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DEVELOPMENT AND ANALYSIS OF NOVEL INTEGRABLE NONLINEAR DYNAMICAL SYSTEMS ON QUASI-ONE-DIMENSIONAL LATTICES. PARAMETRICALLY DRIVEN NONLINEAR SYSTEM OF PSEUDO-EXCITATIONS ON A TWO-LEG LADDER LATTICE

Following the main principles of developing the evolutionary nonlinear integrable systems on quasi-one-dimensional lattices, we suggest a novel nonlinear integrable system of parametrically driven pseudo-excitations on a regular two-leg ladder lattice. The initial (prototype) form of the system is derivable in the framework of semi-discrete zero-curvature equation with the spectral and evolution operators specified by the properly organized 3 × 3 square matrices. Although the lowest conserved local densities found via the direct recursive method do not prompt us the algebraic structure of system's Hamiltonian function, but the heuristically substantiated search for the suitable two-stage transformation of prototype field functions to the physically motivated ones has allowed to disclose the physically meaningful nonlinear integrable system with time-dependent longitudinal and transverse inter-site coupling parameters. The time dependencies of inter-site coupling parameters in the transformed system are consistently defined in terms of the accompanying parametric driver formalized by the set of four homogeneous ordinary linear differential equations with the time-dependent coefficients. The physically meaningful parametrically driven nonlinear system permits its concise Hamiltonian formulation with the two pairs of field functions serving as the two pairs of canonically conjugated field amplitudes. The explicit example of oscillatory parametric drive is described in full mathematical details.

Keywords: nonlinear dynamics, integrable system, two-leg ladder lattice, parametric drive, Hamiltonian dynamics.

1. Introduction

Since the middle of the last century the trend to switch the physical and mathematical consideration of multi-component physical systems beyond the purely linear descriptions became more and more pronounced. Here it is worth noticing the pioneering nonlinear approach initiated by Landau and Pekar on the polaron theory [1] as well as the near nonlinear consideration suggested by Bogolyubov on the adiabatic perturbation theory in the problem of particle interaction with a quantum field [2]. These and a number of other forthcoming researches [3–6] have given rise to the very generative concept of completely integrable nonlinear Schrödinger models both in their differential-differential (continuous) [7–9] and differential-difference (semi-discrete) [9–16] embodiments.

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Due to the recent technological progress in synthesizing the low-dimensional nanoscale superstructures [17–21], treated as metamaterials prospective for the microelectronic devices, we believe the semi-discrete completely integrable models of nonlinear excitations and nonlinear pseudo-excitations on quasi-one-dimensional lattices will be in a considerable applicative demand.

In the present paper, we suggest a novel parametrically driven nonlinear integrable system of pseudo-excitations on a two-leg ladder lattice. To proceed with this task, we rely upon the basic principles of developing the semi-discrete nonlinear integrable systems on quasi-one-dimensional lattices, scrupulously listed in our previous article [22]. Additionally, we outline the main steps for generating the infinite hierarchy of local conservation laws in terms of direct recurrence technique [13, 23–26]. The heuristic peculiarities concerning the proper adjustment of originally unfixed sampling functions are also elucidated.

It is remarkable, that the ultimate form of developed semi-discrete nonlinear integrable system admits a concise Hamiltonian dynamical formulation characterized by the standard Poisson bracket. However, the very procedure of standardization turned out to be absolutely distinct from that related to the integrable nonlinear Schödinger system on a two-leg ladder lattice with the background-controlled intersite resonant coupling [27, 28].

2. Matrix-Valued Auxiliary Linear Problem and the Appropriate Ansätze for the Spectral and Evolution Matrices

Following the general rules of developing the integrable semi-discrete nonlinear systems on regular quasi-one-dimensional lattices [12, 22, 29], let us start with the set of two auxiliary matrix-valued equations

$$X(n+1|\lambda) = L(n|\lambda)X(n|\lambda) \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}X(n|\lambda) = A(n|\lambda)X(n|\lambda), \tag{2.2}$$

that are linear with respect to the auxiliary matrix-function $X(n|\lambda)$. Here, the symbol n denotes the discrete spatial coordinate variable running from minus infinity to plus infinity. The symbol τ stands for the continuous time variable. The symbol λ marks the time-independent spectral parameter. For our present purpose, all three involved entities $X(n|\lambda)$, $L(n|\lambda)$, $A(n|\lambda)$ are assumed to be 3×3 square matrices. The spectral equation (2.1) is governed by the spectral operator $L(n|\lambda)$, while the evolutionary equation (2.2) is governed by the evolution operator $A(n|\lambda)$.

The set of auxiliary linear equations (2.1)–(2.2) is overdetermined. To ensure the compatibility of this overdetermined set, the operation of differentiation over the time variable τ and the operation of shifting along the spatial variable n as applied to the auxiliary matrix-function $X(n|\lambda)$ within the auxiliary linear set (2.1)–(2.2) must commutate, *i.e.* [12]

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau}X(m|\lambda)\right]_{m=n+1} = \frac{\mathrm{d}}{\mathrm{d}\tau}X(n+1|\lambda). \tag{2.3}$$

As a consequence of such a commutative procedure, we come to the matrix-valued semi-discrete zero-curvature condition [29, 30]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}L(n|\lambda) = A(n+1|\lambda)L(n|\lambda) - L(n|\lambda)A(n|\lambda) \tag{2.4}$$

on the permissible forms of spectral $L(n|\lambda)$ and evolution $A(n|\lambda)$ operators.

Below, we propose one of successful variants invented to satisfy the zero-curvature condition (2.4). Namely, our suggestion is based upon the following ansätze

$$L(n|\lambda) = \begin{pmatrix} f_{11}(n) & f_{12}(n) & f_{13}(n) \\ f_{21}(n) & f_{22}(n)+\lambda & f_{23}(n) \\ f_{31}(n) & f_{32}(n) & f_{33}(n) \end{pmatrix} \tag{2.5}$$

and

$$A(n|\lambda) = \begin{pmatrix} a_{11}(n) & a_{12}(n) & a_{13}(n) \\ a_{21}(n) & a_{22}(n)+b_{22}(n)\lambda & a_{23}(n) \\ a_{31}(n) & a_{32}(n) & a_{33}(n) \end{pmatrix} \tag{2.6}$$

for the spectral $L(n|\lambda)$ and evolution $A(n|\lambda)$ operators under the nonsingularity condition $\det L(n|\lambda) \neq$ $\neq 0$. Here, the space-and-time-dependent ingredients $f_{jk}(n) = f_{jk}(n|\tau)$ of the spectral matrix $L(n|\lambda)$ should be treated as the prototype field functions of future nonlinear integrable system encoded in the zero-curvature equation (2.4). On the other hand, the space-and-time-dependent ingredients $a_{jk}(n) =$ $= a_{jk}(n|\tau)$ and $b_{22}(n) = b_{22}(n|\tau)$ of the evolution matrix $A(n|\lambda)$ must be specifiable relying upon the zero-curvature condition (2.4) and on certain reasonable assumptions about the physically meaningful local conservation laws.

3. Integrable Semi-Discrete Nonlinear System in Terms of Prototype Field Functions

Thus, having inserted the above suggested ansätze (2.5)–(2.6) into the zero-curvature equation (2.4) and having collected the terms with the same powers of the spectral parameter λ within each of equation's matrix element, we are able to specify six ingredients

$$a_{22}(n) = a_{22}, \tag{3.1}$$

$$b_{22}(n) = b_{22}, \tag{3.2}$$

$$a_{21}(n) = b_{22} f_{21}(n), \tag{3.3}$$

$$a_{12}(n+1) = f_{12}(n) b_{22}, \tag{3.4}$$

$$a_{23}(n) = b_{22} f_{23}(n), \tag{3.5}$$

$$a_{32}(n+1) = f_{32}(n) b_{22} \tag{3.6}$$

of the evolution matrix $A(n|\lambda)$. Here, each of parameters a_{22} and b_{22} can be time-dependent. Another four ingredients $a_{11}(n)$, $a_{13}(n)$, $a_{33}(n)$, $a_{31}(n)$, referred to as the sampling functions, remain unfixed for the time being. In addition, we recover the set of nine semi-discrete nonlinear equations

$$\dot{f}_{11}(n) = a_{11}(n+1) f_{11}(n) + a_{13}(n+1) f_{31}(n) - \\ - f_{11}(n) a_{11}(n) - f_{13}(n) a_{31}(n), \tag{3.7}$$

$$\dot{f}_{13}(n) = a_{11}(n+1) f_{13}(n) + a_{13}(n+1) f_{33}(n) - \\ - f_{11}(n) a_{13}(n) - f_{13}(n) a_{33}(n), \tag{3.8}$$

$$\dot{f}_{33}(n) = a_{31}(n+1) f_{13}(n) + a_{33}(n+1) f_{33}(n) - \\ - f_{31}(n) a_{13}(n) - f_{33}(n) a_{33}(n), \tag{3.9}$$

$$\dot{f}_{31}(n) = a_{31}(n+1) f_{11}(n) + a_{33}(n+1) f_{31}(n) - \\ - f_{31}(n) a_{11}(n) - f_{33}(n) a_{31}(n), \tag{3.10}$$

$$\dot{f}_{22}(n) = b_{22} f_{21}(n+1) f_{12}(n) + b_{22} f_{23}(n+1) f_{32}(n) - \\ - f_{21}(n) f_{12}(n-1) b_{22} - f_{23}(n) f_{32}(n-1) b_{22}, \tag{3.11}$$

$$\dot{f}_{21}(n) = b_{22} f_{21}(n+1) f_{11}(n) + a_{22} f_{21}(n) + \\ + b_{22} f_{23}(n+1) f_{31}(n) - f_{21}(n) a_{11}(n) - \\ - f_{22}(n) b_{22} f_{21}(n) - f_{23}(n) a_{31}(n), \tag{3.12}$$

$$\dot{f}_{12}(n) = a_{11}(n+1) f_{12}(n) + f_{12}(n) b_{22} f_{22}(n) + \\ + a_{13}(n+1) f_{32}(n) - f_{11}(n) f_{12}(n-1) b_{22} - \\ - f_{12}(n) a_{22} - f_{13}(n) f_{32}(n-1) b_{22}, \tag{3.13}$$

$$\dot{f}_{23}(n) = b_{22} f_{21}(n+1) f_{13}(n) + a_{22} f_{23}(n) + \\ + b_{22} f_{23}(n+1) f_{33}(n) - f_{21}(n) a_{13}(n) - \\ - f_{22}(n) b_{22} f_{23}(n) - f_{23}(n) a_{33}(n), \tag{3.14}$$

$$\dot{f}_{32}(n) = a_{31}(n+1) f_{12}(n) + f_{32}(n) b_{22} f_{22}(n) + \\ + a_{33}(n+1) f_{32}(n) - f_{31}(n) f_{12}(n-1) b_{22} - \\ - f_{32}(n) a_{22} - f_{33}(n) f_{32}(n-1) b_{22}, \tag{3.15}$$

referred to as the prototype semi-discrete nonlinear integrable system of our interest. The overdot in each of above written equations (3.7)–(3.9) stands for the differentiation with respect to the time variable τ.

In view of its representability in a concise matrix-valued form of zero-curvature equation (2.4) the obtained semi-discrete nonlinear system (3.7)–(3.9) acquires the status of a system integrable in the Lax sense. As a rule, this fact also supports the integrability of a semi-discrete nonlinear system in the Liouville sense [30]. At any rate, the Lax integrability ensures the strict methods for obtaining the system's exact analytic solutions, as well as for generating an infinite hierarchy of local conservation laws.

4. Main Steps in Generating the Local Conservation Laws

By the definition, any particular local conservation law related to a certain semi-discrete system on a quasi-one-dimensional regular lattice is writable in the following form:

$$\dot{\rho}(n) = J(n) - J(n+1), \tag{4.1}$$

where the functions $\rho(n) = \rho(n|\tau)$ and $J(n) = J(n|\tau)$ denote the local density and local current, respectively.

The most straightforward way to generate at least some of the local conservation laws for an integrable semi-discrete system on a quasi-one-dimensional lattice is based on the universal local conservation law

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \ln [\det L(n|\lambda)] = \mathrm{Sp} A(n+1|\lambda) - \mathrm{Sp} A(n|\lambda) \tag{4.2}$$

readily obtainable from the system's zero-curvature representation (2.4) by virtue of identity

$$\mathrm{Sp}\left(L^{-1} \frac{\mathrm{d}}{\mathrm{d}\tau} L\right) = \frac{\mathrm{d}}{\mathrm{d}\tau} \ln(\det L) \tag{4.3}$$

valid for any nonsingular ($\det L \neq 0$) square matrix L.

Thus, for the spectral $L(n|\lambda)$ and evolution $A(n|\lambda)$ operators specified by the earlier suggested formulas (2.5) and (2.6)–(3.6) the recipe based on the universal local conservation law (4.2) yields only two local conservation laws

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\left[W_0(n)\right] = a_{11}(n+1) + a_{22}(n+1) - \\ - a_{11}(n) - a_{22}(n), \tag{4.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\left[W_1(n)\right] = a_{11}(n+1) + a_{22}(n+1) - \\ - a_{11}(n) - a_{22}(n), \tag{4.5}$$

where the local densities $\ln[W_0(n)]$ and $\ln[W_1(n)]$ are decoded by the expressions

$$W_0(n) = f_{21}(n)f_{13}(n)f_{32}(n) + f_{23}(n)f_{31}(n)f_{12}(n) - \\ - f_{21}(n)f_{33}(n)f_{12}(n) - f_{23}(n)f_{11}(n)f_{32}(n) + \\ + [f_{11}(n)f_{33}(n) - f_{13}(n)f_{31}(n)]f_{22}(n) \tag{4.6}$$

and

$$W_1(n) = f_{11}(n)f_{33}(n) - f_{13}(n)f_{31}(n), \tag{4.7}$$

respectively.

Fortunately, there exist several technically different but basically equivalent systematic approaches for generating the hierarchy of local conservation laws recursively [13, 23–26, 28] without any reference on the scattering data of auxiliary spectral problem, as well as on the Hamiltonian structure underlying the hierarchy of integrable systems linked with the adopted spectral operator.

For example, the first step of our own approach [23–25, 28] consists in recovering the adequate recursive representations for the auxiliary quantities $\Gamma_{jk}(n|\lambda)$ subjected to the following restrictions

$$\Gamma_{ji}(n|\lambda)\Gamma_{ik}(n|\lambda) = \Gamma_{jk}(n|\lambda). \tag{4.8}$$

To succeed, with this task, the auxiliary quantities $\Gamma_{jk}(n|\lambda)$ should be expanded in certain proper series with respect to spectral parameter λ or inverse spectral parameter $1/\lambda$. Then they should be inserted into the fundamental set of spatial Riccati equations

$$\Gamma_{jk}(n+1|\lambda)\sum_{i=1}^{3} L_{ki}(n|\lambda)\Gamma_{ik}(n|\lambda) = \\ = \sum_{i=1}^{3} L_{ji}(n|\lambda)\Gamma_{ik}(n|\lambda) \tag{4.9}$$

governed exclusively by the matrix elements $L_{jk}(n|\lambda)$ of the spectral operator $L(n|\lambda)$. Therefore, each expansion coefficient of any auxiliary quantity $\Gamma_{jk}(n|\lambda)$ is obliged to emerge as a certain combined expression consisting of prototype field functions.

Once the required precision in the recursive representations of the auxiliary quantities $\Gamma_{jk}(n|\lambda)$ has been achieved the obtained truncated series should be substituted into the collection of three ($j = 1, 2, 3$) generating equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\left[M_{jj}(n|\lambda)\right] = B_{jj}(n+1|\lambda) - B_{jj}(n|\lambda), \tag{4.10}$$

whose functional structures are seen to duplicate the functional structure of a typical local conservation law (4.1). For this reason, the quantities $M_{jj}(n|\lambda)$ and $B_{jj}(n|\lambda)$, defined by formulas

$$M_{jj}(n|\lambda) = \sum_{i=1}^{3} L_{ji}(n|\lambda)\Gamma_{ij}(n|\lambda) \tag{4.11}$$

and

$$B_{jj}(n|\lambda) = \sum_{i=1}^{3} A_{ji}(n|\lambda)\Gamma_{ij}(n|\lambda), \tag{4.12}$$

are served to generate the hierarchy of local densities and the hierarchy of local currents, respectively. Here, the quantities $A_{jk}(n|\lambda)$ stand for the matrix elements of evolution operator $A(n|\lambda)$.

Having collected the terms with the same powers of spectral parameter λ in each of three ($j = 1, 2, 3$) generating equations (4.10) it is possible to recover any required number of local conservation laws from their infinite hierarchy.

The application of the above described generating scheme to each of three generating equations (4.10) specified by the spectral $L(n|\lambda)$ and evolution $A(n|\lambda)$ matrices of our present interest (2.5) and (2.6)–(3.6) reveals that at least several lowest conserved densities related to the first ($j = 1$) generating equation and to the third ($j = 3$) generating equation are the conserved densities of rather trivial form $F(n+1) - F(n)$ absolutely useless from the physical point of view. For this reason, we trace here only the key calculations related to the second ($j = 2$) generating equation.

First of all, we observe that the explicit expressions (4.11) and (4.12) taken for the composite quantities $M_{22}(n|\lambda)$ and $B_{22}(n|\lambda)$ operate only with two unknown auxiliary quantities $\Gamma_{12}(n|\lambda)$ and $\Gamma_{32}(n|\lambda)$, inasmuch as $\Gamma_{22}(n|\lambda) = 1$ by virtue of fundamental restrictions (4.8). Hence, it is sufficient to deal recursively merely with two

$$\Gamma_{12}(n+1|\lambda) \times$$
$$\times [f_{21}(n)\Gamma_{12}(n|\lambda) + \lambda + f_{22}(n) + f_{23}(n)\Gamma_{32}(n|\lambda)] =$$
$$= f_{11}(n)\Gamma_{12}(n|\lambda) + f_{12}(n) + f_{13}(n)\Gamma_{32}(n|\lambda), \quad (4.13)$$
$$\Gamma_{32}(n+1|\lambda) \times$$
$$\times [f_{21}(n)\Gamma_{12}(n|\lambda) + \lambda + f_{22}(n) + f_{23}(n)\Gamma_{32}(n|\lambda)] =$$
$$= f_{31}(n)\Gamma_{12}(n|\lambda) + f_{32}(n) + f_{33}(n)\Gamma_{32}(n|\lambda) \quad (4.14)$$

of six original Riccati equations (4.9).

The set of two above written nonlinear Riccati equations (4.13)–(4.14) turns out to be solvable recursively by the use of following two expansions

$$\Gamma_{12}(n|\lambda) = \sum_{k=0}^{\infty} \gamma_{12}(n|k)\lambda^{-k-1} \quad (4.15)$$

and

$$\Gamma_{32}(n|\lambda) = \sum_{k=0}^{\infty} \gamma_{32}(n|k)\lambda^{-k-1} \quad (4.16)$$

for the two involved auxiliary quantities $\Gamma_{12}(n|\lambda)$ and $\Gamma_{32}(n|\lambda)$ with $|\lambda| \to \infty$. As a result of elementary algebraic calculations, the lowest expansion coefficients were found to be

$$\gamma_{12}(n|0) = f_{12}(n-1), \quad (4.17)$$
$$\gamma_{32}(n|0) = f_{32}(n-1), \quad (4.18)$$
$$\gamma_{12}(n|1) = f_{11}(n-1)f_{12}(n-2) +$$
$$+ f_{13}(n-1)f_{32}(n-2) - f_{12}(n-1)f_{22}(n-1), \quad (4.19)$$
$$\gamma_{32}(n|1) = f_{33}(n-1)f_{32}(n-2) +$$
$$+ f_{31}(n-1)f_{12}(n-2) - f_{32}(n-1)f_{22}(n-1). \quad (4.20)$$

Consequently, the generator $\ln[M_{22}(n|\lambda)]$ of the local densities, having being presented via the composite quantity $M_{22}(n|\lambda)$ (see formula (4.11)) with the use of expansions (4.15) and (4.16) for $\Gamma_{12}(n|\lambda)$ and $\Gamma_{32}(n|\lambda)$ supplemented by the explicit formulas (4.17)–(4.20) for the lowest expansion coefficients, yields the following expressions

$$\rho_{22}(n|1) = f_{22}(n), \quad (4.21)$$
$$\rho_{22}(n|2) = f_{21}(n)f_{12}(n-1) - \frac{1}{2}f_{22}(n)f_{22}(n) +$$
$$+ f_{23}(n)f_{32}(n-1), \quad (4.22)$$
$$\rho_{22}(n|3) = f_{21}(n)f_{11}(n-1)f_{12}(n-2) +$$
$$+ f_{21}(n)f_{13}(n-1)f_{32}(n-2) +$$
$$+ f_{23}(n)f_{33}(n-1)f_{32}(n-2) +$$
$$+ f_{23}(n)f_{31}(n-1)f_{12}(n-2) -$$
$$- f_{22}(n)[f_{21}(n)f_{12}(n-1) + f_{23}(n)f_{32}(n-1)] -$$
$$- [f_{21}(n)f_{12}(n-1) + f_{23}(n)f_{32}(n-1)]f_{22}(n-1) +$$
$$+ \frac{1}{3}f_{22}(n)f_{22}(n)f_{22}(n) \quad (4.23)$$

for the lowest local conserved densities $\rho_{22}(n|1)$, $\rho_{22}(n|2)$, $\rho_{22}(n|3)$. The status of these three quantities (4.21)–(4.23) as the local conserved densities has been verified by the direct calculation of their time derivatives $\dot{\rho}_{22}(n|1)$, $\dot{\rho}_{22}(n|2)$, $\dot{\rho}_{22}(n|3)$ with the use of semi-discrete nonlinear equations (3.7)–(3.9) for the suggested prototype nonlinear integrable system.

The expression for the local current $J_{22}(n|1)$ related to the local density $\rho_{22}(n|1)$ (4.21) is evident from the semi-discrete nonlinear equation (3.11) for $\dot{f}_{22}(n)$. The expression for the local current $J_{22}(n|2)$ related to the local density $\rho_{22}(n|2)$ (4.22) is given by formula

$$J_{22}(n|3) = f_{21}(n)f_{12}(n-1)b_{22}f_{22}(n-1) +$$
$$+ f_{23}(n)f_{32}(n-1)b_{22}f_{22}(n-1) -$$
$$- f_{21}(n)f_{11}(n-1)f_{12}(n-2)b_{22} -$$
$$- f_{21}(n)f_{13}(n-1)f_{32}(n-2)b_{22} -$$
$$- f_{23}(n)f_{33}(n-1)f_{32}(n-2)b_{22} -$$
$$- f_{23}(n)f_{31}(n-1)f_{12}(n-2)b_{22}. \quad (4.24)$$

The expression for the local current $J_{22}(n|3)$ related to the local density $\rho_{22}(n|3)$ (4.23) is so long that we decided to omit it for the brevity sake.

5. Intuitive Fixation of Sampling Functions and Intermediate Form of Semi-Discrete Nonlinear Integrable System

Looking at expressions (4.21)–(4.23) for the obtained local conserved densities $\rho_{22}(n|1)$, $\rho_{22}(n|2)$, $\rho_{22}(n|3)$, we see that neither of them could be treated as the density of Hamiltonian function or the density of excitations related to the prototype semi-discrete nonlinear integrable system under study (3.7)–(3.9).

The crucial step to overcome this serious obstacle is to fix the arbitrary sampling functions $a_{11}(n)$, $a_{13}(n)$, $a_{33}(n)$, $a_{31}(n)$ by a certain reasonably motivated demand. After the scrupulous analysis of our prototype integrable semi-discrete nonlinear system (3.7)–(3.9) we decided to press the function

$$\varrho_{22}(n) = f_{21}(n)f_{12}(n) + f_{23}(n)f_{32}(n) \tag{5.1}$$

into a Procrustean bed of local conservation law

$$\dot{\varrho}_{22}(n) = \mathcal{J}_{22}(n) - \mathcal{J}_{22}(n+1). \tag{5.2}$$

Our intuitive reasoning supported by elementary analytic calculations has provided the following results:

$$a_{11}(n) = a_{11}, \tag{5.3}$$

$$a_{13}(n) = a_{13}, \tag{5.4}$$

$$a_{33}(n) = a_{33}, \tag{5.5}$$

$$a_{31}(n) = a_{31} \tag{5.6}$$

and

$$f_{11}(n) = f_{11}, \tag{5.7}$$

$$f_{13}(n) = f_{13}, \tag{5.8}$$

$$f_{33}(n) = f_{33}, \tag{5.9}$$

$$f_{31}(n) = f_{31}. \tag{5.10}$$

In another words, the functions $a_{11}(n)$, $a_{13}(n)$, $a_{33}(n)$, $a_{31}(n)$ and $f_{11}(n)$, $f_{13}(n)$, $f_{33}(n)$, $f_{31}(n)$ must be independent of the spatial coordinate variable n. Nevertheless, each of these functions can be time-dependent.

As for the local current $\mathcal{J}_{22}(n)$ related to the local density $\varrho_{22}(n)$ (5.1), it acquires the form

$$\begin{aligned}\mathcal{J}_{22}(n) = &-b_{22}f_{21}(n)f_{11}f_{12}(n-1) - \\ &- b_{22}f_{21}(n)f_{13}f_{32}(n-1) - b_{22}f_{23}(n)f_{31}f_{12}(n-1) - \\ &- b_{22}f_{23}(n)f_{33}f_{32}(n-1).\end{aligned} \tag{5.11}$$

Although expression (5.1) for $\varrho_{22}(n)$ looks as the local density of excitations or the local density of charge, however, this naïve interpretation turns out to be delusive.

Meanwhile, due to the spatial independence of sampling functions a_{11} and a_{33} (see formulas (5.3) and (5.5)) the local conservation laws (4.4) and (4.5) are converted into the two differential constraints

$$\dot{W}_0(n) = 0, \tag{5.12}$$

$$\dot{W}_1(n) = 0 \tag{5.13}$$

on the functional expressions (4.6) and (4.7) for $W_0(n)$ and $W_1(n)$, respectively.

The second differential constraint (5.13) implies that the space-independent expression $f_{11}f_{33} - f_{13}f_{31}$ must be time-independent too, *i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[f_{11}f_{33} - f_{13}f_{31}\right] = 0. \tag{5.14}$$

As for the first differential constraint (5.12), we prefer to convert it into the sheer identity by means of substitution

$$\begin{aligned}f_{22}(n) = &\, h_{22}(n) + \\ &+ \frac{f_{21}(n)f_{33}f_{12}(n) + f_{23}(n)f_{11}f_{32}(n)}{f_{11}f_{33} - f_{13}f_{31}} - \\ &- \frac{f_{21}(n)f_{13}f_{32}(n) + f_{23}(n)f_{31}f_{12}(n)}{f_{11}f_{33} - f_{13}f_{31}}\end{aligned} \tag{5.15}$$

implying that the function $f_{22}(n)$ has lost its status of independent field function. Here, $h_{22}(n)$ is the time-independent integration function. In order to preserve the uniformity of space, we assume its independence of the spatial coordinate n as well. Thus, we have

$$h_{22}(n) = h_{22}, \tag{5.16}$$

where

$$\dot{h}_{22} = 0. \tag{5.17}$$

Having taken into account the findings of this fifth Section we come to the intermediate form of semi-discrete nonlinear integrable system, which reads as follows

$$\dot{f}_{11} = a_{13}f_{31} - f_{13}a_{31}, \tag{5.18}$$

$$\dot{f}_{13} = a_{11}f_{13} + a_{13}f_{33} - f_{11}a_{13} - f_{13}a_{33}, \tag{5.19}$$

$$\dot{f}_{33} = a_{31}f_{13} - f_{31}a_{13}, \tag{5.20}$$

$$\dot{f}_{31} = a_{31}f_{11} + a_{33}f_{31} - f_{31}a_{11} - f_{33}a_{31}, \tag{5.21}$$

$$\begin{aligned}\dot{f}_{21}(n) = &\, b_{22}f_{21}(n+1)f_{11} + a_{22}f_{21}(n) + \\ &+ b_{22}f_{23}(n+1)f_{31} - f_{21}(n)a_{11} - \\ &- f_{22}(n)b_{22}f_{21}(n) - f_{23}(n)a_{31},\end{aligned} \tag{5.22}$$

$$\dot{f}_{12}(n) = a_{11}f_{12}(n) + f_{12}(n)b_{22}f_{22}(n) + \\ + a_{13}f_{32}(n) - f_{11}f_{12}(n-1)b_{22} - \\ - f_{12}(n)a_{22} - f_{13}f_{32}(n-1)b_{22}, \quad (5.23)$$

$$\dot{f}_{23}(n) = b_{22}f_{21}(n+1)f_{13} + a_{22}f_{23}(n) + \\ + b_{22}f_{23}(n+1)f_{33} - f_{21}(n)a_{13} - \\ - f_{22}(n)b_{22}f_{23}(n) - f_{23}(n)a_{33}, \quad (5.24)$$

$$\dot{f}_{32}(n) = a_{31}f_{12}(n) + f_{32}(n)b_{22}f_{22}(n) + \\ + a_{33}f_{32}(n) - f_{31}f_{12}(n-1)b_{22} - \\ - f_{32}(n)a_{22} - f_{33}f_{32}(n-1)b_{22}. \quad (5.25)$$

Here, we should remember that the function $f_{22}(n)$ is specified by the previously obtained formulas (5.15)–(5.17).

Without the loss of generality, we simplify our forthcoming consideration by assuming that each of the two time-dependent parameters a_{22} and h_{22} is equal to zero

$$a_{22} = 0, \quad (5.26)$$

$$h_{22} = 0. \quad (5.27)$$

This assumption is readily justifiable by the gauge transformation from the original $f_{21}(n)$, $f_{12}(n)$, $f_{23}(n)$, $f_{32}(n)$ to the transformed $F_{21}(n)$, $F_{12}(n)$, $F_{23}(n)$, $F_{32}(n)$ field functions specified by formulas

$$f_{21}(n) = F_{21}(n)\exp[+A_{22} - C_{22}], \quad (5.28)$$

$$f_{12}(n) = F_{12}(n)\exp[-A_{22} + C_{22}], \quad (5.29)$$

$$f_{23}(n) = F_{23}(n)\exp[+A_{22} - C_{22}], \quad (5.30)$$

$$f_{32}(n) = F_{32}(n)\exp[-A_{22} + C_{22}], \quad (5.31)$$

$$\dot{A}_{22} = a_{22}, \quad (5.32)$$

$$\dot{C}_{22} = b_{22}h_{22}. \quad (5.33)$$

Another simplification consists in introducing the rescaled time-dependent parameters a_{11}, a_{13}, a_{31}, a_{33} and the rescaled time variable $\mathcal{T}$ by means of formulas

$$a_{jk} = b_{22}\mathrm{a}_{jk} \quad (5.34)$$

and by means of differential equality

$$\mathrm{d}\mathcal{T} = b_{22}\mathrm{d}\tau, \quad (5.35)$$

respectively. This observation allows us to specify the parameter b_{22} by the simple equality

$$b_{22} = 1 \quad (5.36)$$

in the spatiotemporal part (5.22)–(5.25) of the obtained intermediate semi-discrete nonlinear system (5.18)–(5.25).

Eventually, the three adopted simplifications (5.26), (5.27), (5.36) do not discard the system's parametric driving sources manifested through the permissible time dependencies of parameters a_{11}, a_{13}, a_{33}, a_{31}, as well as through the time dependencies of spatially-independent driving functions f_{11}, f_{13}, f_{33}, f_{31}. As a matter of fact, it is reasonable to associate the parametrically driven system as such only with the last four (5.22)–(5.25) of the obtained equations, while to consider the first four (5.18)–(5.21) of the obtained equations as the main parametric driver.

6. Semi-Discrete Nonlinear Integrable System in Terms of Physically Motivated Field Functions

The main idea to convert the intermediate semi-discrete nonlinear integrable system (5.18)–(5.25) into the habitual Hamiltonian form is to make an appropriate transformation of its original field functions $f_{21}(n)$, $f_{12}(n)$, $f_{23}(n)$, $f_{32}(n)$ to the suitable new ones $g_{21}(n)$, $g_{12}(n)$, $g_{23}(n)$, $g_{32}(n)$ relying upon a certain physically understandable condition.

To realize this pertinent idea, we start with the following set of transformation formulas

$$g_{21}(n) = f_{21}(n)e_{11} + f_{23}(n)e_{31}, \quad (6.1)$$

$$g_{12}(n) = e_{11}f_{12}(n) + e_{13}f_{32}(n), \quad (6.2)$$

$$g_{23}(n) = f_{21}(n)e_{13} + f_{23}(n)e_{33}, \quad (6.3)$$

$$g_{32}(n) = e_{31}f_{12}(n) + e_{33}f_{32}(n) \quad (6.4)$$

supplemented by the condition

$$[g_{21}(n)g_{12}(n) + g_{23}(n)g_{32}(n)]\sqrt{f_{11}f_{33} - f_{13}f_{31}} = \\ = f_{21}(n)f_{33}f_{12}(n) + f_{23}(n)f_{11}f_{32}(n) - \\ - f_{21}(n)f_{13}f_{32}(n) - f_{23}(n)f_{31}f_{12}(n), \quad (6.5)$$

whose the right-hand-side part has been prompted by the explicit expression (5.15) for the local conserved density $\rho_{22}(n|1) = f_{22}(n)$ (4.21) with $h_{22}(n) = 0$.

The elementary algebraic manipulations with the transformation formulas (6.1)–(6.4) and with the adopted condition (6.5) give rise to the set of nonlinear algebraic equations

$$e_{33}^2 + e_{31}e_{13} = F_{11}, \tag{6.6}$$

$$(e_{33} + e_{11})e_{13} = -F_{13}, \tag{6.7}$$

$$e_{11}^2 + e_{13}e_{31} = F_{33}, \tag{6.8}$$

$$(e_{11} + e_{33})e_{31} = -F_{31} \tag{6.9}$$

allowing to determine the unknown time-dependent coefficients e_{11}, e_{13}, e_{33}, e_{31} in terms of the driving functions f_{11}, f_{13}, f_{33}, f_{31}. Here, the short-hand notation

$$F_{jk} = \frac{f_{jk}}{\sqrt{f_{11}f_{33} - f_{13}f_{31}}} \tag{6.10}$$

with $j \neq 2$ and $k \neq 2$ has been adopted.

The result of calculation is given by formulas

$$e_{11} = e + d, \tag{6.11}$$

$$e_{13} = -\frac{F_{13}}{2e}, \tag{6.12}$$

$$e_{33} = e - d, \tag{6.13}$$

$$e_{31} = -\frac{F_{31}}{2e}, \tag{6.14}$$

where

$$e^2 = \frac{1}{2} + \frac{1}{4}(F_{11} + F_{33}), \tag{6.15}$$

$$d = \frac{1}{4e}(F_{33} - F_{11}). \tag{6.16}$$

In so doing, the identity

$$e_{11}e_{33} - e_{13}e_{31} \equiv 1 \tag{6.17}$$

is taken place.

To perform all necessary transformations with the intermediate nonlinear integrable system (5.22)–(5.25) we are obliged to consider the inverse transformation formulas

$$f_{21}(n) = g_{21}(n)d_{11} + g_{23}(n)d_{31}, \tag{6.18}$$

$$f_{12}(n) = d_{11}g_{12}(n) + d_{13}g_{32}(n), \tag{6.19}$$

$$f_{23}(n) = g_{21}(n)d_{13} + g_{23}(n)d_{33}, \tag{6.20}$$

$$f_{32}(n) = d_{31}g_{12}(n) + d_{33}g_{32}(n) \tag{6.21}$$

too. Here the time-dependent coefficients d_{11}, d_{13}, d_{31}, d_{33} are related to the time-dependent coefficients e_{11}, e_{13}, e_{31}, e_{33} by the simple formulas

$$d_{11} = e_{33}, \tag{6.22}$$

$$d_{13} = -e_{13}, \tag{6.23}$$

$$d_{33} = e_{11}, \tag{6.24}$$

$$d_{31} = -e_{31}. \tag{6.25}$$

Despite of its elementary background the actual procedure of system's reformulation in terms of new physically motivated field functions $g_{jk}(n)$ turns out to be rather cumbersome due to the pronounced time dependencies of driving functions f_{jk} and transformation coefficients e_{jk}, as well as due to the possible time dependencies of parameters a_{jk}. Therefore we prefer to present only the final formulas encompassing the dynamical features of the transformed semi-discrete nonlinear integrable system. The obtained equations of motion read as follows

$$\begin{aligned} +\dot{g}_{21}(n) &= g_{21}(n+1)f_{11} + g_{23}(n+1)f_{31} - \\ &- g_{21}(n)a_{11} - g_{23}(n)a_{31} - \\ &- \frac{g_{21}(n)g_{12}(n) + g_{23}(n)g_{32}(n)}{\sqrt{f_{11}f_{33} - f_{13}f_{31}}}\, g_{21}(n), \end{aligned} \tag{6.26}$$

$$\begin{aligned} -\dot{g}_{12}(n) &= f_{11}g_{12}(n-1) + f_{13}g_{32}(n-1) - \\ &- a_{11}g_{12}(n) - a_{13}g_{32}(n) - \\ &- g_{12}(n)\,\frac{g_{21}(n)g_{12}(n) + g_{23}(n)g_{32}(n)}{\sqrt{f_{11}f_{33} - f_{13}f_{31}}}, \end{aligned} \tag{6.27}$$

$$\begin{aligned} +\dot{g}_{23}(n) &= g_{23}(n+1)f_{33} + g_{21}(n+1)f_{13} - \\ &- g_{23}(n)a_{33} - g_{21}(n)a_{13} - \\ &- \frac{g_{21}(n)g_{12}(n) + g_{23}(n)g_{32}(n)}{\sqrt{f_{11}f_{33} - f_{13}f_{31}}}\, g_{23}(n), \end{aligned} \tag{6.28}$$

$$\begin{aligned} -\dot{g}_{32}(n) &= f_{33}g_{32}(n-1) + f_{31}g_{12}(n-1) - \\ &- a_{33}g_{32}(n) - a_{31}g_{12}(n) - \\ &- g_{32}(n)\,\frac{g_{21}(n)g_{12}(n) + g_{23}(n)g_{32}(n)}{\sqrt{f_{11}f_{33} - f_{13}f_{31}}}. \end{aligned} \tag{6.29}$$

Here, we have taken into account the already announced simplifications (5.16), (5.27) and (5.26), (5.36) which assert that $h_{22}(n) = 0$ and $a_{22} = 0$, $b_{22} = 1$ without the loss of generality. Moreover one must remember that the driving functions f_{jk} are governed by the set of equations (5.18)–(5.21) listed in the fifth Section.

The above written semi-discrete nonlinear equations (6.26)–(6.29) declare that the quantity $\rho_{22}(n)$ given by formula

$$\rho_{22}(n) = g_{21}(n)g_{12}(n) + g_{23}(n)g_{32}(n) \tag{6.30}$$

has the sense of the local conserved density. Presently, we suspect that this local conserved density $\rho_{22}(n)$ does not preserves its sign as a function of coordinate n and time τ. For this reason, we are inclined to treat it as the local density of charge similarly to the terminology adopted in other our papers [31–37].

The local current $J_{22}(n)$ in the local conservation law

$$\dot{\rho}_{22}(n) = J_{22}(n) - J_{22}(n+1) \tag{6.31}$$

related to the charge local density (6.30) is determined by formula

$$J_{22}(n) =$$
$$= -g_{21}(n)f_{11}g_{12}(n-1) - g_{21}(n)f_{13}g_{32}(n-1) -$$
$$- g_{23}(n)f_{33}g_{32}(n-1) - g_{23}(n)f_{31}g_{12}(n-1). \tag{6.32}$$

The system under study (6.26)–(6.29) should be treated as the system of two coupled pseudo-excitonic subsystems. Each of subsystems is described by its own pair of field functions. These two pairs of functions are as follows $g_{21}(n)$, $g_{12}(n)$ and $g_{23}(n)$, $g_{32}(n)$. Each pair of functions is prescribed to a separate one-dimensional regular chain. Therefore, each of subsystems is settled exclusively on the sites of its own separate chain. The inter-site linear coupling along one separate chain is described by the parameter f_{11}. The inter-site linear coupling along another separate chain is described by the parameter f_{33}. The inter-site linear coupling along a particular chain is seen to be extremely asymmetric (one-sided) in contrast to the symmetric (two-sided) inter-site linear coupling along a particular chain typical of the conventional molecular excitons [38]. For this reason, the intra-site excitations of our system are referred to as the pseudo-excitonic ones. The linear coupling parameters f_{31}, a_{31} and f_{13}, a_{13} between the field functions of distinct subsystems characterize the linear interaction between the sites of two distinct chains. This transverse linear interaction effectively establishes the two leg ladder configuration of underlying regular lattice.

7. Hamiltonian Formulation of the Semi-Discrete Nonlinear Integrable System in Terms of Physically Motivated Field Functions

As we already mentioned in fifth Section, the standardly obtained lowest local conserved densities $\rho_{22}(n|1)$, $\rho_{22}(n|2)$ (see (4.21)–(4.22)) can not be taken for the density of Hamiltonian function of the semi-discrete nonlinear integrable system under study. This situation appears to share some similarity with the case of multi-component integrable semi-discrete nonlinear Schrödinger systems where the knowledge of basic local conservation laws does not open the routes to construct the exact Hamiltonian representation in physically meaningful terms [39, 40].

Fortunately, the equations of motion for the semi-discrete nonlinear integrable system of our interest represented in terms of physically motivated field functions (6.26)–(6.29) permit being rewritten in concise Hamiltonian form revealable by the purely heuristic consideration. As a result, we come to the canonical Hamiltonian dynamic equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_{21}(n) = -\frac{\partial H}{\partial g_{12}(n)}, \tag{7.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_{12}(n) = +\frac{\partial H}{\partial g_{21}(n)}, \tag{7.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_{23}(n) = -\frac{\partial H}{\partial g_{32}(n)}, \tag{7.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_{32}(n) = +\frac{\partial H}{\partial g_{23}(n)} \tag{7.4}$$

with the Hamiltonian function given by the expression

$$H = \sum_{m=-\infty}^{\infty} [g_{21}(m)a_{11}g_{12}(m) - g_{21}(m+1)f_{11}g_{12}(m)] +$$
$$+ \sum_{m=-\infty}^{\infty} [g_{21}(m)a_{13}g_{32}(m) - g_{21}(m+1)f_{13}g_{32}(m)] +$$
$$+ \sum_{m=-\infty}^{\infty} [g_{23}(m)a_{33}g_{32}(m) - g_{23}(m+1)f_{33}g_{32}(m)] +$$
$$+ \sum_{m=-\infty}^{\infty} [g_{23}(m)a_{31}g_{12}(m) - g_{23}(m+1)f_{31}g_{12}(m)] +$$
$$+ \sum_{m=-\infty}^{\infty} \frac{[g_{21}(m)g_{12}(m) + g_{23}(m)g_{32}(m)]^2}{2\sqrt{f_{11}f_{33} - f_{13}f_{31}}}. \tag{7.5}$$

Thus, we clearly see that the field functions $g_{21}(n)$ and $g_{12}(n)$ acquire the meaning of canonically conjugated dynamical field amplitudes settled on the one leg of a ladder lattice, while the field functions $g_{23}(n)$ and $g_{32}(n)$ acquire the meaning of canonically conjugated dynamical field amplitudes settled on the another leg of a ladder lattice.

In general, the obtained Hamiltonian system (7.1)–(7.5) does not conserve its total energy due to the permissible time dependencies of coupling parameters a_{11}, a_{13}, a_{33}, a_{31} and f_{11}, f_{13}, f_{33}, f_{31}. This statement is in line with the fundamental rule proved to exclude the total energy from the list of conserved quantities of any time-dependent (parametrically driven) Hamiltonian system [41, 42].

8. Explicit Example of Accompanying Parametric Drive

Now let us demonstrate one of the feasible explicit realization of parametric drive (5.18)–(5.21) accompanying either the intermediate semi-discrete nonlinear integrable system (5.22)–(5.25) or the physically motivated semi-discrete nonlinear integrable system (6.26)–(6.29) on an equal footing.

For this purpose, we decompose both the driving functions f_{11}, f_{13}, f_{33}, f_{31} and the driving parameters a_{11}, a_{13}, a_{33}, a_{31} into the suitable time-independent and time-dependent parts. Namely, we set up f_{jk} and a_{jk} by formulas

$$f_{jk} = u_{jk} + v_{jk}, \tag{8.1}$$
$$a_{jk} = u_{jk} - v_{jk}, \tag{8.2}$$

where the summands u_{jk} are time-independent

$$\dot{u}_{jk} = 0, \tag{8.3}$$

while the time-dependent summands v_{jk} are governed by the set of ordinary linear differential equations with the constant coefficients

$$\dot{v}_{11} = 2u_{13}v_{31} - 2v_{13}u_{31}, \tag{8.4}$$
$$\dot{v}_{13} = 2u_{11}v_{13} - 2v_{11}u_{13} + 2u_{13}v_{33} - 2v_{13}u_{33}, \tag{8.5}$$
$$\dot{v}_{33} = 2u_{31}v_{13} - 2v_{31}u_{13}, \tag{8.6}$$
$$\dot{v}_{31} = 2u_{31}v_{11} - 2v_{31}u_{11} + 2u_{33}v_{31} - 2v_{33}u_{31}. \tag{8.7}$$

The above homogeneous ordinary linear differential equations (8.4)–(8.7) emerge via the direct substitution of the adopted decomposition formulas (8.1)–(8.2) into the original equations for the parametric driver (5.18)–(5.21).

In what follows, we impose the restriction

$$(u_{11} - u_{33})^2 + 4u_{13}u_{31} < 0. \tag{8.8}$$

allowing to treat the set of homogeneous ordinary linear differential equations (8.4)–(8.7) as the oscillatory one characterized by the eigenfrequency

$$\omega = 2\sqrt{-(u_{11} - u_{33})^2 - 4u_{13}u_{31}}. \tag{8.9}$$

As a consequence, the purely oscillatory solutions to the set of driving equations (8.4)–(8.7) are representable in the form

$$v_{jk} = c_{jk}\cos(\omega\tau + \varphi) + s_{jk}\sin(\omega\tau + \varphi), \tag{8.10}$$

where φ is an arbitrary constant parameter. Here the time-independent coefficients c_{jk} and s_{jk} are specified by formulas

$$c_{11} = +c, \tag{8.11}$$
$$c_{13} = \frac{c}{2u_{31}}(u_{33} - u_{11}) - \frac{s\,\omega}{4u_{31}}, \tag{8.12}$$
$$c_{33} = -c, \tag{8.13}$$
$$c_{31} = \frac{c}{2u_{13}}(u_{33} - u_{11}) + \frac{s\,\omega}{4u_{13}}, \tag{8.14}$$
$$s_{11} = +s, \tag{8.15}$$
$$s_{13} = \frac{s}{2u_{31}}(u_{33} - u_{11}) + \frac{c\,\omega}{4u_{31}}, \tag{8.16}$$
$$s_{33} = -s, \tag{8.17}$$
$$s_{31} = \frac{s}{2u_{13}}(u_{33} - u_{11}) - \frac{c\,\omega}{4u_{13}}, \tag{8.18}$$

where c and s are free time-independent parameters.

Meanwhile, the expression (8.9) for the eigenfrequency ω prompts us to parameterize the background constants u_{jk} by formulas

$$u_{11} = u + \frac{\omega}{4}\sinh(r), \tag{8.19}$$
$$u_{13} = +\frac{\omega}{4}\cosh(r)\exp(+2q), \tag{8.20}$$
$$u_{33} = u - \frac{\omega}{4}\sinh(r), \tag{8.21}$$
$$u_{31} = -\frac{\omega}{4}\cosh(r)\exp(-2q), \tag{8.22}$$

where each of three introduced parameters u, r, q is a time-independent one. As a consequence, for the time-independent parameters c_{13}, c_{31} and s_{13}, s_{31}, we obtain the following parameterized expressions

$$c_{13} = +c\tanh(r)\exp(+2q) + \\ + s\,\mathrm{sech}(r)\exp(+2q), \tag{8.23}$$

$$c_{31} = -c\tanh(r)\exp(-2q) + + s\,\mathrm{sech}(r)\exp(-2q), \quad (8.24)$$

$$s_{13} = +s\tanh(r)\exp(+2q) - - c\,\mathrm{sech}(r)\exp(+2q), \quad (8.25)$$

$$s_{31} = -s\tanh(r)\exp(-2q) - - c\,\mathrm{sech}(r)\exp(-2q). \quad (8.26)$$

At last, the expression for the time-independent normalization factor $f_{11}f_{33} - f_{13}f_{31}$ acquires rather perceptive representation

$$f_{11}f_{33} - f_{13}f_{31} = u^2 + \frac{\omega^2}{16} - (c^2 + s^2)\,\mathrm{sech}^2(r) \quad (8.27)$$

allowing to establish the physically informative criterion of its positive determinedness

$$u^2 + \frac{\omega^2}{16} > (c^2 + s^2)\,\mathrm{sech}^2(r). \quad (8.28)$$

As a matter of fact, the parameter q turns out to be absolutely surplus for the practical consideration inasmuch as it can be safely eliminated from the equations of motion (6.26)–(6.29) by the simple transformation

$$g_{21}(n) = G_{21}(n)\exp(-q), \quad (8.29)$$

$$g_{12}(n) = G_{21}(n)\exp(+q), \quad (8.30)$$

$$g_{23}(n) = G_{23}(n)\exp(+q), \quad (8.31)$$

$$g_{32}(n) = g_{32}(n)\exp(-q) \quad (8.32)$$

to the rescaled field amplitudes $G_{21}(n)$, $G_{12}(n)$ and $G_{23}(n)$, $G_{32}(n)$ physically equivalent to the previous ones $g_{21}(n)$, $g_{12}(n)$ and $g_{23}(n)$, $g_{32}(n)$ with q taken to zero. Thus, we assume

$$q = 0 \quad (8.33)$$

without the loss of generality.

9. Physically Motivated Semi-Discrete Nonlinear Integrable System under the Explicitly Given Parametric Drive

For the sake of convenience let us consider the unveiled Hamiltonian formulation of suggested physically motivated semi-discrete nonlinear integrable system under the explicitly given parametric drive in some details.

First of all, the detailed Hamiltonian function $\mathcal{H}$ based on the previously presented formulas (7.5), (8.1), (8.2), (8.27) reads as follows

$$\begin{aligned}\mathcal{H} = &\sum_{m=-\infty}^{\infty} g_{21}(m)(u_{11} - v_{11})g_{12}(m) - \\ &- \sum_{m=-\infty}^{\infty} g_{21}(m+1)(u_{11} + v_{11})g_{12}(m) + \\ &+ \sum_{m=-\infty}^{\infty} g_{21}(m)(u_{13} - v_{13})g_{32}(m) - \\ &- \sum_{m=-\infty}^{\infty} g_{21}(m+1)(u_{13} + v_{13})g_{32}(m) + \\ &+ \sum_{m=-\infty}^{\infty} g_{23}(m)(u_{33} - v_{33})g_{32}(m) - \\ &- \sum_{m=-\infty}^{\infty} g_{23}(m+1)(u_{33} + v_{33})g_{32}(m) + \\ &+ \sum_{m=-\infty}^{\infty} g_{23}(m)(u_{31} - v_{31})g_{12}(m) - \\ &- \sum_{m=-\infty}^{\infty} g_{23}(m+1)(u_{31} + v_{31})g_{12}(m) + \\ &+ \sum_{m=-\infty}^{\infty} \frac{[g_{21}(m)g_{12}(m) + g_{23}(m)g_{32}(m)]^2}{2\sqrt{u^2 + \omega^2/16 - (c^2 + s^2)\,\mathrm{sech}^2(r)}}. \end{aligned} \quad (9.1)$$

Here the time-independent background parameters u_{jk} (see (8.19)–(8.22)) specified by the equality $q = 0$ are given by formulas

$$u_{11} = u + \frac{\omega}{4}\sinh(r), \quad (9.2)$$

$$u_{13} = +\frac{\omega}{4}\cosh(r), \quad (9.3)$$

$$u_{33} = u - \frac{\omega}{4}\sinh(r), \quad (9.4)$$

$$u_{31} = -\frac{\omega}{4}\cosh(r). \quad (9.5)$$

In turn, the time-dependent driving functions v_{jk} (see (8.10), (8.11), (8.13), (8.15), (8.17), (8.23)–(8.26) specified by the equality $q = 0$ are given by formulas

$$v_{11} = +c\cos(\omega\tau + \varphi) + s\sin(\omega\tau + \varphi), \quad (9.6)$$

$$v_{13} = +[c\tanh(r) + s\,\mathrm{sech}(r)]\cos(\omega\tau + \varphi) + + [s\tanh(r) - c\,\mathrm{sech}(r)]\sin(\omega\tau + \varphi), \quad (9.7)$$

$$v_{33} = -c\cos(\omega\tau + \varphi) - s\sin(\omega\tau + \varphi), \tag{9.8}$$

$$v_{31} = -\left[c\tanh(r) - s\,\mathrm{sech}(r)\right]\cos(\omega\tau + \varphi) + \\ -\left[s\tanh(r) + c\,\mathrm{sech}(r)\right]\sin(\omega\tau + \varphi). \tag{9.9}$$

Of course, the concise record of Hamiltonian dynamic equations related to the detailed Hamiltonian function (9.1) preserves its standard canonical form

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_{21}(n) = -\frac{\partial \mathcal{H}}{\partial g_{12}(n)}, \tag{9.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_{12}(n) = +\frac{\partial \mathcal{H}}{\partial g_{21}(n)}, \tag{9.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_{23}(n) = -\frac{\partial \mathcal{H}}{\partial g_{32}(n)}, \tag{9.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_{32}(n) = +\frac{\partial \mathcal{H}}{\partial g_{23}(n)}. \tag{9.13}$$

Due to the explicit parametric drive ensured by the time-dependent parts v_{jk} of coupling parameters this specified dynamic Hamiltonian system (9.1)–(9.13) does not conserve its total energy in a complete accordance with the similar property of its general parametrically driven predecessor (7.1)–(7.5).

In contrast, the system's total charge

$$Q = \sum_{m=-\infty}^{\infty} \left[g_{21}(m)g_{12}(m) + g_{23}(m)g_{32}(m)\right] \tag{9.14}$$

is conserved provided the local current on the one side of infinite lattice coincides with the local current on the another side of infinite lattice

$$J_{22}(-\infty) = J_{22}(+\infty). \tag{9.15}$$

This statement is based on the elementary consideration of the respective local conservation law (6.31) supplemented by the expression (6.30) for the local conserved density $\rho_{22}(n)$.

10. Conclusion

One of the objectives of our research was to complement the basic principles for the development of semi-discrete nonlinear integrable systems formulated in our previous paper [22] by certain delicate nuances important for the actual implementation of novel parametrically driven integrable dynamical systems.

Having started with the appropriately constructed ansätze for the auxiliary spectral and evolution operators specified by the 3×3 square matrices, we have managed to develop the novel prototype semi-discrete nonlinear integrable system with the unfixed sampling functions. The obtained prototype semi-discrete nonlinear integrable system has been reduced to the novel semi-discrete nonlinear integrable system of parametrically driven pseudo-excitations on a two-leg ladder lattice both in its intermediate and physically motivated incarnations.

We have recovered several local conservation laws related to the general (prototype) semi-discrete nonlinear integrable system and have revealed that neither of the obtained conserved densities could not be taken as the density of Hamiltonian function either for the intermediate nonlinear system or for the physically motivated one.

Nevertheless, the physically motivated system turns out to be the Hamiltonian dynamical system characterized by the two pairs of canonically conjugated field amplitudes. Despite of its nontrivial complexity the two-stage procedure of transformation from the prototype system to the physically motivated one has been rewarded by the unusual splitting into the true physically motivated dynamical system and the coordinate independent parametric driver formalized by the set of four homogeneous ordinary linear differential equations with the time-dependent coefficients.

We have comprehensively demonstrated one particular realization of the purely oscillatory parametric drive and formulated the criteria of its validity in terms of time-independent background values of inter-site coupling parameters.

Due to its Lax integrability the suggested semi-discrete nonlinear system permits the exact analytical solutions, obtainable in the framework of modern mathematical methods such as the method of inverse scattering transform [9, 12–15, 29] and the method of Darboux–Bäcklund transformation [16, 28, 31, 34]. Nevertheless, the actual procedure of system's explicit analytical integration is expected to be substantially complicated by the noncommutativity of spatially-independent spectral and evolution seed operators caused by the nontrivial action of inherent parametric drive.

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ПОБУДОВА ТА АНАЛІЗ НОВИХ ІНТЕҐРОВНИХ НЕЛІНІЙНИХ ДИНАМІЧНИХ СИСТЕМ НА КВАЗІОДНОВИМІРНИХ ҐРАТКАХ. ПАРАМЕТРИЧНО УРУХОМЛЮВАНА НЕЛІНІЙНА СИСТЕМА ПСЕВДОЗБУДЖЕНЬ НА ДВОНІЖКОВІЙ ДРАБИНЧАТІЙ ҐРАТЦІ

Спираючись на засадничі принципи побудови інтеґровних еволюційних нелінійних систем на квазіодновимірних ґратках запропоновано нову нелінійну інтеґровну систему параметрично урухомлюваних псевдоекситонів на регулярній двоніжковій драбинчатій ґратці. Початкова (прототипна) форма системи є виводжуваною в термінах напівдискретного рівняння нульової кривини зі спектральним та еволюційним операторами, заданими спеціально підлаштованими 3×3 квадратовими матрицями. Хоча найнижчі збережні локальні густини, знайдені нами прямим рекурсивним методом, і не вказали на можливу алгебричну будову Гамільтонової функції системи, проте евристично обґрунтований пошук вдалого двоступеневого перетворення прототипних польових функцій до фізично вмотивованих дав фізично змістовну нелінійну інтеґровну систему з часозалежними повздовжніми та поперечними параметрами міжвузлових зв'язків. Часові залежності параметрів міжвузлових зв'язків трансформованої системи є послідовно означеними в термінах супутнього параметричного урухомлювача, формалізованого чотирма звичайними однорідними лінійними диференційними рівняннями з часозалежними коефіцієнтами. Фізично змістовна параметрично урухомлювана нелінійна система допускає компактне Гамільтонове формулювання, в якому дві пари польових функцій набувають сенсу двох пар канонічно спряжених польових амплітуд. Насамкінець розлого висвітлено математичні властивості явного параметричного урухомлювання коливного типу.

Ключові слова: нелінійна динаміка, інтеґровна система, двоніжкова драбинчата ґратка, параметричне урухомлювання, Гамільтонова динаміка.