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A. KAMENSHCHIK,[1,2] F. MUSCOLINO[3,4]

[1] Dipartimento di Fisica e Astronomia, Università di Bologna
(via Irnerio 46, 40126 Bologna, Italy; e-mail: kamenshchik@bo.infn.it)
[2] I.N.F.N., Sezione di Bologna
(viale Berti Pichat 6/2, 40127 Bologna, Italy)
[3] Department of Science and High Technology, Università dell'Insubria
(via Valleggio 11, IT-22100 Como, Italy; e-mail: federica.muscolino@gmail.com)
[4] I.N.F.N., Sezione di Milano
(via Celoria 16, IT-20133 Milano, Italy)

LOOKING FOR CARROLL PARTICLES IN THE TWO-TIME SPACETIME [1]

We make an attempt to describe Carroll particles with a non-vanishing value of energy (i.e., the Carroll particles which always stay in rest) in the framework of two-time physics, developed in the series of papers by I. Bars and his co-authors. In the spacetime with one additional time dimension and one additional space dimension, where one can localize the symmetry which exists between generalized coordinates and their conjugate momenta. Such a localization implies the introduction of the gauge fields, which, in turn, implies the appearance of some first-class constraints. Choosing different gauge-fixing conditions and solving the constraints, we obtain different time parameters, Hamiltonians, and, generally, physical systems in the standard one-time spacetime. We find a set of gauge fixing conditions which gives the description of a Carroll particle in the one-time world. We construct the quantum theory of such a particle using an unexpected correspondence between our parametrization and that obtained by Bars for the hydrogen atom in 1999.

Keywords: two-time spacetime, Carroll group, particles.

1. Introduction

Theories with extra spatial dimensions have become quite traditional since the times, when they were put forward in the famous works by Kaluza and Klein. Theories with more than one time dimensions look much less intuitive and plausible. However, an impressive series of papers devoted to the so-called two-time or 2T physics was produced by I. Bars and his co-authors beginning from 1996 [1–6]. Classical and quantum physics of simple systems such as non-relativistic particle, massive and massless relativistic particles, harmonic oscillator, and hydrogen-like atoms can be described in the framework of 2T-physics from a unifying point of view. In the book [6], it is explained that different one-time physical systems arise as some "shadows" in the Plato's cave, which is nothing but the world with one additional temporal dimension and one additional spatial dimension. The language of the two-time physics is quite adapted also for the description of field theories [7] and of the gravity [8]. A new approach to cosmology, inspired by two-time physics has opened a way to an interesting treatment of the problem of passing through the cosmological singularities [9]. Relations between the two-time physics and the Carroll symmetry were not explored before. It was done in our paper [10].

The Carroll symmetry was discovered in 1965 [11] as an another Inönu–Wigner type contraction [12] of the Poincarè algebra, alternative to the contraction which results in the appearance of the Lie algebra of

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the Galilei group. Independently, it was discovered by Sen Gupta in 1966 [13]. While the Galilei group arises when the velocity of light tends to infinity, the Carroll symmetry corresponds to the vanishing velocity of light. Recently, the Carroll symmetry has become very popular and has found some interesting and unexpected applications [14–17].

In the rest of the Introduction, we shall describe briefly the main ideas of the two-time physics and of the Carroll symmetry. In the second section, we describe the classical theory of the Carroll particle arising from two-time spacetime. The third section is devoted to its quantum theory, while the last section contains some concluding remarks.

From the point of view of the 2T Physics, usual physical systems living in a one-time world represent projections from the spacetime with one additional temporal dimension and one additional spatial dimension. These additional dimensions are introduced to construct a new gauge theory, based on the localization of the phase-space symmetry described by the symplectic group $\mathrm{Sp}(2,\mathbb{R})$. The usual physics with 1T is obtained by means of the gauge fixing.

The phase-space coordinates for the two-time world have form

$$X^M=\left(X^{0'},X^{1'},X^{\mu}\right),\quad P^M=\left(P^{0'},P^{1'},P^{\mu}\right).$$

Here, the indices $0'$ and $1'$ label an extra time and an extra space dimensions. The extra space dimension is necessary to get the right number of degrees of freedom in the 1T theory. The index $\mu=0,...,d-1$ labels the usual coordinates in one-time world. It is convenient to write

$$X_i^M=\left(X^M,P^M\right),$$

where $i=1,2$ labels mean the position and momentum respectively.

The two types of phase variables can be mixed through $\mathrm{Sp}(2,\mathbb{R})$ transformations.

The worldline action for a free particle in a flat two-time spacetime

$$S=\frac{1}{2}\int d\tau\ \epsilon^{ij}\eta_{MN}\ \partial_\tau X_i^M X_j^N,$$

where $\eta_{MN}=\mathrm{Diag}(-1,1,-1,1,...,1)$ is the flat metric, with signature $(2,d)$, and ϵ^{ij} is the antisymmetric tensor with $\epsilon^{12}=1$, τ is a proper time parameter.

The action is invariant under the global $\mathrm{Sp}(2,\mathbb{R})$ transformations

$$\delta_\omega X_i^M=\epsilon_{ij}\omega^{jk}X_k^M.$$

The transformation parameters ω^{jk} are symmetric in j,k. When $\omega^{ij}\to\omega^{ij}(\tau)$, we need to introduce a connection that accounts for the new gauge symmetry. The covariant derivative is

$$\partial_\tau X_i^M\to D_\tau X_i^M=\partial_\tau X_i^M-\epsilon_{ij}A^{jk}(\tau)X_k^M,$$

where $A^{jk}(\tau)$ is symmetric in the indices i,j and belongs to the adjoint representation of the Lie algebra of $\mathrm{Sp}(2,\mathbb{R})$ (that we call $\mathfrak{sp}(2,\mathbb{R})$). It transforms as a gauge field under the $\mathrm{Sp}(2,\mathbb{R})$ group:

$$\delta_\omega A^{ij}(\tau)=\partial_\tau\omega^{ij}+\omega^{ik}\epsilon_{kl}A^{lj}+\omega^{jk}\epsilon_{kl}A^{li}.$$

The worldline action invariant under these gauge transformations is

$$S=\frac{1}{2}\int d\tau\ \epsilon^{ij}\eta_{MN}D_\tau X_i^M X_j^N=$$
$$=\int d\tau\ \left[\eta^{MN}\partial_\tau X_M P_N-A^{ij}(\tau)Q_{ij}\right],$$

where

$$Q_{11}=\frac{1}{2}X\cdot X,\quad Q_{22}=\frac{1}{2}P\cdot P,$$
$$Q_{12}=Q_{21}=\frac{1}{2}X\cdot P$$

are the $\mathfrak{sp}(2,\mathbb{R})$ conserved currents of constraints.

The gauge fields A^{ij} are not dynamical and play the role of Lagrange multipliers. When a gauge is chosen, the following constraints must be satisfied

$$X\cdot X=0,$$
$$X\cdot P=0,$$
$$P\cdot P=0.$$

These constraints lead to a non-trivial parameterization of the 1T spacetime, only when the starting theory has more than one timelike dimension.

When the gauge is fixed and the constraints are satisfied, one gets the right number of 1T variables $X_i^M(\tau)=X_i^M(\mathbf{x}(\tau),\mathbf{p}(\tau))$. The action is

$$S=\int d\tau\,(\dot{\mathbf{x}}\cdot\mathbf{p}-H),$$

where H is the Hamiltonian of the 1T theory. Different gauge fixings correspond to different choices of the Hamiltonian (and different choices of the time). Thus, the different systems in the 1T physics are described by a unique two-time model. These systems are dual to each other under local $\mathrm{Sp}(2,\mathbb{R})$ transformations.

Let us describe now the Carroll symmetry. It is well known that the Poincaré group possesses the contraction, obtained by sending the speed of light to infinity $c \to \infty$. This limit leads to the Galilean group that describes non-relativistic dynamics.

What does happen when we consider the opposite limit: $c \to 0$? Let us define the new variables:

$$t = \frac{1}{c}x_0, \quad \hat{\mathbf{v}} = \frac{1}{c}\boldsymbol{\beta}, \quad b = \frac{1}{c}a_0,$$

requiring that they remain constant after c is sent to zero. We get the following transformations

$$\begin{cases} t' = t + \hat{\mathbf{v}} \cdot (R\mathbf{x}) + b, \\ \mathbf{x}' = R\mathbf{x} + \mathbf{a}. \end{cases}$$

These transformations form the Carroll group [11,13].

The Carroll Lie algebra has the following form:

$$[L^{ij}, L^{kl}] = \delta^{ik}L^{jl} + \delta^{jl}L^{ik} - \delta^{il}L^{jk} - \delta^{jk}L^{il},$$
$$[L^{ij}, P^k] = \delta^{ik}P^j - \delta^{jk}P^i,$$
$$[L^{ij}, B^k] = \delta^{ik}B^j - \delta^{jk}B^i,$$
$$[L^{ij}, H] = 0,$$
$$[P^i, P^j] = 0,$$
$$[P^i, B^j] = \delta^{ij}H,$$
$$[P^i, H] = 0,$$
$$[B^i, B^j] = 0,$$
$$[B^i, H] = 0,$$

where L^{ij} are spatial rotations, P^k are spatial translations, B^k are boosts and H is the generator of the time translations.

2. Carroll Particle in the Two-Time Spacetime: Classical Theory

It is known (see, e.g., [17]) that the Carroll particle with non-zero energy should be always in rest. On the other hand, the Carroll particle with zero energy is always moving. These two cases are not connected. The reason for this phenomenon consists in the fact that the Carroll boosts do not change the value of the energy in contrast to the Lorentzian and Galilean boosts. Indeed, the Lie algebra generators corresponding to the Lorentz boosts have the form:

$$B^x_{\text{Lorentz}} = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t},$$

the Galilean boosts can be represented by the generators

$$B^x_{\text{Galilei}} = t\frac{\partial}{\partial x},$$

while the Carroll boosts are

$$B^x_{\text{Carroll}} = x\frac{\partial}{\partial t}.$$

The Hamiltonian is always proportional to the operator

$$H = \frac{\partial}{\partial t}.$$

The Carroll boost commutes with the Hamiltonian in contrast to the Lorentz boost and Galilei boost. One cannot change the value of the energy making a boost in the Carroll world and should treat the cases of the vanishing and non-vanishing energy separately. The conservation of the energy-momentum tensor yields the disappearance of the flux of energy, if the energy is different from zero. The action for the Carroll particle can be represented as

$$S = -\int d\tau \left\{\dot{t}E - \dot{x} \cdot p - \lambda\left(E - E_0\right)\right\},$$

where τ is the proper time, t is the physical time, E represents the classical Hamiltonian and x^i and p^i are the space coordinates and the momenta, for $i = 1, ..., d-1$. Then $E_0 \neq 0$ represents the rest energy of the Carroll particle, and λ plays the role of a Lagrange multiplier. This action is invariant under the transformations generated by

$$L^{ij} = x^i p^j - x^j p^i, \quad B^i = Ex^i, \quad p^i \text{ and } E.$$

The Poisson brackets of these generators satisfy the Carroll algebra. The equations of motion are

$$\dot{t} = \lambda, \quad \dot{E} = 0,$$
$$\dot{x}^i = 0, \quad \dot{p}^i = 0.$$

We would like to obtain this action from the 2T action. Let us introduce the light cone coordinates

$$X^+ = \frac{1}{2}\left(X^{1'} + X^{0'}\right), \quad X^- = \frac{1}{2}\left(X^{1'} - X^{0'}\right).$$

We fix the gauge fields as

$$A_{11} = A_{12} = 0, \quad A_{22} = \lambda = \text{const}.$$

The two-time coordinate and momenta are

$$\begin{aligned}
&X^+ = E_0 t,\\
&X^- = \frac{x_i p^i}{E_0} + \frac{t}{E_0}\left(E - E_0 + \frac{p_i p^i}{2}\right),\\
&X^0 = \sqrt{x_i x^i},\\
&X^i = x^i + t p^i,\\
&P^+ = E_0,\\
&P^- = \frac{1}{E_0}\left(E - E_0 + \frac{p_i p^i}{2}\right),\\
&P^0 = 0,\\
&P^i = p^i.
\end{aligned}$$

In terms of this parametrization (gauge-fixing) the constraints have the form

$$\begin{aligned}
&X \cdot X = -2t^2(E - E_0),\\
&X \cdot P = -2t(E - E_0),\\
&P \cdot P = -2(E - E_0)
\end{aligned}$$

and are satisfied, if and only if $E = E_0$.

Substituting the above parametrization into the two-time action, we obtain one-time action for a Carroll particle. The system possesses also the symmetry with respect to $SO(2, d)$ two-time Lorentz group. The generators of the group $SO(2, d)$ are

$$L^{MN} = X^M P^N - X^N P^M,$$

and they are invariant under $Sp(2, \mathbb{R})$ transformations.

Written in terms of our reparametrization, they have the form

$$\begin{aligned}
&L^{ij} = x^i p^j - x^j p^i,\\
&L^{0i} = \sqrt{x^j x_j}\; p^i,\\
&L^{+i} = -E_0 x^i\\
&L^{-i} = -\frac{E - E_0}{E_0} x^i - \frac{p_j p^j}{2E_0} x^i + \frac{p^j x_j}{E_0} p^i,\\
&L^{+-} = -p^i x_i,\\
&L^{-0} = -\sqrt{x_i x^i}\left(\frac{E - E_0}{E_0} + \frac{p_i p^i}{2E_0}\right),\\
&L^{+0} = -E_0 \sqrt{x_i x^i}.
\end{aligned}$$

The Poisson brackets of these generators do not form an $\mathfrak{so}(2, d)$ algebra, unless the constraint $E - E_0 = 0$ is satisfied. A direct computation shows that

$$\{L^{-i}, L^{-j}\} = -2\frac{E - E_0}{E_0} L^{ij},$$

which is a new element of the algebra. When $E - E_0 = 0$, these Poisson brackets vanish, and all the generators form the $\mathfrak{so}(2, d)$ algebra, described by

$$\{L^{MN}, L^{RS}\} = \eta^{MR} L^{NS} + \eta^{NS} L^{MR} - \eta^{MS} L^{NR} - \eta^{NR} L^{MS}.$$

Concluding this section, we write down (for comparison) the two-time parametrization which produces the one-time action for a non-relativistic massive particle, following the paper [5]:

$$\begin{aligned}
&X^+ = t,\\
&X^- = \frac{\mathbf{r} \cdot \mathbf{p}}{m} - t\frac{H}{m},\\
&X^0 = \sqrt{\mathbf{r}^2 - 2t\frac{\mathbf{r} \cdot \mathbf{p}}{m} + 2\frac{H}{m} t^2},\\
&X^i = \mathbf{r}^i,\\
&P^+ = 0,\\
&P^- = H,\\
&P^0 = 0,\\
&P^i = \mathbf{p}^i.
\end{aligned}$$

One can see that the finding of a proper parametrization for the transition from two-time spacetime to a relatively simple one-time system is not an easy task, and the results are not obvious.

3. Carrol Particle in the Two-Time Spacetime: Quantum Theory

Let us try to construct the quantum theory of a Carroll particle following the recipe presented in the paper [4]. The commutation relations for the position and momentum operators in the standard $(d - 1)$-dimensional space are

$$[x^i, p^j] = i\,\delta^{ij}.$$

When we quantize some classical functions of these operators, the problem of the choice of the ordering arises. All the operators should be Hermitian, but this requirement is not sufficient. For example,

$$p^2 r \to p_i r p^i,$$

which is clearly Hermitian. However, this ordering is not unique ordering providing the Hermiticity. We can choose another form of the operator

$$p^2 r \to r p_i r^{-1} p^i r = p_i r p^i - \frac{d-3}{2r}.$$

In a more general case

$$p^2 r \to r^\lambda p_i r^{1-2\lambda} p^i r^\lambda = p_i r p^i + \frac{\lambda(\lambda - d + 2)}{2r}.$$

We have to resort to the covariant quantization in the 2T spacetime. That means that the generators L^{MN} which become operators should constitute the Lie algebra with respect to the commutators.

This requirement also does not define the ordering in the quantum generators in a unique way, and one should use also the properties of the Casimir operators of the unitary representations of both the groups $SO(2,d)$ and $\mathrm{Sp}(2,\mathbb{R})$. The constraints play the role of the generators of the symmetry with respect to the $\mathrm{Sp}(2,\mathbb{R})$ group. They should be applied to the acceptable quantum states of the system according to the prescription of the Dirac quantization of systems with first-class constraints:

$$Q|\Psi\rangle = 0.$$

The same should be valid also for the Casimir operators.

If we choose the basis of the Hermitian quantum generators of the $\mathrm{Sp}(2,\mathbb{R})$ group as

$$J_0 = \frac{1}{4}(P^2 + X^2),\ J_1 = \frac{1}{4}(P^2 - X^2),$$
$$J_2 = \frac{1}{4}(X \cdot P + P \cdot X),$$

then

$$[J_0, J_1] = iJ_2,\ [J_0, J_2] = -iJ_1,$$
$$[J_1, J_2] = -iJ_0.$$

The quadratic Casimir operator is defined as

$$C_2(\mathrm{Sp}(2,\mathbb{R})) = J_0^2 - J_1^2 - J_2^2.$$

Using the commutation rules

$$[X^M, P^N] = i\eta^{MN},$$

we can show that

$$C_2(\mathrm{Sp}(2,\mathbb{R}) = \\ = \frac{1}{4}\left(X^M P^2 X_M - (X \cdot P)(P \cdot X) + \frac{d^2}{4} - 1\right).$$

On the other hand, we define the quadratic Casimir operator for the $SO(2,d)$ group as

$$C_2(SO(2,d)) = \frac{1}{2} L_{MN} L^{MN},$$

and the direct calculation shows for that

$$C_2(SO(2,d)) = 4C_2(\mathrm{Sp}(2,\mathbb{R}) + 1 - \frac{d^2}{4}.$$

If the generators of the $\mathrm{Sp}(2,\mathbb{R})$ select quantum states, and their quadratic Casimir operator should be equal to zero, the quadratic Casimir operator on the same quantum states treated as belonging to a representation of the $SO(2,d)$ group should be equal to $1 - \frac{d^2}{4}$. It is this requirement that fixes the ordering in the generators of the $SO(2,d)$ group. In paper [4], this technique was implemented to reproduce the quantization scheme and the spectrum for the hydrogen-like atom. It is well known that, at the dawn of the development of quantum mechanics in twenties, this spectrum was obtained by two different methods: by solution of the corresponding Schrödinger equation and by purely algebraic method, using the enlarged group symmetry of the system. Remarkably, using the technique, sketched above, the author of [4] has managed to generalize the standard formulae for the hydrogen spectrum for the case of the $((d-1)+1)$-dimensional spacetime, starting from the $(d+2)$-dimensional spacetime.

The procedure of fixing of the ordering of the operators at the transition from a classical theory to its quantum counterpart has an interesting feature: if we manage to fix the ordering in the generators of $SO(2,d)$ group at an initial moment $\tau = 0$, then the same ordering will be conserved. Here, we can note rather an unusual fact: our parametrization of the variables X^M, P^M at $\tau = 0$ coincides with that used for the description of the hydrogen atom in [4] provided we have already put $E = E_0$. It is amazing, because these physical systems are quite different, and their actions are also different. Indeed, when we calculate a one-time action starting from a certain parametrization of the two-time variables, the derivatives with respect to the proper time parameter come into the play. Thus, while we calculate the action, the time dependence present in the time parametrization or, in other words, in the gauge-fixing procedure, is important. However, for the derivation of the rules of quantization the time dependence is not important. That means that having two quite different parametrizations, which coincide at some moment, we can use similar quantization procedures for quite different systems. Thus, we can use the results of the paper [4] to quantize our Carroll particle in rest. Here, we give the formulae which explicitly show how the ordering of the operators in the quantum generators of the group $SO(2,d)$ is chosen:

$$L^{ij} = x^i p^j - x^j p^i,$$
$$L^{0i} = \frac{1}{2}\left(r\ p^i + p^i r\right),$$
$$L^{+i} = -E_0 x^i,$$
$$L^{-i} = -\frac{1}{2E_0} p_j x^i p^j + \frac{1}{2E_0}\left(p \cdot x p^i + p^i x \cdot p\right) + \frac{x^i}{8E_0 r^2},$$
$$L^{+-} = -\frac{1}{2}\left(x \cdot p + p \cdot x\right),$$
$$L^{-0} = -\frac{1}{2E_0} p_i r p^i - \frac{5-2d}{8E_0 r},$$
$$L^{+0} = -E_0 r.$$

One can have an impression that the results of such a quantization can be rather counterintuitive. Indeed, the spectrum of the hydrogen atom is discrete, and the Carroll particle in rest does not look as a system which can have a discrete spectrum. However, the fact that we use the correspondence between these two quantum systems does not mean that we shall obtain the discrete spectrum for a Carroll particle in rest. Indeed, the quantization procedure tells us that the combination of the squared momentum and the inverse radius, which gives the Hamiltonian of the hydrogen atom, has a discrete spectrum. However, the same combination does not represent the Hamiltonian of the Carroll particle. Indeed, this Hamiltonian is disconnected from both the spatial coordinates and momenta. Moreover, the momentum is not connected with the velocity of the Carroll particle (which is always equal to zero).

What is the role of the momentum? As for standard quantum mechanical systems, the momenta enter into the commutation relations, with the coordinates and, hence, the Heisenberg inequality of uncertainties

$$\Delta x^i \cdot \Delta p^j \geq \frac{1}{4}\delta^{ij}$$

is valid. That implies the appearance of another question: does it mean that we are unable to localize a Carroll particle in rest? Again, the answer is negative. In contrast to the standard non-relativistic quantum mechanics, we can choose the quantum states with a dispersion of the coordinate Δx as small as we wish, because the growth of the dispersion of the momentum Δp is not important. Thus, a particle can be localized with an arbitrary high precision.

4. Concluding Remarks

The Carroll group and Carroll symmetry have acquired a great popularity during last years. Their properties rather interesting from a mathematical point of view have found numerous and sometimes unexpected physical applications. The two-time physics is less known, but its results are also very interesting, because they permit one to see quite different physical systems and phenomena from a unified point of view. As far as we know, there were no studies devoted to a description of Carroll particles in the two-time physics. We have made such an attempt in our paper [10]. We have limited ourselves by investigation of a relatively simple case of Carroll particles which have non-vanishing energy and should stay at rest in a $(d-1)+1$ spacetime. For this case, we have found such gauge-fixing conditions in the enlarged $d+2$ dimensional spacetime (which possesses two-time variables) which together with the constraints of the theory give a parametrization of the phase space variables in the enlarged spacetime that produces Carroll particle in the standard one-time spacetime.

Remarkably, if we treat our phase variables as the quantum operators, then, at the moment, when the proper time parameter is equal to zero, our parametrization coincides with that obtained in [4] for the hydrogen atom up to some coefficients. That permits us to follow the quantization scheme developed in [4] for this case. The equations which we obtain are very close to those obtained there, while their physical sense and interpretation are quite different. The roots of this difference lie in the fact that our Hamiltonian does not depend neither on the momenta nor on the coordinates of the system. Moreover, the momenta are not connected with the velocities in contrast with the traditional formulas to which one is accustomed working with Lorentz or Galilei symmetric systems. The role of momenta consists in the fact that they obey the standard commutation relations with the operators of position and, hence, the Heisenberg indeterminacy inequality is valid. However, the role of the dispersions of the coordinates and of the dispersions of the momenta are different. While the dispersion of the coordinate characterizes the localization of the particle, the dispersion of the momenta does not have a direct physical sense, and one can choose the physical state with an arbitrary high degree of the space localization which is compatible with the fact that classical Carroll particle should always stay in rest. All said above concerns the particles with a non-zero value of energy. The case with zero energy, when the Carroll particles are always in motion, is more complicated, and we hope to present an analysis of this case in the future. Another possible and interesting direction of the research is to look for field-theoretical systems with the Carroll symmetry in the two-time world.

Concluding we would like to say that it was a special pleasure to present our work at the XII Bolyai–Gauss–Lobachevsky conference (BGL-2024): Non-Euclidean Geometry in Modern Physics and Mathematics. We have tried in our work to combine two rather unusual spacetime geometries: the geometry with the spacetime which has more than one time direction and the geometry, where instead of Galilei or Poincaré symmetry, the much more counter-intuitive Carroll symmetry is present. At first glance, these two geometries are not so close to works made by Bolyai, Gauss, and Lobachevsky. However, both these spacetime geometries hardly could be invented, if the idea of the possibility of the existence of the non-Euclidean geometries were not discovered, elaborated, and defended by these scientists. We know that, in the nineteenth century, the acceptance of the non-Euclidean geometry was not easy and the destinies of Lobachevsky and, especially, Bolyai were difficult (see, e.g., the book [18]). We can believe that it is due their efforts, now we can easily treat not only different kinds of Riemannian geometries, but also the spacetimes with torsion, Finsler geometries, and some even more exotic geometries, part of which was mentioned during this conference. Coming back to the topic of our presentation, we can say that the geometry of the spacetime with two time-like directions can hardly be reconciled with our intuition. Indeed, it seems quite obvious that the time is something linear and one-dimensional. On the other hand, in the framework of modern physics, specially in quantum gravity and cosmology, one encounters the so-called "problem of time". Indeed, due to the reparametrization invariance of the general relativity theory, the Hamiltonian of a closed universe is proportional to a linear combination of the first class constraints. Then, if we implement the Dirac procedure of quantization of the system with constraints, we see that applying the Hamiltonian to an accepted quantum state of the universe, we obtain zero. It gives us the famous Wheeler–DeWitt equation [19], which can be seen as a Schrödinger equation without time. The time disappears and to force it emerge again, one has to apply rather an involved procedure of gauge-fixing, separation of degrees of freedom, playing the role of time parameter, construction of the Hilbert space with an inner product, which provides a reasonable probability interpretation of the reduced wave function, *etc.* We have mentioned this problem to show that the situations, when the time disappears and then appears again is not so unusual in modern physics. Moreover, even in non-relativistic quantum mechanics, one can encounter this problem in a similar form (see, e.g., the book [20]). Indeed, if we have a closed quantum system which finds itself in an energy eigenstate, the time parameter enters into the solution of the corresponding Schrödinger equation as a simple phase factor $\exp(-iEt/\hbar)$, and it seems that the evolution disappears. Thus, one should make the procedures similar to those applied in quantum cosmology to make the time reappear. In this case, we see some transitions between the physics without time dimensions and the physics in one-time dimension. From this point of view, the transition between the world with two time dimensions and the world with one time dimension does not look so bizzare.

As far as concerned the history of the Carroll symmetry, it is described in the recent preprint by Lévy-Leblond [21]. The physics of the world, where the velocity of light is equal to zero, was considered as a mathematical curiosity for a long time, and then useful applications of this idea were discovered and studied. We can add here that, in different branches of physics, one encounters the situations, when the velocity of transmission of signals is smaller than the velocity of particles and the Carrol world can represent an extremal case of such a phenomenon.

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А. Каменщик, Ф. Мусколіно

ПОШУК ЧАСТИНОК КЕРРОЛЛА
У ДВОЧАСОВОМУ ПРОСТОРІ-ЧАСІ

Ми намагаємось описати частинки Керролла з ненульовим значенням енергії (тобто частинки Керролла, які завжди перебувають у спокої) у рамках двочасової фізики, розробленої в серії робіт І. Барса і його співавторів. У просторі-часі з одним додатковим виміром часу і одним додатковим виміром простору можна локалізувати симетрію, яка існує між узагальненими координатами та їх сполученими імпульсами. Така локалізація передбачає введення калібрувальних полів, що, у свою чергу, передбачає появу деяких обмежень першого класу. Вибираючи різні умови калібрувальної фіксації і вирішуючи обмеження, можна отримати різні параметри часу, гамільтоніани і загалом фізичні системи в стандартному одночасовому просторі-часі. Ми знаходимо набір умов фіксації калібрування, який відображає частинки Керролла в одночасовому світі. Крім того, ми будуємо квантову теорію такої частинки, використовуючи несподівану відповідність між нашою параметризацією і параметризацією, отриманою Барсом для атома водню в 1999 році.

Ключові слова: двочасовий простір-час, група Керролла, частинки.