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STATIONARY CYCLOTRON RESONANCE IN NON-NEUTRAL PLASMA

This work deals with the condition required for a cyclotron resonance between a charged particle and an axially asymmetric zero-frequency electromagnetic wave to take place in a non-neutral plasma with an excess of electrons in the laboratory reference frame. The values of the parameter $q = 2\omega_{pe}^2/\omega_{ce}^2$ were calculated at which the condition of stationary cyclotron resonance is obeyed for electrons, positrons, and positive and negative ions. Relationships between the azimuthal number m and the resonance multiplicity n, which are necessary for a stationary cyclotron resonance to occur, have been determined. We have found the integral of the drift motion equations which determines the variations of the particle trajectory parameters in the non-neutral plasma subjected to the action of a small-amplitude electrostatic wave under the cyclotron resonance conditions. These variations can be substantial and force the particles to go outside the plasma. It is shown that, under the cyclotron resonance conditions, the considered drift motions can invoke an anomalous radial transport of particles in non-neutral-plasma devices.

Keywords: non-neutral plasma, cyclotron resonance, drift motion.

1. Introduction

In a neutral plasma, the condition of the cyclotron resonance between an electromagnetic wave and a non-relativistic charged particle that is located in a longitudinal uniform magnetic field and moves in a plane oriented transversely to the magnetic field has the well-known form

$$\omega = n\omega_{c\alpha}, \tag{1}$$

where ω is the wave frequency, $\omega_{c\alpha} = e_\alpha B/(M_\alpha c)$ is the cyclotron frequency of the particle, e_α and M_α are the particle's charge (including the sign) and mass, respectively, B is the magnetic field induction, and $n = \pm1, \pm2, ...$ is the resonance multiplicity.

In a non-neutral plasma, besides the longitudinal magnetic field, there also exists an electric field, which is a result of an uncompensated space charge of electrons and ions. In a plasma with the shape of a long cylinder – for example, plasma in the Malmberg–Penning trap – the electric field is directed along the radius. In these fields, in the absence of collisions, the particles rotate in the azimuth direction and oscillate along the radius (see (28)).

In a uniform non-neutral plasma, the cyclotron resonance condition for a particle and a wave traveling in the azimuth direction looks like [1]

$$\omega' = n\Omega_\alpha, \tag{2}$$

where $\omega' = \omega - m\omega_{\rm rot}^\alpha$ is the wave frequency in the reference frame, where a charged particle of the α-th

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kind slowly rotates in the crossed fields,

$$\omega^{\alpha}_{\text{rot}} = \frac{1}{2}(-\omega_{c\alpha} + \Omega_{\alpha}), \tag{3}$$

$$\Omega_{\alpha} = \operatorname{sgn}(e_{\alpha}) \sqrt{\omega^2_{c\alpha} - \frac{4e_{\alpha}E_r}{M_{\alpha}r}} \tag{4}$$

is the "modified" cyclotron frequency of the particle in the crossed fields (it is equal to the frequency of radial particle oscillations), and m is the azimuthal wave number. In the case of an excess of electrons, which is considered in this work, the radial electric field is directed toward the plasma axis, i.e., it is negative,

$$\frac{E_r}{r} = \frac{M_e}{2e_e}\omega^2_{pe}(1-f) = \text{const} < 0.$$

Here, r is the distance from the plasma cylinder axis, $f = n_i/n_e < 1$ is the charge neutralization coefficient of plasma consisting of electrons and positive ions, $n_{e,i} = \text{const}$ are the uniform concentrations of non-neutral plasma components, and $\omega^2_{pe} = 4\pi e^2 n_e/m_e$ is the square of the Langmuir frequency of electrons. In a uniform non-neutral plasma, all characteristic frequencies of the particle ($\omega^{\alpha}_{\text{rot}}$, Ω_{α}) and the plasma itself (ω_{pe}) do not depend on the radius.

Condition (2) of cyclotron resonance in the non-neutral plasma is also well-known for a long time. The author of book [1] refers to the work [2] which is published in 1965 and deals with beams, and the authors of work [3] refer to the article [4] published in 1967 and related to plasma rotation in crossed fields.

The author of this paper is aware of works [5–10], carried out as long ago as in 1943–1945, published in 1948, and related to the development of the magnetron theory. In these works, such quantities are used as the "modified" cyclotron frequency Ω_{α} (4), the slow rotation frequency $\omega^{\alpha}_{\text{rot}}$ (3), and the fast rotation frequency $\omega^{\alpha}_{\text{rot}} = \frac{1}{2}(-\omega_{c\alpha} - \Omega_{\alpha})$.

In the laboratory frame of reference, the cyclotron resonance condition (2) takes the form

$$\omega = m\omega^{\alpha}_{\text{rot}} + n\Omega_{\alpha}. \tag{5}$$

The resonance frequencies ω in this formula can be positive or negative. For certain strengths of the electric and magnetic fields, as well as the particle (e_{α}, M_{α}) and wave (m and n) parameters, the resonance frequency ω (5) can be equal to zero,

$$\omega = 0 = m\omega^{\alpha}_{\text{rot}} + n\Omega_{\alpha} \quad (m \neq 0,\ n \neq 0). \tag{6}$$

This interesting specific case of cyclotron resonance for a charged particle in a zero-frequency wave attracted attention in works [11, 12]. Resonances satisfying condition (6) can naturally be called stationary or static cyclotron resonances. We prefer the first term.

To satisfy the stationary resonance condition (6), the radial electric and longitudinal magnetic fields must be such that the second equality in (6) would be satisfied, and axially asymmetric waves with the frequencies $\omega = 0$ in the laboratory reference frame would be present in the plasma (the first equality in (6)).

Axially asymmetric waves of zero frequency include static axially asymmetric electric, $\mathbf{E}^{\sim}$, and/or magnetic, $\mathbf{H}^{\sim}$, fields. Maxwell's equations, which these fields satisfy, form two pairs of independent equations: the electrostatic equations

$$\operatorname{rot}\mathbf{E}^{\sim} = 0, \quad \operatorname{div}\mathbf{E}^{\sim} = 4\pi\rho^{\sim}, \tag{7}$$

and the magnetostatic equations

$$\operatorname{div}\mathbf{H}^{\sim} = 0, \quad \operatorname{rot}\mathbf{H}^{\sim} = \frac{4\pi}{c}\mathbf{j}^{\sim}. \tag{8}$$

(Here $\rho^{\sim}$ is the axially asymmetric distribution of the static space charge density, which is the source of the field $\mathbf{E}^{\sim}$, and $\mathbf{j}^{\sim}$ is the static current density, which generates the field $\mathbf{H}^{\sim}$, $\operatorname{div}\mathbf{j}^{\sim} = 0$). That is, the static fields $\mathbf{E}^{\sim}$ and $\mathbf{H}^{\sim}$ can be present in the non-neutral plasma independently of each other, or they can be present simultaneously, with their magnitudes not being related to each other. The zero-frequency waves $\mathbf{E}^{\sim}$ and $\mathbf{H}^{\sim}$ will be called static perturbations or simply perturbations.

As one can see from (6), the condition of stationary cyclotron resonance does not depend on the disturbance origin and the type of the electrodynamic structure where the resonance is realized. It depends only on the magnitudes of the main crossed fields E_r and B, in which the charged particle is located. The corresponding resonance values of these fields are determined and analyzed in this work.

Of course, the result of the charged particle interaction with a static disturbance under the cyclotron resonance conditions (6) depends on the disturbance topography, which, in turn, depends on the disturbance nature and the electrodynamic structure where this interaction is realized. The zero-frequency disturbance in (6) can or cannot be an eigenmode of the non-neutral plasma. It is known that, in a waveguide partially filled with cold uniform plasma, the modes with zero frequency can be lower hybrid and diocotron modes with finite values of the longitudinal wave vector ($k_z \neq 0$) [13–16]. In neutral plasma, there are no eigenmodes with zero frequency.

In Eq. (6), as static perturbations that are not eigenmodes of the non-neutral plasma (i.e., not the solutions of the dispersion equation for the non-neutral plasma oscillations) can be axially asymmetric static perturbations, intentional or unintentional, of the external electric, E_r, or magnetic, B, field (field errors). Such perturbations can be caused by electrodes or diagnostic probes that are either introduced into the plasma or located near its surface. Static field disturbances can also arise due to inaccurate manufacturing of electrodes, their inaccurate orientation along the magnetic field [17–19], and so forth. Disturbances of the main uniform magnetic field can be created by conductors with permanent currents located near the plasma surface. Surely, such small disturbances of external electric or magnetic fields always exist in plasma. It is attractive that they can be created experimentally, making no use of the RF equipment.

The current interest concerning static disturbances in the non-neutral plasma is associated with the fact that they are considered as an origin of an anomalous transport of charged particles along the radius and the losses of particles from traps in general. It is believed that this is a result of static disturbances (with the participation of collisions) that are zero-frequency eigenmodes of the non-neutral plasma [13–15]. But, for the condition of stationary cyclotron resonance (6), it does not matter whether the disturbance is an eigenmode of non-neutral plasma or not.

In the models considered, condition (6) of stationary cyclotron resonance for particles is not assumed. However, this is just under cyclotron resonance conditions (including the stationary resonance) that the perturbation effect on the motion of charged particles is much stronger than under non-resonance conditions [20]. Therefore, the transport of particles along the radius should be accelerated under the cyclotron resonance conditions. Furthermore, the common change in the particle trajectory parameters (the parameters R and ρ in Eq. (28); see below) under cyclotron resonance conditions even at a perturbation with a small amplitude is much larger than under non-resonance conditions and can reach the size of the plasma itself. In the course of this process, particles can go beyond the plasma boundaries without the "help" of collisions.

In the theory of non-neutral plasma [1], the following parameter is introduced:

$$q = 2\frac{\omega_{pe}^2}{\omega_{ce}^2} < \frac{1}{1-f}, \tag{9}$$

which characterizes the ratio between the electric and magnetic fields in plasma. The quantities entering the expression for q are measured and can be changed in the experiment. The characteristic frequencies of the charged particles of all types that compose the plasma are expressed in terms of the parameter q. The inequality in formula (9) determines the interval of q-values, where the equilibrium of electrons takes place, and the existence of the non-neutral plasma with an excess of electrons as a whole is possible.

In Sect. 2, the values of the parameter q [Eq. (9)] and the perturbation parameters satisfying the stationary cyclotron resonance condition (6) for particles of various masses and charge signs in a non-neutral plasma with an excess of electrons are determined. In Sect. 3, in the drift approximation, the law describing the change in the elements of the trajectories of charged particles under the conditions of cyclotron resonance with a potential wave of zero or non-zero frequency is determined.

2. Resonance Values of Parameter q

2.1. Let us determine the values of the parameter q [Eq. (9)] at which the stationary cyclotron resonance condition (6) is satisfied for electrons, positrons, and positive and negative ions. From Eq. (6), it follows that this condition is obeyed, if

$$\omega_{\rm rot}^{\alpha} = -\frac{n}{m}\Omega_\alpha. \tag{10}$$

The (slow) rotation frequency of charged particle ω_{rot}^{α} in Eqs. (5) and (6) is determined by expression (3). In the case of excess of electrons ($E_r < 0$), the rotation frequency is positive ($\omega_{\rm rot}^{\alpha} > 0$) for both positively and negatively charged particles. The sign of the "modified" cyclotron frequency of the particle Ω_α[Eq. (4)] is determined by the sign of the particle charge e_α. For Eq. (10) to be satisfied and the stationary cyclotron resonance be achieved, it is necessary that the following inequality be satisfied:

$$-nm\,\mathrm{sgn}\,(e_\alpha) > 0. \tag{11}$$

Substituting Eq. (3) into Eq. (6), we find the relationship between $\omega_{c\alpha}$ and Ω_α under stationary resonance conditions,

$$\Omega_\alpha = \frac{m}{m+2n}\omega_{c\alpha}. \tag{12}$$

Since Ω_α and $\omega_{c\alpha}$ have the same sign in Eq. (12), the necessary condition for achieving the resonance is also the fulfillment of the inequality

$$m\,(m+2n) > 0. \tag{13}$$

That is, we have two necessary conditions on m and n, under which the resonance is possible. These are conditions (11) and (13).

Substituting Eq. (12) into Eq. (10), we find the relationship between ω^α_{rot} and $\omega_{c\alpha}$ under the conditions of stationary cyclotron resonance,

$$\omega^\alpha_{\mathrm{rot}} = -\frac{n}{m+2n}\omega_{c\alpha}. \tag{14}$$

For Eq. (14) to hold, it is necessary to satisfy conditions (11) and (13).

2.2. For electrons (the index $\alpha = e$), the quantities Ω_e and ω_{ce} are negative ($\Omega_e < 0$, $\omega_{ce} < 0$). Then it follows from Eq. (11) that Eq. (6) can be fulfilled, only if m and n have the same sign (i.e., $mn > 0$). For unambiguity, we assume that the azimuthal number m is positive, $m > 0$. Then the stationary cyclotron resonance is possible for electrons, only if $n > 0$. In this case, condition (13) is also fulfilled.

The quantity Ω_e for electrons, being expressed in terms of the parameter q, looks like [1]

$$\Omega_e = [1 - q(1-f)]^{1/2}\,\omega_{ce}, \quad q < 1/(1-f). \tag{15}$$

By combining Eq. (12) and (15), we obtain the resonance q-values for electrons,

$$q = q^e_{\mathrm{res}} = \frac{4n\,(m+n)}{(m+2n)^2}\,\frac{1}{(1-f)} \quad (m > 0, n > 0). \tag{16}$$

The stationary cyclotron resonance for electrons can be achieved for all $(n,m) > 0$. Since in Eq. (16), the coefficient $4n\,(m+n)\,/(m+2n)^2 < 1$, then for all $(n,m) > 0$, the resonance values q^e_{res} satisfy the equilibrium condition for the non-neutral plasma with an excess of electrons (9), (15), and $q^e_{\mathrm{res}} < 1/(1-f)$.

For illustration, Table 1 shows the numerical values of q^e_{res}, Ω_e, ω^e_{rot} for electrons under the stationary resonance condition (6), the multiplicity $n = 1$, and various values of the azimuthal number m are quoted; they were calculated by formulas (16), (15) (or (12)), and (14). One can see that electrons with a low rotation frequency in the non-neutral plasma [see Eq. (3)] can be in resonance with a static perturbation with lower m- and n-values, if the value of the parameter q of the order of the Brillouin value, $q \sim 1/(1-f)$, i.e., in electric fields with the magnitude order of extremely large permissible values.

2.3. For positive ions (the index $\alpha = i^+$), from Eqs. (3) and (4), we have

$$\begin{aligned} &\omega^{i+}_{\mathrm{rot}} = (-\omega_{ci+} + \Omega_{i+})/2 > 0, \\ &\omega_{ci+} > 0, \quad \Omega_{i+} > 0. \end{aligned} \tag{17}$$

From Eq. (11), it follows that the stationary resonance condition (6) can be satisfied, only if m and n have different signs (i.e., $mn < 0$). Since we assume $m > 0$, then n must be negative, $n < 0$. In this case, condition (13) must also be satisfied.

For positive ions, expression (4) for Ω_{i+} in terms of the parameter q, has the form

$$\begin{aligned} &\Omega_{i+} = \omega_{ci+}\,[1 + q\,(M_{i+}/M_e)\,(1-f)]^{1/2}, \\ &q < 1/(1-f). \end{aligned} \tag{18}$$

Substituting this expression into Eq. (12), we get the q-values at which the stationary cyclotron resonance condition (6) for positive ions in a non-neutral plasma

Table 1. **Numerical values of the parameters q^e_{res}, Ω_e, and ω^e_{rot} for electrons under the conditions of stationary cyclotron resonance (6) with multiplicity $n = 1$ and various azimuthal numbers $m > 0$. Calculations were made by formulas (16), (15), and (14)**

m	q^e_{res}	Ω_e/ω_{ce}	$\omega^e_{\mathrm{rot}}/\lvert\omega_{ce}\rvert$
1	$(8/9)/(1-f)$	1/3	1/3
2	$(3/4)/(1-f)$	1/2	1/4
3	$(16/25)/(1-f)$	3/5	1/5

is satisfied,

$$q = q_{\rm res}^{i+} \equiv \frac{M_e}{M_{i+}} \frac{(-4n)\,(m+n)}{(m+2n)^2} \frac{1}{(1-f)} \qquad (19)$$
$$(n < 0, m + 2n > 0).$$

At stationary resonance condition (6), the $q_{\rm res}^{i+}$-values are different for ions with different masses. Thus, a selective influence on ions with different masses is possible in a non-neutral plasma under stationary cyclotron resonance conditions, as at cyclotron resonance (5) in a non-neutral plasma ($\omega \neq 0$) and "ordinary" cyclotron resonance in a neutral plasma ($\omega = n\omega_{c\alpha}$).

In Table 2, the numerical values of $q_{\rm res}^{i+}$, Ω_{i+}, and $\omega_{\rm rot}^{i+}$ are shown for positive ions under stationary resonance conditions (6) with multiplicity $n = -1$ and various permissible values of the azimuthal number m. As follows from Eq. (13), a resonance of a positive ion with a static perturbation is possible at sufficiently large values of the azimuthal number m ($m > -2n$). In particular, resonances of multiplicity $n = -1, -2, -3, ...$ are possible only at $m \geq$ $\geq +3, +5, +7, ...$, respectively.

Owing to the coefficient M_e/M_{i+}, the resonance $q_{\rm res}^{i+}$-values for positive ions [Eq. (19)] turn out small as compared to the resonance $q_{\rm res}^{e}$-values for electrons [Eq. (16)]. For example, for argon ($M_{i+} = 40$ a.u.), we have $M_e/M_{i+} \approx 1.36 \times 10^{-5}$. However, the radial electric field acting on the ions and corresponding to these small $q_{\rm res}^{i+}$-values remains strong in comparison with the magnetic field. Thus, at $m = 3$, the ratio between the electric and magnetic fields, which is described by the second term in the square brackets in Eq. (18), is equal to 8 (see the second column in Table 2). At $m = 4$, this ratio equals 3.

Table 2. **Numerical values of the parameters $q_{\rm res}^{i+}$, Ω_{i+}, and $\omega_{\rm rot}^{i+}$ for positive ions under the conditions of stationary cyclotron resonance (6) with multiplicity $n = -1$ and various admissible azimuthal numbers $m > 0$. Calculations were made by formulas (19), (18) [or (12)], and (14)**

m	$q_{\rm res}^{i+}$	Ω_{i+}/ω_{ci+}	$\omega_{\rm rot}^{i+}/\omega_{ci+}$
3	$8(M_eM_{i+})/(1-f)$	3	1
4	$3(M_eM_{i+})/(1-f)$	2	1/2
5	$(16/9)(M_eM_{i+})/(1-f)$	5/3	1/3

2.4. If, along with electrons and positive ions, a non-neutral plasma contains a small number of negative ions (the index $\alpha = i^-$), we have the following expression for their frequency Ω_{i-}, which is similar to expression (18):

$$\Omega_{i-} = \omega_{ci-}\sqrt{1 - q\frac{M_{i-}}{M_e}(1-f)} < 0. \qquad (20)$$

This formula (20) differs from Eq. (18) by the sign of the second term under the square root and the sign of ω_{ci-}. The motion of negative ions is finite within the radius only in a narrow (as compared to the interval of admissible q-values for electrons (15)) interval of the values of the parameter q,

$$q < \frac{M_e}{M_{i-}}\frac{1}{1-f}. \qquad (21)$$

Negative ions are held within the radius if the radial electric field is sufficiently weak. If inequality (21) is not satisfied, the ions are ejected from the plasma along the radius. In this interval of q-values, the non-neutral plasma is a filter for heavy negative ions with the masses

$$M_{i-} > \frac{M_e}{q\,(1-f)}. \qquad (22)$$

This situation is analogous to the situation that occurs in the Archimedes filter for positive ions [21]. In this filter, the radial electric field is positive, and heavy positive ions with masses M_{i+} satisfying inequalities (22) are not confined within the radius and ejected from the plasma, thus, being separated from light ions.

For negative ions, the stationary cyclotron resonance condition (6) can be obeyed only if the signs of m and n are identical ($mn > 0$). The resonance $q_{\rm res}^{i-}$-values at which the stationary cyclotron resonance is achieved for negative ions in a non-neutral plasma are as follows:

$$q = q_{\rm res}^{i-} \equiv \frac{M_e}{M_{i-}} \frac{4n\,(m+n)}{(m+2n)^2} \frac{1}{(1-f)} \qquad (23)$$
$$(n > 0,\ m > 0).$$

This expression for $q_{\rm res}^{i-}$ satisfies inequalities (21) at all $(n, m) > 0$, because the coefficient $4n(m+n)/$ $/(m+2n)^2 < 1$.

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Table 3 contains the numerical values of $q_{\rm res}^{i-}$, Ω_{i-}, $\omega_{\rm rot}^{i-}$ calculated formulas (23), (20), and (14) for negative ions under stationary cyclotron resonance conditions (6) for multiplicity $n = 1$ and various values of the azimuthal number m. The $q_{\rm res}^{i-}$-values in Eq. (23) and Table 3 differ from the $q_{\rm res}^{e}$-values for electrons in formula (16) and Table 1 by the factor M_e/M_{i-}, whereas the ratios Ω_{i-}/ω_{ci-} and Ω_e/ω_{ce} in Tables exactly coincide. Owing to the coefficient M_e/M_{i-}, the resonance $q_{\rm res}^{i-}$-values for negative ions (23) are much smaller than the resonance $q_{\rm res}^{e}$-values for electrons (16). They also turn out smaller than the resonance $q_{\rm res}^{i+}$-values for positive ions (19).

2.5. If, in addition to electrons and positive ions, there is a small number of positrons (the index $\alpha = e^+$) in a non-neutral plasma, the expression for their resonance frequency Ω_{e+} looks like

$$\Omega_{e+} = \omega_{ce+}\sqrt{1 + q(1 - f)} > 0 \quad \big(\omega_{ce+} > 0,\, \omega_{\rm rot}^{e+} > 0\big). \tag{24}$$

Note that for positrons, the Ω_{e+}-value varies from the minimum value ω_{ce+}, which is reached at $q = 0$, to the maximum value, which equals only $\Omega_{e+} = \omega_{ce+}\sqrt{2}$ at the Brillouin value of the parameter q for electrons, $q = 1/(1 - f)$. At the same time, the Ω_{i+}-value for positive ions varies from ω_{ci+} to $\Omega_{i+} \approx \omega_{ci+}\sqrt{M_{i+}/M_e}$, which can be hundreds of times larger than ω_{ci+}.

The stationary cyclotron resonance (6) for positrons can be achieved if m and n have different signs (i.e., $mn < 0$). Condition (13) must also be satisfied. From Eq. (24) and (12), we find the resonance q-value for positrons,

$$q = q_{\rm res}^{e+} \equiv \frac{(-4n)\,(m + n)}{(m + 2n)^2}\,\frac{1}{(1 - f)}, \quad n < 0, \quad m + 2n > 0. \tag{25}$$

This expression will coincide with the expression for the resonance $q_{\rm res}^{i+}$-value for positive ions (19), if we make the substitution $M_{i+} \to M_{e+} = M_e$ in the latter.

Direct calculations using formula (25) show that, for $n = -1$ and azimuthal numbers $m = 3$, 4, 5, 6, the $q_{\rm res}^{e+}$-values turn out larger than the maximally admissible value of the parameter q for the equilibrium of a non-neutral plasma with an excess of electrons, $q_{\rm res}^{e+} > 1/(1 - f)$. This fact means that stationary cyclotron resonances for positrons at the indicated m- and n-values cannot be realized in a non-neutral plasma with an excess of electrons. Only if $m \geq 7$, $q_{\rm res}^{e+}$-value (25) decreases to an acceptable value $q_{\rm res}^{i+} < 1/(1 - f)$, and the stationary resonance for positrons can be realized.

For arbitrary values $n < 0$, the q_{res}^{i+}-value becomes smaller than the maximum permissible value for electrons, $q_{\rm res}^{e+} < 1/(1 - f)$, and the stationary cyclotron resonance becomes possible for m-values satisfying the condition

$$m > (4 + 2\sqrt{2})\,|n| \approx 6.83\,|n|. \tag{26}$$

Whence it follows that the resonance of a positron with a static perturbation is possible for sufficiently large values of the azimuthal number m. Resonances of multiplicity $n = -1$, -2, $-3, \dots$ are possible only if $m \geq +7$, $+14$, $+21, \dots$, respectively.

In Table 4, the numerical values of $q_{\rm res}^{e+}$, Ω_{e+}, and $\omega_{\rm rot}^{e+}$ are shown for positrons under the stationary cyclotron resonance conditions (6) at $n = -1$ and various permissible values of the azimuthal number m. From formula (25) and Table 4, one can see that the magnitudes of the resonance $q_{\rm res}^{e+}$-values for

Table 3. **Numerical values of the parameters $q_{\rm res}^{i-}$, Ω_{i-}, and $\omega_{\rm rot}^{i-}$ for negative ions under conditions of stationary cyclotron resonance (6) with multiplicity $n = 1$ and various azimuthal numbers $m > 0$. Calculations were made by formulas (23), (20) [or (12)], and (14)**

m	$q_{\rm res}^{i-}$	Ω_{i-}/ω_{ci-}	$\omega_{\rm rot}^{i-}/\lvert\omega_{ci-}\rvert$
1	$(8/9)(M_e/M_{i-})(1 - f)$	1/3	1/3
2	$(3/4)(M_e/M_{i-})/(1 - f)$	1/2	1/4
3	$(16/25)(M_e/M_{i-})/(1 - f)$	3/5	1/5

Table 4. **Numerical values of the parameters $q_{\rm res}^{e+}$, Ω_{e+}, and $\omega_{\rm rot}^{e+}$ for positrons under conditions of stationary cyclotron resonance (6) with multiplicity $n = -1$ and various admissible azimuthal numbers $m > 0$. Calculations were made by formulas (25), (12), and (14)**

m	$q_{\rm res}^{e+}$	Ω_{e+}/ω_{ce+}	$\omega_{\rm rot}^{e+}/\omega_{ce+}$
7	$(24/25)/(1 - f)$	7/5	1/7
8	$(28/36)/(1 - f)$	8/6	1/8
9	$(32/49)/(1 - f)$	9/7	1/9

positrons are of an order of the maximum permissible values for electrons, $q_{\rm res}^{e+} \sim 1/(1-f)$.

3. Drift Motion of Charged Particles under Cyclotron Resonance Conditions

3.1. Let us determine how the parameters of the trajectory of a particle located in a non-neutral plasma with an excess of electrons change under the conditions of cyclotron resonance of the general [Eq. (5)] or stationary [Eq. (6)] form with a potential wave of small amplitude. In both cases, the law according to which the elements of the charged particle trajectory change remains the same.

Note that, in the case of stationary cyclotron resonance, the total transverse energy of a charged particle, which consists of the kinetic and potential energies, is conserved in the laboratory reference frame, because the main electric, E_r, and magnetic, B, fields, as well as the $\mathbf{E}^{\sim}$ and $\mathbf{H}^{\sim}$ fields of static perturbation, do not depend on time. Separately, the kinetic and potential energies of the particle can change substantially.

In the field of an axially asymmetric static perturbation, the generalized azimuthal moment P_φ of the particle changes. Let us use the results obtained in works [11, 12, 22], while solving the equations of motion for a charged particle in crossed fields and in the field of a potential wave with a small amplitude that travels in the azimuth direction and has the form

$$\phi\,(r,\, m\varphi - \omega t). \tag{27}$$

The solutions were obtained under the cyclotron resonance conditions (5) in the drift approximation. Provided the stationary cyclotron resonance conditions (6), the wave potential (27) takes the form $\phi\,(r,\, m\varphi)$, i.e., the form of a static perturbation.

In the absence of wave (27), the solutions of the equations of motion of a charged particle in a plane that is transverse to the magnetic field have the following complex form:

$$r \exp{(i\varphi)} = \\ = \exp{(i\omega_{\rm rot}^{\alpha} t)}\,[R \exp{(i\theta)} + \rho \exp{(i\vartheta - \Omega_\alpha t)}]. \tag{28}$$

In a neutral plasma ($f = 1$, $\omega_{\rm rot}^{\alpha} = 0$, $\Omega_\alpha = \omega_{c\alpha}$), the quantities R and θ are the cylindrical coordinates of the Larmor center of the particle, ρ is its Larmor radius, and θ is the phase of the particle on the Larmor circle at the initial time moment $t = 0$.

In the general case ($\omega_{\rm rot}^{\alpha} \neq 0$), the quantities R, θ and ρ, ϑ in Eq. (28) are generalizations of the cylindrical coordinates of a charged particle in a neutral plasma to the case of non-neutral plasma. In a reference frame rotating with a frequency of $\omega_{\rm rot}^{\alpha}$, the quantities R, θ and ρ, ϑ have the same meaning as for a neutral plasma. This can be seen from Eq. (28).

In the absence of the wave, the quantities R, θ and ρ, ϑ are constants; they are integrals of motion. The motion of a particle along the radius according to Eq. (28) occurs within the interval $|R - \rho| \leq r \leq R + \rho$. The condition that during such motion the particle will remain within a plasma column of radius a has the form $R + \rho \leq a$. The line $R + \rho = a$ corresponds to particles that, during their motion (28), touch the plasma boundary (line *1* in Figs. 1 and 2), but do not cross it.

In the presence of a small-amplitude wave under cyclotron resonance conditions (5), the trajectory parameters R, θ and ρ, ϑ change slowly. In this case, the following combination remains unchanged [11, 12, 22]:

$$\frac{\rho^2}{n} - \frac{R^2}{m+n} = \text{const}. \tag{29}$$

This is one of the integrals of the drift motion of a particle in the field of a potential wave with a small amplitude (27), which travels in the azimuth direction, under cyclotron resonance conditions (5). Integral (29) determines the trajectories of the drift motion of the particle in the $R - \rho$ plane. Equation (29) is valid for any resonance frequencies (5), including the case of stationary cyclotron resonance (6), any small or large initial values of the particle parameters R and ρ, any dependence of the wave potential (27) on the radius r, and any periodic dependence on the azimuthal angle φ, for any azimuthal numbers m and resonance multiplicities n, including $m = 0$ and/or $n = 0$.

Under the action of wave (27), the particles perform periodic motions along drift trajectories (29). The extent to which the parameters R and ρ change during the drift process and how far the particle moves from the initial R- and ρ-values are determined by the drift equations for $\dot{R}$, $\dot{\theta}$ and $\dot{\rho}$, $\dot{\vartheta}$ [11, 12, 22], and ultimately by the specific dependence of the wave potential (27) on the radius r and the azimuthal angle φ. These issues are not dealt in this paper. The consideration is reduced to the analysis of the particle drift on the basis of the integral of motion (29).

3.2. For electrons, the stationary cyclotron resonance (6) is achieved under the condition $m \cdot n > 0$ (see Section 2.2). In this case, the electron drift trajectory in the $R-\rho$ plane (29) is a hyperbolaa with the asymptote $\rho = R\sqrt{n(m+n)}$. The electron drift trajectories in the $R-\rho$ plane and the indicated asymptote are presented in Fig. 1. The tangent of the asymptote slope (line *3*) equals $\sqrt{n(m+n)} < 1$. Electrons with (R, ρ) in the region below this asymptote (i.e., with $\rho < R\sqrt{n(m+n)}$), when drifting under the action of wave (27) along hyperbolaas (29), continue to remain in this region. These electrons do not encircle the system axis during the Larmor rotation. The corresponding condition looks like $\rho < R$ (the region below line *2* in Fig. 1).

Electrons with (R, ρ) in the region above the asymptote (i.e., with $\rho > R\sqrt{n(m+n)}$), continue to remain in this region. In this case, they may encircle (if $\rho > R$) or not (if $\rho < R$) the plasma axis during the Larmor rotation. Moreover, in the process of their drift under the action of the wave, they can transit from one state to the other one, if they cross the line $\rho = R$ (line *2* in Fig. 1).

As can be seen from Fig. 1, all drift trajectories of electrons inside the plasma (in the region $R + \rho \leq \leq a$) cross the plasma boundary, line *1*, for which $R + \rho = a$. This means that electrons located in the non-neutral plasma in the $R-\rho$ plane on any trajectory, provided the favorable phases of the wave and the particle (the angles θ, ϑ and the suitable dependence of potential ϕ (27) on the radius r), can cross the plasma boundary and drift under the wave action out of the plasma (into the region $\rho + R > a$), thus leading to the electron transfer along the radius.

3.3. For positive ions, stationary resonance (6) can be achieved under the condition $(n < 0, m + 2n > 0)$ (see Section 2.3). In this case, the drift trajectories of ions in the $R-\rho$ plane (29) are ellipses with the ratio of the semiaxes along ρ and R equal to $\sqrt{|n|/(m+n)}$ (Fig. 2). This ellipse is oblate along the ρ-axis, since, according to Eq. (13), $|n| < (m+n)$. For example, for $m = 3$ and $n = -1$, the semiaxis ratio eqauls $\sqrt{|n|/(m+n)} = 1/\sqrt{2} \approx 0.7$.

For ions at any drift trajectories, there is a fundamental possibility of their transition from a state in which they encircle the plasma axis during the Larmor rotation ($\rho > R$) to a state in which they do not encircle this axis ($\rho < R$), and vice versa, if they cross the line $\rho = R$ (line *2* in Fig. 2). Only those positive ions can cross the plasma boundary and quit the plasma, which are located at the drift trajectories (ellipses) in the $R-\rho$ plane that cross the line $\rho + R = a$ (line *1* in Fig. 2). These are ellipses whose semiaxes along R and ρ are larger than $a\sqrt{(m+n)/n}$ and $a\sqrt{|n|/m}$, respectively.

Ellipses with smaller semiaxes do not intersect the line $\rho + R = a$. When drifting along them, positive ions approach the plasma surface or move away from it, without reaching the plasma surface itself.

3.4. For negative ions, as it was for electrons, the stationary resonance condition (6) can be fulfilled provided the same sign of m and n, i.e., for $m > 0$ and $n > 0$ (section 2.4). All conclusions of Section 3.2 and

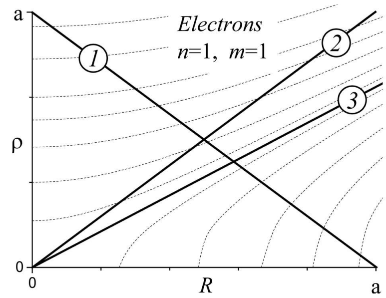

Fig. 1. Drift trajectories of electrons (hyperbolas) in the $R-\rho$ plane under conditions of cyclotron resonance with multiplicity $n = 1$ with potential wave (27) with azimuthal number $m = 1$. Straight line *1* denotes the dependence $R + \rho = a$, where a is the plasma radius, which corresponds to the plasma boundary. Line *2* denotes the dependence $\rho = R$, which separates particles that encircle and do not encircle the system axis during the Larmor rotation. Line *3* is the asymptote of the family of hyperbolasaas $\rho = R\sqrt{n/(m+n)}$

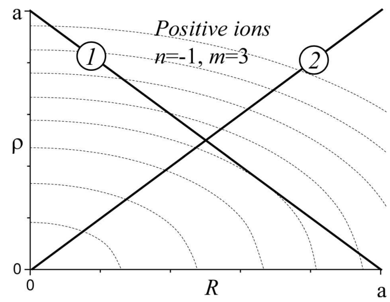

Fig. 2. Drift trajectories of positive ions (ellipses) in the $R-\rho$ plane under conditions of cyclotron resonance with multiplicity $n = -1$ with potential wave (27) with azimuthal number $m = = 3$. Lines *1* and *2* denote the same as in Fig. 1

Fig. 1 are also applicable to negative ions. It should be borne in mind that stationary cyclotron resonances for negative ions are realized at much smaller values of the parameter q than for electrons (cf. Tables 1 and 3).

The drift trajectories of negative ions in the $R-\rho$ plane (29) are hyperbolasas with the asymptote $\rho = R\sqrt{n/(m+n)}$ (see Fig. 1). Negative ions located in the $R-\rho$ plane in the region below the asymptote ($\rho < R\sqrt{n/(m+n)}$) remain in this region when drifting in the wave field. They do not encircle the axis of the system during the Larmor rotation.

Negative ions located on the $R-\rho$ plane in the region above the asymptote ($\rho > R\sqrt{n/(m+n)}$) remain in this region when drifting. They can encircle or not the axis of the system. During the drift process, negative ions can transit from one state to the other one.

Negative ions located at any drift trajectory in the $R-\rho$ plane, if they are in favorable phases of the wave and the particle, can cross the plasma surface and drift outward under the wave action, which leads to the transport of particles along the radius.

3.5. For positrons, the condition of stationary resonance (6) in a non-neutral plasma can be achieved under the condition $n < 0$ and $m > \left(4+2\sqrt{2}\right)|n|$. The drift trajectories of positrons in the $R-\rho$ plane (29) are ellipses with the ratio of the semiaxes along ρ and R equal to $\sqrt{|n|/(m+n)}$. The same semiaxis ratio is valid for positive ions (see Section 3.3). However, since stationary resonances for positrons are possible only at larger values of the azimuthal numbers m ($m+n \gg |n|$), these ellipses are more oblate along the ρ-axis than the ellipses in Fig. 2. So, for $n=-1$ and $m=+7$, the ratio of the semiaxes along ρ and R is equal to $\sqrt{|n|/(m+n)} = 1/\sqrt{6} \approx 0.41$.

All conclusions made in Section 3.3 for positive ions remain valid for positrons.

4. Conclusions

4.1. It has been shown that a stationary cyclotron resonance between charged particles in a non-neutral plasma with an excess of electrons ($E_r < 0$) and a static axially asymmetric perturbation of electric and/or magnetic fields is possible in the laboratory reference frame. The zero frequency is a resonance one for all types of charged particles, but this resonance is realized at various values of the fields E_r and B.

The values of the parameter q at which the stationary cyclotron resonance is realized for electrons (16), positive (19) and negative (23) ions, and positrons (25) have been calculated. For light particles (electrons and positrons), the resonance values of q are smaller, but they remain of the order of the maximum possible value $1/(1-f)$ for a non-neutral plasma with an excess of electrons. For positive and negative ions, the resonance values of q are much smaller. They are of the order of $(M_e/M_i)/(1-f)$. For positive ions, they can be several times larger; nevertheless, the resonance values of the parameter q are still small for ions, $q^i_{\text{res}} \ll 1/(1-f)$.

Relationships between the azimuthal number m (11) and the resonance multiplicity n (13) when the stationary cyclotron resonance (6) is possible have been determined.

4.2. The integral of the drift motion equations (29) is pointed out, which determines the change in the parameters R and ρ of the particle's transverse trajectory in a non-neutral plasma under the influence of an electrostatic wave with a small amplitude. The integral is valid both in the general case of cyclotron resonance (5) and in the special case of stationary cyclotron resonance (6).

The drift trajectories of negatively charged particles (electrons, negative ions) have the form of hyperbolaas in the plane of the parameters R and ρ. The trajectories of positively charged particles (positive ions, positrons) have the shape of ellipses. The parameters of the ellipses and hyperbolas are determined by the azimuthal wave number m and the cyclotron resonance multiplicity n. The change $(\Delta R, \Delta\rho)$ of the parameters R and ρ as a result of interaction with the wave under cyclotron resonance conditions can be substantial, even of the order of the plasma radius a itself ($\Delta R \sim a$, $\Delta\rho \sim a$).

Negatively charged particles (electrons and negatively charged ions) have greater ability to reach the plasma surface and go beyond its boundaries. Every hyperbolaa along which they drift in the $R-\rho$ plane intersects the line $R+\rho = a$. In this case, the particle goes beyond the plasma boundaries during the Larmor rotation.

Among positively charged particles (positive ions, positrons), only those particles can reach the plasma surface and go beyond its boundaries that are located in the $R-\rho$ plane within ellipses with the semi-axes along R and ρ greater than $a\sqrt{(m+n)/m}$ and $a\sqrt{|n|/m}$, respectively.

The considered drift motions under cyclotron resonance conditions may be the origin of the anomalous transport of particles along the radius (without collisions), which is observed in experiments with the non-neutral plasma.

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СТАТИЧНИЙ ЦИКЛОТРОННИЙ
РЕЗОНАНС У ЗАРЯДЖЕНІЙ ПЛАЗМІ

У роботі розглянуто умову циклотронного резонансу частинки зарядженої плазми з надлишком електронів з аксіально несиметричною електромагнітною хвилею, яка у лабораторній системі відліку має нульову частоту. Вираховано значення параметра $q \equiv 2\omega_{pe}^2/\omega_{ce}^2$, при яких виконується умова статичного циклотронного резонансу для електронів, позитивних і негативних іонів, позитронів. Визначено співвідношення між азимутальним числом хвилі m і кратністю резонансу n, при яких може бути досягнутий статичний циклотронний резонанс. Вказано інтеграл рівнянь дрейфового руху, що визначає зміни параметрів траєкторії частинки у зарядженій плазмі під дією електростатичної хвилі малої амплітуди в умовах циклотронного резонансу. Ці зміни можуть бути суттєвими та вести частинки за межі плазми. Дрейфові рухи в умовах статичного циклотронного резонансу можуть бути причиною аномального переносу частинок по радіусу в пристроях із зарядженою плазмою.

Ключові слова: заряджена плазма, циклотронний резонанс, дрейфовий рух.