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DECOMPOSITION OF ELECTROMAGNETIC POTENTIALS IN PARTIAL FUNCTIONS OF DISPERSIVE ELECTRODYNAMIC LINES

The utilization of partial functions, or oscillets, as the basis functions localized in all spatial coordinates, is proposed for the expansion of non-stationary, non-harmonic electromagnetic potentials within lengthy three-dimensional dispersive electrodynamic systems, such as electrodynamic lines (ELs). These functions are derived as linear transformations of the manifold of EL eigenfunctions, aiming to minimize the spatial extension of each oscillet. Emphasis is placed on the adoption of these new functions in electrodynamic and electronic computations, particularly in the optimization of irregular ELs found in various microwave and optical sources, including those with open-ended configurations featuring a continuous spectrum of eigenfunctions. An illustrative example showing the utility of partial functions in the electrodynamic calculation of a longitudinally inhomogeneous EL is provided.

K e y w o r d s: dispersive electrodynamic system, electromagnetic potential, Fourier series, eigenfunction, partial function.

1. Introduction

There exists a persistent demand for the development of powerful and highly efficient microwave sources capable of operating within the terahertz and subterahertz (up to infrared) frequency bands [1]. Such sources are crucial for great number of essential technical and scientific applications, including:

• heating and diagnosing plasma in next-generation controlled thermonuclear fusion facilities;

• particle acceleration;

• rapid terahertz spectroscopy and magnetometry;

• experimental investigation of and control over various media, including biological samples;

• utilizing high-power sub-terahertz radiation pulses as undulators for short-wavelength free electron lasers.

These applications span across diverse fields, encompassing plasma physics (e.g., the development of intense ultraviolet radiation sources through terahertz discharge in plasma), space communications, and beyond. Notably, in the 1970s and 1980s, the developers made significant strides in enhancing the efficiency of traveling wave tubes (TWTs) and drift space klystrons, thereby bestowing, upon them, a decisive advantage over crossed-field devices (CFDs). The key factor underlying this progress lay in the optimization of the process of interaction between waves and beams along the longitudinal direction.

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In contrast to the methodologies of the 1970s and 1980s, contemporary optimization processes heavily rely on computer simulations. Consequently, there is a pressing need for specialized mathematical tools adapted for both electrodynamic ("cold") and electronic ("hot") simulations. These tools facilitate the calculation of arbitrary non-stationary, non-monochromatic fields within extensive dispersive three-dimensional (3D) electrodynamic systems such as electrodynamic lines (ELs), with or without electron beams. Examples of such systems encompass periodic delay structures utilized in TWTs and CFDs, as well as smooth bore waveguides employed in fast wave devices, among others.

The exploration of the electromagnetic (EM) wave propagation within intricate and irregular ELs, particularly those containing nonlinear media, remains a perennially pertinent and challenging endeavor in the realms of radio physics, theory of dynamic systems, and numerical modeling of wave processes [2–9]. While methods such as finite difference in time domain (FDTD) and finite element in time domain (FETD) [10–13] are universally recognized and well-suited for addressing these challenges, they often pose significant computational burdens, particularly when applied to ELs with complex boundary conditions (BCs).

Given that the electron beam typically occupies only a small fraction of the total volume of the device, the variable separation (Fourier) method presents a viable alternative in certain cases. This approach reduces computational overhead by employing complex BCs solely for the initial calculation of an EL eigenfunction manifold, which is then stored exclusively within the beam space. However, ELs with open-ended or matched characteristics, featuring continuous spectra of their eigenfunctions, may necessitate considerable computational storage resources.

The objective of this paper is to introduce a novel adaptation of the classical Fourier method, aimed at replacing the continuous set of eigenfunctions of open-ended or matched ELs with a finite discrete set of newly defined spatially localized functions, termed as "partial modes of EL" or "oscillets". This innovative approach not only streamlines the simulations of irregular ELs along their axes, but also holds promise for enhancing the computational efficiency in the analysis of such systems.

2. Notations and Abbreviations

In this paper, we adopt the four-dimensional (4D) pseudo-Euclidean formalism with imaginary time ("1, 1, 1, *i*") as the default spacetime geometry framework. The Cartesian coordinate system is utilized, where x, y, and z denote real-valued spatial coordinates, while t represents the temporal coordinate with dimension of imaginary length [14].

In Minkowski spacetime, four-vectors denoted with arrows (e.g., $\vec{a}$) possess mixed-valued components, comprising real spatial and imaginary temporal components. Column vectors within the Hilbert L^2 space of arbitrary dimension are represented in bold (e.g., $\mathbf{a}$). These notations can be combined with one another, as demonstrated by expressions like $\vec{\mathbf{a}}$. Braces are utilized to aggregate scalar values into vectors or column vectors, as exemplified by $\vec{a} = \{a_x, a_y, a_z, a_t\}$ and $\mathbf{a} = \{a_0, a_1, ..., a_{N-1}\}$. The scalar product of four-vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z + a_t b_t$, while the scalar product of column vectors $\mathbf{a}$ and $\mathbf{b}$ follows a similar formulation. The squared four-vector $\vec{a}$ is expressed as $(\vec{a})^2 = a_x^2 + a_y^2 + a_z^2 + a_t^2$. Matrices and tensors are enclosed within square brackets, such as $[c]$. Notably, the mixed spatio-temporal components of "physical" four-tensors (e.g., c_{xt}) are imaginary, while the remaining components are real-valued. Completing the 4D pseudo-Euclidean mathematical toolkit are operators like the four-gradient $\vec{\nabla} = \{\partial/\partial x, \partial/\partial y, \partial/\partial z, \partial/\partial t\}$, the four-divergence $\vec{\nabla} \cdot \vec{a} = \partial a_x/\partial x + \partial a_y/\partial y + \partial a_z/\partial z + \partial a_t/\partial t$, and the D'Alembert operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 + \partial^2/\partial t^2$.

Generic symbols are employed in lieu of explicit spatio-temporal coordinates: τ represents a generic symbol for x, y, z, or t, while ξ signifies a generic symbol for x, y, or z. The index ξ associated with any of the four-operators denotes a similar operator with an identical zero temporal component or time derivative. An abstract four-vector aleph-function $\vec{\aleph}(x, y, z, t)$ [14] serves as a generalization of the electromagnetic potential four-vector $\vec{A}(x, y, z, t)$ under the constraint of the Lorenz gauge $\vec{\nabla} \cdot \vec{\aleph} \equiv 0$. This function exhibits mixed-valued components and is square-integrable within the spatial volume V of the EL, possessing non-predefined but equal dimensions for all its components.

3. The "Old" Mathematical Physics vs. Computational Electrodynamics

When addressing and generalizing specific electrodynamic problems, the foremost consideration often turns to the FDTD and FETD computational methods [10–13]. Undoubtedly, these methods stand as the most versatile and potent tools in the realm of electrodynamics. However, their universality comes at a significant cost in terms of computational resources. Conversely, classical mathematical physics offers solutions to some important electrodynamic problems without resorting to numerical methods. While such solutions may not align with modern engineering simulations, the brute force approach of "number-crunching" is also deemed suboptimal when more intellectually-driven methodologies exist.

Therefore, the quest for a "golden mean" arises, especially when tackling electronic engineering problems akin to those aforementioned. Could leveraging known analytic methods from classical mathematical physics help alleviate the computational burden? This question becomes particularly pertinent to experts in mathematical physics. Specifically, regarding the integration of the inhomogeneous wave equation

$$\nabla^2 \vec{\aleph} = \vec{j}, \tag{1}$$

where $\vec{\aleph}(x, y, z, t)$ represents a four-vector of the current density, which of the two possible approaches – Bernoulli or D'Alembert – might prove suitable? The Bernoulli approach, rooted in Fourier's variable separation method [15], entails expressing solutions as superpositions of standing waves (spatial eigenmodes of the EL with time-dependent coefficients). Conversely, the D'Alembert approach involves non-separable solutions of the wave equation, represented as superpositions of traveling waves.

For narrowband potentials, particularly in a harmonic basis, the classical Fourier method proves to be quite applicable. It involves separating a quasi-harmonic factor, dependent on time (or time and the longitudinal coordinate of the EL), from the overall solution when integrating the wave equation. However, with the advancement of ultra-wideband (UWB) techniques, greater emphasis is placed on original mathematical methods that incorporate both separable and non-separable solutions of the wave equation [16–20].

To delve deeper into the physical underpinnings of this trend, let us consider a scenario within an EL, where a lumped, long-term external influence – an exciting current – commences at a specific spatial point and moment. In the immediate aftermath, an observer would perceive traveling waves, akin to partial waves in Brillouin's concept, propagating in all directions throughout the line's internal volume. Prior to encountering the conductive boundaries of the EL, where they reflect and interfere, these waves are most naturally described by the D'Alembert approach. They manifest themselves as spherical solutions of the wave equation, advancing at a constant speed (assuming the dispersion of the filling medium is negligible).

Conversely, as the time progresses, a stationary regime emerges within the line. From the perspective of the same observer, this regime is more aptly characterized by the Bernoulli approach. It appears as a superposition of standing waves – eigenmodes of the EL – each oscillating with amplitudes and frequencies determined by the enduring external influence.

From a mathematical standpoint, both the transitional and steady states of the EL are adequately described by both approaches. For instance, the superposition of all excited normal modes initially forms expanding spherical waves. However, understanding the physical aspect of this process is less straightforward.

Consider the concept of "short-range interaction". How do we explain that immediately after the application of a concentrated influence, the values of the EL eigenmodes are already non-zero throughout the entire line? This paradox mirrors the challenges encountered in Fourier analysis of finite functions and can be attributed to the non-local nature of the Fourier basis employed. Similar issues are encountered in other domains, leading to the widespread adoption of wavelet analysis. One manifestation of this challenge is the slow convergence of Fourier series in terms of EL eigenfunctions.

The question arises: is there a "third" mathematical approach to solving the wave equation that effectively captures both transient and stationary states of a distributed oscillatory system? Such an approach would ideally amalgamate the advantages of the variable separation method, such as its simplicity in accommodating EL dispersion, with the strengths of non-separable solutions, which facilitate the mod-

eling of limit problems. This hypothetical approach would offer a comprehensive framework for describing the behavior of complex systems undergoing dynamic transitions.

A basic illustration of such an approach involves transitioning from the eigenfunctions of a periodic EL to new functions, achieved through discrete and continuous Fourier transforms with respect to the longitudinal wave number parameter [21]. These functions are localized in the longitudinal direction, meaning they attenuate rapidly when moving away from their maximum in both directions. However, it is important to note that this transformation is only applicable to periodic or longitudinally homogeneous (regular) ELs.

As a more general alternative, another algorithm for generating longitudinally localized functions of periodic or regular ELs, known as partial modes, is outlined in [21]. This method involves a one-dimensional linear transform of the EL eigenfunction manifold using a designated "eigenmode form-matrix". Notably, this algorithm is independent of the specific EL design, as it relies solely on determining the optimal form of the form-matrix. Physically, partial modes can be envisioned as "clouds" of the EM potential inside the EL, interconnected and constrained longitudinally. Each of these "clouds" oscillates with its own phase, enabling the application of matrix theory to describe oscillatory systems with lumped parameters and N degrees of freedom, such as a chain of N coupled oscillators (bandpass filter) [22].

The concept of partial modes offers a clear framework for understanding both transient and steady-state processes within the system under external influence. For instance, when one of the N oscillators is excited, significant oscillations initially manifest themselves only in the cells nearest to it. Subsequently, these oscillations propagate along the chain, reflect from its ends, and interfere. Standing waves also find a straightforward interpretation in terms of partial modes. Notably, all aforementioned concepts can be readily applied to partial modes of EL, where the instantaneous values are regarded as independent generalized coordinates of the system with N degrees of freedom.

The term "partial modes" derives from the concept that each of them can be viewed as an eigenfunction of a specific partial system, obtained by fixing $N-1$ independent coordinates in the original oscillatory system with N degrees of freedom [22]. Unlike chains of lumped elements, establishing clear boundaries for the partial systems in distributed ELs is impractical. However, partial modes are not only precisely defined from a mathematical standpoint, but can also be observed physically as a superposition of free oscillations of the N lowest normal modes of the EL, each with suitable amplitudes and phases at a given moment in time.

The advantages of partial modes over normal modes, particularly their longitudinal localization, are detailed in [21]. In addition to providing a physically clear solution to the wave equation, partial modes offer practical independence of the EM potential structure and electrodynamic parameters of each mode from the properties of the boundaries and the filling of the EL at sufficient distances from the mode's localization. This independence ensures that partial modes maintain discreteness, thereby obviating the need for introducing Fourier integrals when separating variables in limit problems – for instance, in scenarios involving infinite or matched-ended ELs. These advantages are significant enough to warrant attempts to generalize the concept of partial modes to all three spatial coordinates [23].

The central concept of the matrix theory of distributed oscillatory systems in EM studies, as discussed below, revolves around employing the method of variable separation to expand non-stationary, non-harmonic EM potentials of ELs into a series of spatially localized basic coordinate functions. These functions are associated with coefficients that are arbitrarily dependent on time, effectively interpreting these potentials as a superposition of oscillations originating from a collection of electrodynamically coupled partial oscillators.

4. The Matrix Theory of Distributed Oscillatory Systems

A non-stationary, non-harmonic EM potential is analyzed within closed and simply connected ELs of arbitrary geometry with walls made of an ideal conductor. For simplicity, we assume the EL to be empty, filled only with vacuum, where free charges (currents) $\vec{j}$ may be in motion. For the subsequent discussion, ELs with periodic BCs can be treated as closed lines along one or several spatial coordinates.

The aleph-function (wave function) $\vec{\aleph}$ for the EM potential is a solution of the D'Alembert equation (1)

within the spatial volume V of the EL, with homogeneous BCs on the surface S that encloses this volume (or, in the presence of periodic BCs, on the part of S that interfaces with the metal walls of the EL). The four-vector of an abstract current density $\vec{j}$ is a predefined within V function of the spatio-temporal coordinates. Normalizing factors such as electric ε_0 and magnetic μ_0 constants can be omitted through an appropriate choice of units for $\vec{\aleph}$ and $\vec{j}$.

We will employ the method of separation of variables to solve Eq. (1). First, we separate the time-dependent factor by expressing the aleph-function four-vector as a Fourier series. This series is defined in terms of the complete infinite orthogonal set of spatial eigenfunctions of the EL $\vec{\aleph}_m(x,y,z)$, each with mixed-valued dimensions and dimensionless real-valued coefficients $u_m(t)$:

$$\vec{\aleph}(x,y,z,t) = \sum_{m=0}^{\infty} \vec{\aleph}_m(x,y,z)\, u_m(t). \tag{2}$$

The functions $\vec{\aleph}_m$ (where $m = 0,\ 1,\ 2,\ ...$) represent non-trivial solutions of the eigenvalue problem for the self-conjugate linear Laplace operator $-\nabla_\xi^2 = -\partial^2/\partial x^2 - \partial^2/\partial y^2 - \partial^2/\partial z^2$:

$$\nabla_\xi^2 \vec{\aleph}_m + k_{m\xi}^2 \vec{\aleph}_m = 0, \tag{3}$$

within the volume V, subject to homogeneous BCs of the first kind for the time component $\aleph_{mt}$ of $\vec{\aleph}_m$ and the tangential component $\aleph_{m\xi\|}$ of its spatial component $\aleph_{m\xi}$ on the surface S of the walls:

$$\aleph_{mt}(x,y,z)\big|_S = 0; \quad \aleph_{m\xi\|}(x,y,z)\big|_S = 0, \tag{4}$$

or periodicity conditions for the wave function in one or more spatial coordinates, such as:

$$\vec{\aleph}_m(x,y,0) = \vec{\aleph}_m(x,y,\Delta Z), \tag{5}$$

where ΔZ represents the length of the EL along its longitudinal coordinate z. Conditions (4) and the Lorenz gauge of the wave function imply BCs of the second kind for the normal component $\aleph_{m\xi\perp}$ of $\aleph_{m\xi}$ on S:

$$\partial \aleph_{m\xi\perp}(x,y,z)/\partial\nu\big|_S = 0, \tag{6}$$

where ν is the spatial direction normal to S at the point (x,y,z).

For idealized closed or periodic ELs, the spectrum of the eigenvalue problem (3) is discrete. The eigenvalues (squared spatial wave numbers $k_{m\xi}^2$ of the eigenmodes, which are also their squared eigenfrequencies), are non-negative. We enumerate the eigenfunctions $\vec{\aleph}_m$ so that, as m increases monotonically, the corresponding eigenvalues $k_{m\xi}^2$ form a non-decreasing sequence.

In many cases, the rank of eigenvalues $k_{m\xi}^2$ is greater than one, indicating a degeneracy of eigenmodes. For instance, this occurs for eigenfunctions of TE (H) and TM (E) types in longitudinally homogeneous ELs, volumes V of square or circular cross-sections, or periodic BCs in regular ELs. In such cases, it may be more convenient to consider two or more subsets of the eigenfunction manifold grouped by criteria such as their type, polarization, or phase.

The classical condition of orthogonality of the first kind [23] for the eigenfunctions is expressed as:

$$\frac{1}{2}\int\limits_V dxdydz\, \vec{\aleph}_m \vec{\aleph}_{m'} = \begin{cases} 0 \text{ if } m' \neq m; \\ \tilde{W}_m \text{ if } m' = m, \end{cases} \tag{7}$$

where $\tilde{W}_m$ is the unit pseudo-energy of the m-th eigenmode [21]. All degenerate eigenmodes are orthogonalized according to Eq. (7).

The orthogonality condition of the second kind [23] for eigenfunctions of a self-conjugate operator $-\nabla_\xi^2$ is expressed as:

$$\frac{1}{2}\int\limits_V dxdydz \left(-\nabla_\xi^2 \vec{\aleph}_m\right) \vec{\aleph}_{m'} = \\ = \frac{1}{2}\int\limits_V dxdydz\, \vec{\aleph}_m \left(-\nabla_\xi^2 \vec{\aleph}_{m'}\right) = \begin{cases} 0 & \text{if } m' \neq m; \\ W_m & \text{if } m' = m, \end{cases} \tag{8}$$

where W_m is the unit energy of the m-th normal mode. The Rayleigh relations, derived from Eqs. (3), (7), and (8), are given by:

$$W_m = k_{m\xi}^2 \tilde{W}_m; \quad k_{m\xi}^2 = W_m \tilde{W}_m^{-1}. \tag{9}$$

Considering Eqs. (7) and (8), the manifold of EL eigenfunctions must be regarded as doubly orthogonal [23]. It is worth noting that this should not be confused with the biorthogonality of systems of basic functions.

Let us consider the scenario, where the spectrum of the wave function $\vec{\aleph}$ in the spatial wavenumber domain $\vec{k}_\xi$ is finite at every moment of time:

$$\left|\vec{k}_\xi\right| \leq k_{\xi\,\max} < \infty. \tag{10}$$

In this case, the N-th and all subsequent coefficients in expansion (2) are identically equal to zero (where N is the smallest number such that $k^2_{N\xi} > k^2_{\xi\,\max}$).

From a mathematical perspective, any function $\vec{\aleph}$ satisfying condition (10) belongs to the linear manifold generated by the finite doubly orthogonal system from the subset of N eigenfunctions $\vec{\aleph}_m$ with the smallest eigenvalues ($m = 0, 1, ..., N-1$). Therefore, for any wave function with spectrum (10), this subset retains the property of completeness:

$$\frac{1}{2}\int\limits_V dxdydz\,\vec{\aleph}^2(x,y,z,t) = \sum_{m=0}^{N-1} |u_m(t)|^2\,\tilde{W}_m. \qquad (11)$$

Let us combine the N spatial eigenfunctions $\vec{\aleph}_m$ with the smallest eigenvalues, as well as their corresponding Fourier coefficients u_m in series (2), into column vectors $\vec{\boldsymbol{\aleph}}(x,y,z)$ and $\mathbf{u}(t)$ respectively. The series can now be written in vector form as:

$$\vec{\aleph}(x,y,z,t) = \vec{\boldsymbol{\aleph}}(x,y,z)\mathbf{u}(t). \qquad (12)$$

The eigenvalue problem (3) can be expressed in matrix form as:

$$\nabla^2_\xi\vec{\boldsymbol{\aleph}} + \left[k^2_\xi\right]\vec{\boldsymbol{\aleph}} = 0, \qquad (13)$$

where $\left[k^2_\xi\right]$ is a diagonal matrix of size $N\times N$ containing the N smallest squared eigenfrequencies of the EL in its non-zero elements.

The orthogonality condition of the first kind (7) is redefined as:

$$\frac{1}{2}\int\limits_V dxdydz\,\vec{\boldsymbol{\aleph}}\vec{\boldsymbol{\aleph}}^T = \left[\tilde{W}\right], \qquad (14)$$

where $\left[\tilde{W}\right]$ is a diagonal matrix of size $N\times N$ containing the unit pseudo-energies of the N lowest eigenmodes in its non-zero elements. Here and in the following, the superscript T denotes the transposition of a column vector or matrix.

The orthogonality condition of the second kind (8) is redefined as:

$$\frac{1}{2}\int\limits_V dxdydz\,\left(-\nabla^2_\xi\vec{\boldsymbol{\aleph}}\right)\vec{\boldsymbol{\aleph}}^T =$$

$$= \frac{1}{2}\int\limits_V dxdydz\,\vec{\boldsymbol{\aleph}}\left(-\nabla^2_\xi\vec{\boldsymbol{\aleph}}\right)^T = [W], \qquad (15)$$

where $\nabla^2_\xi\vec{\boldsymbol{\aleph}}$ is a column vector obtained from the first N values of $\nabla^2_\xi\vec{\aleph}_m$, and $[W]$ is a diagonal matrix of size $N\times N$ containing the unit energies of the N lowest eigenmodes in its non-zero elements.

The Rayleigh relations (9) can be expressed in matrix form as:

$$[W] = \left[k^2_\xi\right]\left[\tilde{W}\right]; \quad \left[k^2_\xi\right] = [W]\left[\tilde{W}\right]^{-1}, \qquad (16)$$

where the superscript (-1) in relation to a matrix denotes its inversion. It can be observed from q. (16) that only the doubly orthogonality of the eigenfunctions ensures the diagonalization of the matrix of eigenvalues $\left[k^2_\xi\right]$ (i.e., the electrodynamic disconnectedness of the EL eigenmodes).

The completeness condition (11) can be written in matrix form as:

$$\frac{1}{2}\int\limits_V dxdydz\,\vec{\aleph}^2(x,y,z,t) = \mathbf{u}(t)\left(\left[\tilde{W}\right]\mathbf{u}(t)\right). \qquad (17)$$

The relative magnitudes (normalization) of the elements in the column vector of eigenfunctions $\vec{\boldsymbol{\aleph}}$ importantare, since they are processed together. Four types of normalization can be distinguished:

1. Amplitude normalization. In this type, a characteristic value of all eigenfunctions (for example, the modulus or one of the components of the four-vector $\vec{\aleph}_m$) at their maximum point equals 1:

$$\max\left|\vec{\aleph}_m(x,y,z)\right| = 1; \quad m = 0,\ 1,\ ...,\ N-1.$$

2. Energy normalization of the first kind. In this type, the values of the unit pseudo-energies of all eigenmodes equal 1:

$$\tilde{W}_m = 1; \quad m = 0,\ 1,\ ...,\ N-1.$$

This normalization generates an orthonormal system of eigenfunctions (in the classical sense).

3. Energy normalization of the second kind. In this type, the unit energies of all eigenmodes equal 1:

$$W_m = 1; \quad m = 0,\ 1,\ ...,\ N-1.$$

In contrast to orthogonality, the energy normalization of non-degenerate eigenfunctions can be only single (either of the first or second kind).

4. Special normalization. In this type, the characteristic values of eigenmodes indicated in item 1 depend on $|k_{\xi m}|$ according to some law. For example, a truncated Gaussian law:

$$\max\left|\vec{\aleph}_m(x,y,z)\right| = \exp(-\psi k^2_{\xi m}/k^2_{\xi N});$$
$$m = 0, ..., N-1,$$

where $\psi > 0$ is a normalization parameter. In general, this kind of normalization enables more effective spatial localization of the partial functions.

Thus, the formulas for the decomposition of the aleph-function with a finite spectrum in terms of a finite doubly orthogonal system of EL spatial eigenfunctions $\vec{\aleph}$ (2)–(11) are rewritten in matrix form (12)–(17). In practice, such notation is rarely used, since it makes no sense for eigenmodes to introduce diagonal matrices which limit the number of eigenfunctions taken into account. Instead, it is more convenient to work with independent scalar equations.

Let us define a new column vector of N mixed-valued spatial partial functions of the EL $\vec{\aleph}_p(x,y,z)$ (spatial eigenfunctions of partial oscillators) such that the vector of EL spatial eigenfunctions is its linear transformation:

$$\begin{aligned}&\vec{\aleph}(x,y,z) = [F]\,\vec{\aleph}_p(x,y,z);\\ &\vec{\aleph}_p(x,y,z) = [F]^{-1}\,\vec{\aleph}(x,y,z),\end{aligned} \tag{18}$$

where $[F]$ is a non-singular matrix of size $N\times N$ called the form-matrix of the EL spatial eigenfunctions. Because both the eigenfunctions and partial functions have the same formalism "1, 1, 1, i", this matrix is real-valued.

It is worth noting that all mentioned types of eigenmode normalization can be included in the form-matrix by multiplying its rows by the corresponding normalizing factors.

Since N eigenfunctions from vector $\vec{\aleph}$ allow decomposition in series (12) of an arbitrary function $\vec{\aleph}(x,y,z,t)$ that satisfies condition (10) in volume V, it is evident that vector $\vec{\aleph}_p$ of N linearly independent combinations also allows expansion of that function in a finite series of the form:

$$\vec{\aleph}(x,y,z,t) = \vec{\aleph}_p(x,y,z)\mathbf{u}_p(t), \tag{19}$$

where $\mathbf{u}_p(t)$ is a column vector of N dimensionless real-valued coefficients of the new series. Comparing Eqs. (12) and (19) with the inclusion of (18), we can derive:

$$\mathbf{u}(t) = \left([F]^{-1}\right)^T\mathbf{u}_p(t); \quad \mathbf{u}_p(t) = [F]^T\,\mathbf{u}(t). \tag{20}$$

An analog of the eigenvalue problem (13) for the operator $-\nabla^2_\xi$, in terms of partial functions, is called the intervalue problem, which we formulate as follows. We need to find a solution to the matrix equation:

$$\nabla^2_\xi\vec{\aleph}_p + \left[k^2_{p\xi}\right]\vec{\aleph}_p = 0 \tag{21}$$

inside the volume of the EL, where each element $\vec{\aleph}_{pn}$ of column vector $\vec{\aleph}_p$ (with $n = 0,\ 1,\ \ldots N-1$) is not identically zero and satisfies homogeneous BCs (4)–(6) at the border. Here, $\left[k^2_{p\xi}\right]$ is a symmetric matrix of size $N\times N$ containing N^2 so-called mutual values (intervalues) of the EL. From a physical standpoint, these are the squares of certain mutual wavenumbers of partial oscillators.

Obviously, in this particular formulation, the intervalue problem is not uniquely defined, since the only condition for the existence of a non-trivial solution to Eq. (21) is the equality of all eigenvalues of matrix $\left[k^2_{p\xi}\right]$ to the respective diagonal elements of matrix $\left[k^2_\xi\right]$ for the EL. This problem must be supplemented with some additional conditions.

Matrices $\left[k^2_\xi\right]$ and $\left[k^2_{p\xi}\right]$ are mutually interconnected through similarity transformations (using $[F]$ as the transforming matrix), which correspond to the aforementioned linear transformations of the functional spaces L^2 of eigenfunctions and partial functions of the EL. These can be obtained by substituting (18) into (13) and (21):

$$\begin{aligned}&\left[k^2_\xi\right] = [F]\left[k^2_{p\xi}\right][F]^{-1};\\ &\left[k^2_{p\xi}\right] = [F]^{-1}\left[k^2_\xi\right][F].\end{aligned} \tag{22}$$

Excluding the trivial case of a diagonal form-matrix, matrix $\left[k^2_{p\xi}\right]$ is non-diagonal.

An analogue of the orthogonality condition of the first kind (14) for partial functions is:

$$\frac{1}{2}\int\limits_V dxdydz\,\vec{\aleph}_p\vec{\aleph}^T_p = \left[\tilde{W}_p\right], \tag{23}$$

where $\left[\tilde{W}_p\right]$ is a symmetric matrix of size $N\times N$ containing N^2 unit mutual pseudo-energies of partial oscillators. Matrices $\left[\tilde{W}\right]$ and $\left[\tilde{W}_p\right]$ are interconnected by congruence transformations, which can be obtained by substituting (18) into (14) and (23):

$$\begin{aligned}&\left[\tilde{W}\right] = [F]\left[\tilde{W}_p\right][F]^T;\\ &\left[\tilde{W}_p\right] = [F]^{-1}\left[\tilde{W}\right]\left([F]^{-1}\right)^T.\end{aligned} \tag{24}$$

The orthogonality condition of the second kind (15) for partial functions is expressed as:

$$\begin{aligned}&\frac{1}{2}\int\limits_V dxdydz\,\left(-\nabla^2_\xi\vec{\aleph}_p\right)\vec{\aleph}^T_p =\\ &= \frac{1}{2}\int\limits_V dxdydz\,\vec{\aleph}_p\left(-\nabla^2_\xi\vec{\aleph}_p\right)^T = \left[W_p\right],\end{aligned} \tag{25}$$

where $[W_p]$ is a symmetric matrix of size $N \times N$ containing N^2 unit mutual energies of partial oscillators. It's straightforward to demonstrate that matrices $[W]$ and $[W_p]$ are also related by congruence transformations:

$$\begin{aligned} [W] &= [F]\big[W_p\big][F]^T; \\ \big[W_p\big] &= [F]^{-1}[W]\big([F]^{-1}\big)^T. \end{aligned} \tag{26}$$

The analogs of the Rayleigh relations for partial oscillators follow from Eqs. (21), (23), and (25):

$$\big[W_p\big] = \big[k^2_{p\xi}\big]\big[\tilde{W}_p\big]; \quad \big[k^2_{p\xi}\big] = \big[W_p\big]\big[\tilde{W}_p\big]^{-1}. \tag{27}$$

The completeness condition (17) remains valid for the set of partial functions and is expressed as:

$$\frac{1}{2}\int\limits_V dxdydz\,\vec{\aleph}^2(x,y,z,t) = \mathbf{u}_p(t)\left(\big[\tilde{W}_p\big]\mathbf{u}_p(t)\right). \tag{28}$$

The matrices $\big[\tilde{W}_p\big]$ and $\big[W_p\big]$ are generally non-diagonal, which corresponds to the non-orthogonality of the set of partial functions obtained by using the energy normalizations of both the first and second kinds of the respective eigenfunctions. Therefore, series (19) cannot be considered as a generalized Fourier series. However, unlike the matrix $\big[k^2_{p\xi}\big]$, either of the two aforementioned matrices can retain its diagonal shape not only under a trivial linear transformation (18) with a diagonal matrix. It would suffice, for the matrix $[F]$, to be unitary, and the normalization of the eigenfunctions to be of either the first kind (in this case, the matrix $\big[\tilde{W}_p\big]$ is diagonal) or the second kind (matrix $\big[W_p\big]$ is diagonal in that case). The non-diagonality of at least one of the matrices $\big[\tilde{W}_p\big]$ or $\big[W_p\big]$ leads to the electrodynamic coupling of the partial oscillators with one another.

Introducing partial functions of EL alongside eigenfunctions may seem redundant at first glance, given the advantages of eigenfunctions as a classic basis for decomposing EM potentials. Eigenfunctions offer double orthogonality, ensuring the absence of electrodynamic coupling between eigenmodes in linear conservative systems. They also provide independence of each eigenvalue and eigenfunction taken separately from N, among other benefits.

However, the introduction of partial functions serves a specific purpose, particularly in cases where it is desirable to achieve spatial localization of oscillations. The idea is to choose a form-matrix $[F]$ in such a way that all partial oscillators are essentially localized in space. This localization would be possible for any EL and any $N \geq N_{\min}$ and means that the components of the function $\vec{\aleph}_{pn}$ approximate zero when moving away in all spatial directions from its global extremum. Minimal value $N_{\min}$, which ensures the localization of all partial functions, depends on the number of dimensions of the EM problem. Three-densities of pseudo-energy $\left|\vec{\aleph}_{pn}\right|^2/2$ and energy $\left(-\nabla^2\vec{\aleph}_{pn}\right)\vec{\aleph}_{pn}/2$ of the n-th partial oscillator are also spatially localized at that. This spatial localization is advantageous compared to eigenmodes, whose pseudo-energy and energy are distributed more uniformly throughout volume V, just as the expansion of functions in wavelets is sometimes preferable to their decomposition in harmonic basis.

Analogous to wavelets, partial oscillators can be viewed as "oscillets". These oscillets offer spatially localized oscillations, akin to "clouds" of wave function $\vec{\aleph}$ oscillating in equal phase. This characteristic allows the application of the canonical theory of lumped-parameter oscillatory systems with many degrees of freedom [22] to systems with distributed parameters and finite spectra in the spatial wavenumber domain.

The advantages of partial functions over eigenfunctions become particularly evident in numerical analyses of open-ended or matched ELs. The fundamentally continuous spectrum of eigenvalues and eigenfunctions for such oscillatory systems often complicates their numerical simulations when using the Fourier method with decomposition in series (2). In contrast, the spectrum of partial functions, as well as series (19), remains discrete and finite for all real ELs.

The localization of partial functions introduces an additional constraint that allows for redefining the problem of mutual values of EL, transforming it into a problem of conditional multidimensional optimization with a vector criterion. In this scenario, the matrix elements $\big[k^2_{p\xi}\big]$ serve as N^2 independent "design parameters". For each specific EL, these parameters are uniquely related to the form-matrix $[F]$ by relations (22). Furthermore, it follows from (18) that by varying the N elements of the n-th row of the matrix $[F]^{-1}$, one can localize the n-th oscillet. The only constraints, in addition to the fixed eigenvalues, are the requirements for a sufficiently rapid reduction of matrix elements $\big[k^2_{p\xi}\big]$ and $\big[\tilde{W}_p\big]^{-1}$ corresponding to increasingly distant partial oscillators. These re-

strictions are imposed by the excitation equation for partial modes.

This equation for the instantaneous values of the oscillets $\mathbf{u}_p(t)$ is derived by substituting (1) with (19), considering (21), multiplying the resulting expression by $\vec{\aleph}_p^T$, and subsequently integrating over V, with regard for (23). The resulting equation is as follows:

$$\frac{d^2\mathbf{u}_p}{dt^2} + \left[k_{p\xi}^2\right]^T \mathbf{u}_p = \frac{1}{2}\left[\tilde{W}_p\right]^{-1} \int\limits_V dxdydz\, \vec{\aleph}_p\, \vec{j}. \qquad (29)$$

It is evident that the integral on the right side of (29) is a linear integral transform of the current density with a discrete vector kernel $\vec{\aleph}_p$. This transform performs low-pass filtering of function $\vec{j}$ in the spatial wavenumber domain with a cut-off wavenumber of $\pm k_{\max}$. Thus, series (19) can be viewed as a generalization of the Whittaker–Shannon interpolation formula [24] for vector functions in the 3D space, which also allows the use of a non-orthogonal basis.

5. Types and Properties of Partial Oscillators

In vector manifold theory (see [25] for an example), the solutions to the eigenvalue problem (3) for the wave function inside the spatial volume of an EL with homogeneous or periodic BCs (4)–(6) can be categorized into several subsets, which are doubly orthogonal complements of each other. In regular ELs, the mixed-valued eigenfunctions can be degenerate up to four times for each wavenumber, resulting in four subsets distinguished by the number of non-zero spatio-temporal components of these eigenfunctions. These subsets are well-known in electrodynamics.

In Eqs. (30)–(33), the Hartley transform kernel $\operatorname{cas} x = \cos x + \sin x$ is used as the basis for harmonic decomposition, instead of the complex formulation, to avoid interference with the imaginary parts of four-vectors. A_m is a normalizing factor for the m-th eigenmode, for example, $A_m \equiv 1/\sqrt{2}$ in amplitude normalization.

1. Eigenfunctions of Transverse Electric and Magnetic (TEM) type $\vec{\aleph}_{m\mathrm{TEM}}(x,y,z,t)$, solenoidal in one dimension (x), i.e., $\partial\aleph_{m\mathrm{TEM}x}/\partial x \equiv 0$:

$$\begin{aligned} &\aleph_{m\mathrm{TEM}x} = A_{m\mathrm{TEM}}\operatorname{cas}\left(\vec{k}_m\,\vec{r}\right);\\ &\aleph_{m\mathrm{TEM}y} = \aleph_{m\mathrm{TEM}z} = \aleph_{m\mathrm{TEM}t} = 0, \end{aligned} \qquad (30)$$

where $\vec{r} = \{x,y,z,t\}$ is a coordinate four-vector in pseudo-Euclidean space, and k_{mx} is always zero.

2. Eigenfunctions of Transverse Electric (TE) type $\vec{\aleph}_{m\mathrm{TE}}(x,y,z,t)$, solenoidal in two dimensions (x, y), i.e., $\partial\aleph_{m\mathrm{TE}x}/\partial x + \partial\aleph_{m\mathrm{TE}y}/\partial y \equiv 0$:

$$\begin{aligned} &\aleph_{m\mathrm{TE}x} = -A_{m\mathrm{TE}}\frac{k_{my}k_{mx}}{k_{mx}^2}\operatorname{cas}\left(\vec{k}_m\,\vec{r}\right);\\ &\aleph_{m\mathrm{TE}y} = A_{m\mathrm{TE}}\operatorname{cas}\left(\vec{k}_m\,\vec{r}\right);\\ &\aleph_{m\mathrm{TE}z} = \aleph_{m\mathrm{TE}t} = 0. \end{aligned} \qquad (31)$$

If k_{mx} is zero, k_{my} must also be zero, with $\aleph_{m\mathrm{TE}x}$ being zero at that.

3. Eigenfunctions of Transverse Magnetic (TM) type $\vec{\aleph}_{m\mathrm{TM}}(x,y,z,t)$, solenoidal in three dimensions (x, y, z), i.e., $\partial\aleph_{m\mathrm{TM}x}/\partial x + \partial\aleph_{m\mathrm{TM}y}/\partial y + \partial\aleph_{m\mathrm{TM}z}/\partial z \equiv 0$:

$$\begin{aligned} &\aleph_{m\mathrm{TM}x} = -A_{m\mathrm{TM}}\frac{k_{mz}k_{mx}}{k_{mx}^2+k_{my}^2}\operatorname{cas}\left(\vec{k}_m\,\vec{r}\right);\\ &\aleph_{m\mathrm{TM}y} = -A_{m\mathrm{TM}}\frac{k_{mz}k_{my}}{k_{mx}^2+k_{my}^2}\operatorname{cas}\left(\vec{k}_m\,\vec{r}\right);\\ &\aleph_{m\mathrm{TM}z} = A_{m\mathrm{TM}}\operatorname{cas}\left(\vec{k}_m\,\vec{r}\right); \quad \aleph_{m\mathrm{TM}t} = 0. \end{aligned} \qquad (32)$$

If k_{mx} and k_{my} are both zero, k_{mz} must also be zero, with $\aleph_{m\mathrm{TM}x}$ and $\aleph_{m\mathrm{TM}y}$ being zero at that.

4. Eigenfunctions of Zero Magnetic (ZM) type $\vec{\aleph}_{m\mathrm{ZM}}(x,y,z,t)$ [21], solenoidal in four dimensions, i.e., $\partial\aleph_{m\mathrm{ZM}x}/\partial x + \partial\aleph_{m\mathrm{ZM}y}/\partial y + \partial\aleph_{m\mathrm{ZM}z}/\partial z + \partial\aleph_{m\mathrm{ZM}t}/\partial t \equiv 0$:

$$\begin{aligned} &\aleph_{m\mathrm{ZM}x} = -iA_{m\mathrm{ZM}}\frac{k_{mt}k_{mx}}{k_{mx}^2+k_{my}^2+k_{mz}^2}\operatorname{cas}\left(\vec{k}_m\,\vec{r}\right);\\ &\aleph_{m\mathrm{ZM}y} = -iA_{m\mathrm{ZM}}\frac{k_{mt}k_{my}}{k_{mx}^2+k_{my}^2+k_{mz}^2}\operatorname{cas}\left(\vec{k}_m\,\vec{r}\right);\\ &\aleph_{m\mathrm{ZM}z} = -iA_{m\mathrm{ZM}}\frac{k_{mt}k_{mz}}{k_{mx}^2+k_{my}^2+k_{mz}^2}\operatorname{cas}\left(\vec{k}_m\,\vec{r}\right);\\ &\aleph_{m\mathrm{ZM}t} = iA_{m\mathrm{ZM}}\operatorname{cas}\left(\vec{k}_m\,\vec{r}\right). \end{aligned} \qquad (33)$$

If k_{mx}, k_{my}, and k_{mz} are all zero, k_{mt} must also be zero, with $\aleph_{m\mathrm{ZM}x}$, $\aleph_{m\mathrm{ZM}y}$, and $\aleph_{m\mathrm{ZM}z}$ being zero at that. This systematization is a special case of orthogonal Weyl manifolds in the Hilbert space of square-integrable functions L^2 obtained as solutions to the four-vector eigenvalue problem.

Relations (18) associate each of the three subsets of eigenfunctions (31)–(33) with a separate type of partial functions: TE and TM types (solenoidal in three-space) and ZM type (potential in three-space), respectively. TEM eigenfunctions of a multiply connected EL cannot be converted into partial functions

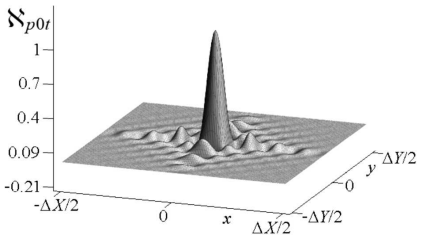

Fig. 1. Scalar 2D oscillet of free space with amplitude normalization

localized over all spatial coordinates. However, this conversion is unnecessary since these eigenmodes do not inherently disperse. TEM partial functions, localized in one direction, form the basis of the Whittaker–Shannon series [24]. In two-dimensional (2D) spatial $(x,\ y)$ problems, there is one solenoidal and one potential subset of eigenfunctions and respective partial functions. Regardless of the orthogonality of the oscillets within each of the types, partial oscillators of different types are always doubly orthogonal. Consequently, none of the EL eigenfunctions can be synthesized with (18) from oscillets of a type that does not correspond to them.

Several simple else fundamental properties of the partial functions follow immediately from the aforementioned definitions:

1. Each oscillet satisfies BCs (4)–(6) on the walls of EL individually.

2. The spectrum of each oscillet in the basis of the EL eigenfunctions is finite, i.e.,

$$\frac{1}{2}\int\limits_V dxdydz\,\vec{\aleph}_{pn}(x,y,z)\vec{\aleph}_m(x,y,z)\equiv 0$$

for $\left|\vec{k}_{m\xi}\right| > k_{\xi\,\max}$.

3. In the general case, amplitudes of all harmonics of the spectrum of each oscillet in the basis of the EL eigenfunctions, corresponding to the type of this oscillet with $\left|\vec{k}_{m\xi}\right| \le k_{\xi\,\max}$, are not equal to zero (for oscillets located near nodal surfaces of the eigenfunctions, the respective harmonics may be absent).

4. Amplitudes of all harmonics of the spectrum of each oscillet in the basis of the EL eigenfunctions, which do not correspond to the type of this oscillet, are identically equal to zero.

5. Matrices of unit mutual pseudo-energies of the partial oscillators $\left[\tilde{W}_p\right]$ of each type are not singular.

Other peculiar properties of the oscillets depend on the specific methods of their synthesis (e.g., the type of normalization used).

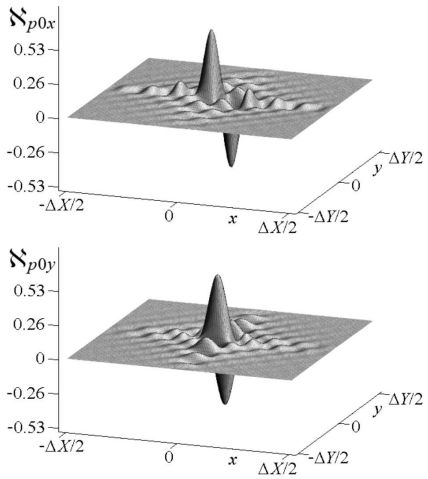

Fig. 2. Potential (divergent) vector 2D oscillet of free space with amplitude normalization

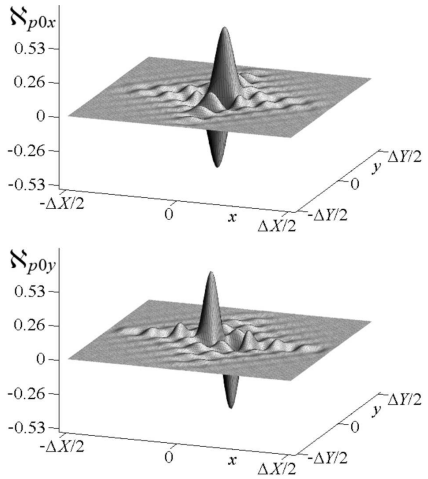

Fig. 3. Solenoidal (vortex) vector 2D oscillet of free space with amplitude normalization

Examples of partial modes of 2D scalar and vector distributed oscillatory systems with periodical BCs on all their bounds (so-called "oscillets of 2D free space") [23] are given in Figs 1–3 using the amplitude normalization. For comparison, similar partial modes, additionally localized by using the truncated Gaussian normalization, are shown in Figs 4–6.

6. Example of the Partial Mode Decomposition of the Wave Function for an Irregular Electrodynamic Line

The definition of the partial modes as a linear transform of the EL eigenmodes using the irregular form-

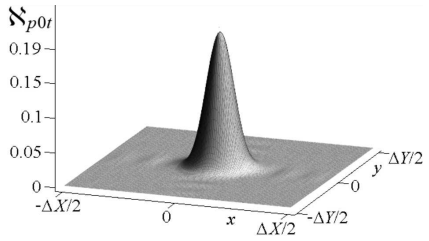

Fig. 4. Scalar 2D oscillet of free space localized with truncated Gaussian normalization

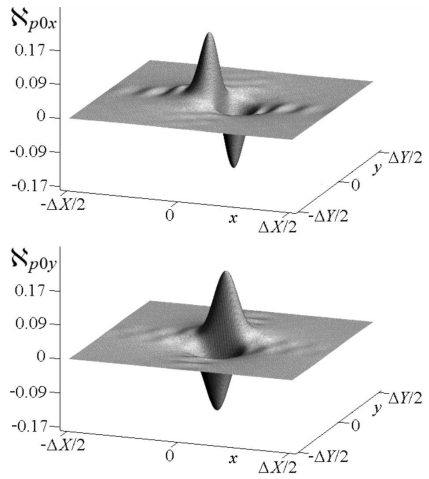

Fig. 5. Potential vector 2D oscillet of free space localized with truncated Gaussian normalization

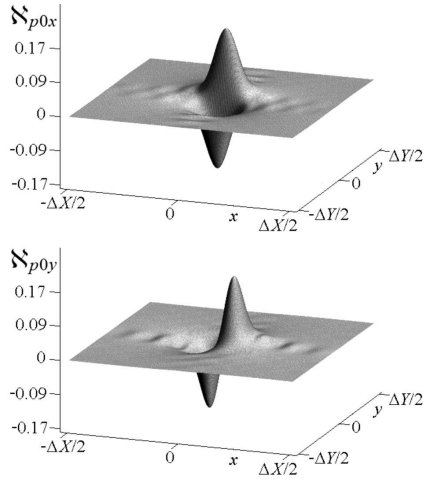

Fig. 6. Solenoidal vector 2D oscillet of free space localized with truncated Gaussian normalization

matrix is more general and flexible compared to the plain discrete Fourier transform of their EM potentials. The "proper" irregularity of $[F]$ can "compensate" the lack of uniformity of EL, resulting in the regularity of the obtained partial modes. For example, let's consider a non-uniform dispersive line in the longitudinal direction with a length of ΔZ and homogeneous BCs at its endpoints $z = 0$ and $z = \Delta Z$. Let's specify the longitudinal dependences of the EL longitudinal wavenumbers $k_{mz}(z)$ for its various eigen-

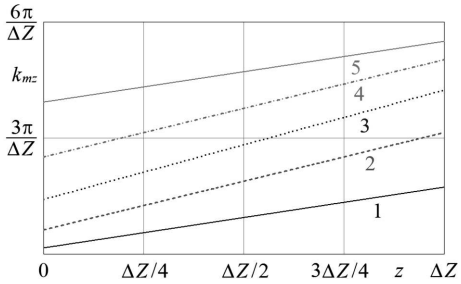

Fig. 7. The longitudinal dependences of the longitudinal wavenumbers $k_{mz}(z)$ for five the lowest eigenmodes of irregular EL

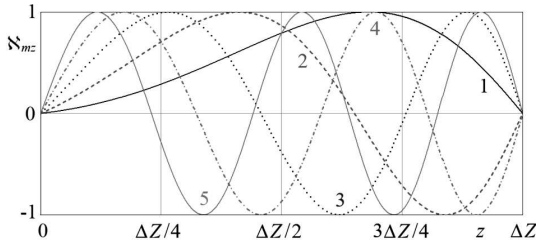

Fig. 8. The longitudinal components of wave functions for five the lowest eigenmodes of irregular EL

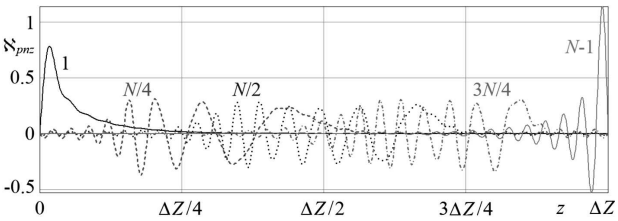

Fig. 9. The longitudinal components of wave functions for five non-regularized partial modes of irregular EL

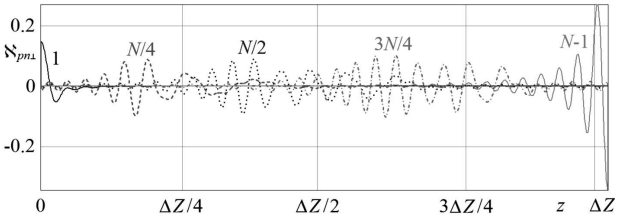

Fig. 10. The transverse components of wave functions for five non-regularized partial modes of irregular EL

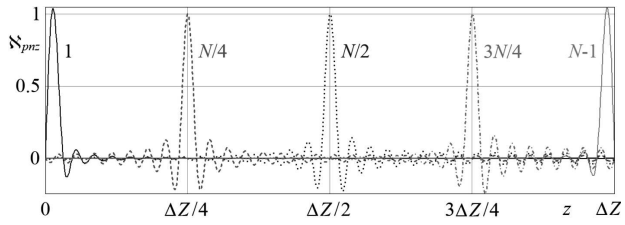

Fig. 11. The longitudinal components of wave functions for five regularized partial modes of irregular EL

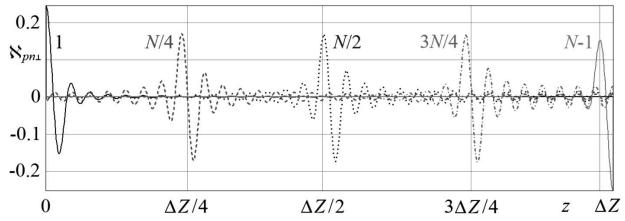

Fig. 12. The transverse components of wave functions for five regularized partial modes of irregular EL

modes and describe them by the formula:

$$k_{mz}(z) = \frac{\pi m}{\Delta Z}\,[1 - (1 - m/N)(1 - 2z/\Delta Z)],$$

where $m = 1, 2, ...$, as shown in Fig. 7 ($N = 6$ is chosen for clarity). We find that the lowest eigenmode (with $m = 1$) is the most longitudinally irregular one, while the highest eigenmode (with $m = 5$) is almost regular (see Fig. 8). When the partial modes of such an EL are synthesized from its eigenmodes using (18) with the regular form-matrix:

$$[F] = \\ = \begin{bmatrix} \sin\Delta\varphi_{11} & \sin\Delta\varphi_{12} & ... & \sin\Delta\varphi_{1(N-1)} \\ \sin\Delta\varphi_{21} & \sin\Delta\varphi_{22} & ... & \sin\Delta\varphi_{2(N-1)} \\ ... & ... & ... & ... \\ \sin\Delta\varphi_{(N-1)1} & \sin\Delta\varphi_{(N-1)2} & ... & \sin\Delta\varphi_{(N-1)(N-1)} \end{bmatrix},$$

where $\Delta\varphi_{mn} = \pi mn/N$ (that is, in fact, the discrete sine Fourier transform), they are poorly longitudinally localized (see Figs 9 and 10). If a non-regular form-matrix:

$$F_{mn} = \sin\frac{\pi mn}{N}\,[1 - (1 - m/N)(1 - n/N)]$$

is used instead, the partial modes become regular (see Figs 11 and 12). We should note that the phase shifts $\Delta\varphi_{mn}$ between different pairs of adjacent partial modes are no longer equal for each EL eigenmode in Figs 11 and 12.

7. Conclusions

The mathematical tool explored in this paper can be identified as a matrix theory of oscillatory systems with distributed parameters. So far, a similar theory has only been applied to lumped systems with many degrees of freedom. Its main advantage lies in replacing the integration of differential equations with partial derivatives by solving systems of ordinary differential equations, which requires fewer computational resources.

Decomposing EM potentials in the basis of EL partial functions offers a "large-scale" alternative to finite difference and finite element methods in the time domain. This is because the spatial distances between neighboring oscillets are determined more by physical reasons (with consideration of the upper limit of the spatial wavenumber domain) than by computational arguments. With further advancements in computational technology, this method may become useful for computer-aided design of new, highly effective microwave and optical sources, including open-ended systems with a continuous spectrum of eigenfunctions.

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РОЗКЛАДАННЯ ЕЛЕКТРОМАГНІТНИХ ПОТЕНЦІАЛІВ ПО ПАРЦІАЛЬНИХ ФУНКЦІЯХ ЕЛЕКТРОДИНАМІЧНИХ ЛІНІЙ З ДИСПЕРСІЄЮ

Пропонується використання парціальних функцій, або осцилетів, як базисних функцій, локалізованих вздовж усіх просторових координат, для розкладання нестаціонарних, негармонічних електромагнітних потенціалів у протяжних тривимірних дисперсійних електродинамічних системах, таких як електродинамічні лінії (ЕЛ). Ці функції отримуються як лінійні перетворення множини власних функцій ЕЛ з метою мінімізації просторової протяжності кожного осцилета. Приділено увагу застосуванню цих нових функцій в електродинамічних і електронних обчисленнях, зокрема в оптимізації нерегулярних ЕЛ, які зустрічаються в різних мікрохвильових і оптичних джерелах, включно з відкритими конфігураціями, що містять неперервний спектр власних функцій. Наведено ілюстративний приклад, що демонструє корисність парціальних функцій в електродинамічному розрахунку поздовжньо неоднорідної ЕЛ.

Ключові слова: електродинамічна система з дисперсією, електромагнітний потенціал, ряд Фур'є, власна функція, парціальна функція.