еволюційних нелінійних систем на квазиодновимірних ґратках подано в ясних математичних та фізичних термінах на противагу до зарозумілих математичних формулювань та комп'ютерно заангажованих викладів. Ці базові принципи підкріплено реальною побудовою нової напівдискретної нелінійної інтеґровної системи, чиї допоміжні спектральний та еволюційний оператори задано 4×4 квадратовими матрицями. Спираючись на наш сукупний попередній досвід, ми детально описуємо процедуру редукування прототипної нелінійної інтеґровної системи з дванадцятьма польовими функціями до фізично осмисленої нелінійної інтеґровної системи з чотирма польовими функціями. Одержана кінцева напівдискретна нелінійна інтеґровна система містить у собі дві підсистеми суттєво відмінного фізичного змісту. Так, одна з підсистем є підсистемою Тодівського типу. Вона характеризується локальним масовим параметром і позитивно визначеним коєфіцієнтом пружности. Навпаки, інша підсистема характеризується просторово розподіленими масовими параметрами і негативно визначеним коєфіцієнтом пружности. Ми вважаємо, що наш ретельний розгляд усіх основних кроків побудови напівдискретних нелінійних інтеґровних систем стане в пригоді для дослідників, необізнаних з численними перепонами, неминучими в такій цікавій і перспективній царині науки як теорія напівдискретних нелінійних інтеґровних систем.

Ключові слова: нелінійна динаміка, інтеґровна система, квазиодновимірна ґратка, стабільні зміщення, нестабільні зміщення.