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Vibroequilibria of acoustically-levitating drops *

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The multi-timing technique is applied to both differential and variational formulations of the problem on the acoustically levitating drops.

Для дифференциальной и вариационной постановок задачи про каплю, которая левитирует в акустическом поле, используется техника разделения быстрых и медленных движений.

Техніка розділення швидких та повільних рухів застосовується до диференціальної та варіаційної постановок задачі про краплю, яка левітує в акустичному полі.

1 Introduction

The acoustic levitation has been developing from the 70-90's as a novel contactless technology in chemical and pharmaceutical industry of ultra-pure smart materials [5, 7]. Along with preventing the liquid reagent contamination, using the acoustic levitation intensifies chemical reactions by increasing the interface area between liquid and ullage gas domains. Furthermore, the acoustic levitators are used in physical measurements of, e.g., the surface tension and the liquid viscosity [13, 16, 19].

An acoustic levitator is shown in Figure 1. It consists of the upper acoustic vibrator and the lower spheric reflector which create, altogether, an almost planar standing acoustic wave whose acoustic radiation pressure counteracts the gravity force and, thereby, holds liquid drops nearby the standing-wave nodes. As long as the vertical drop size $D_0 = 2R_0$ is

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much lower of the acoustic standing-wave length, the acoustic radiation pressure does not deform the drop so that it oscillates relative to the spherical shape as if the drop levitates in the zero-gravity. This case has been extensively studied and we refer interested readers to the papers [6,8,17,18] where theoretical results are reported utilizing the Lagrangian variational formalism. Alternatively, when the vertical drop size and the acoustic wave length are of the same order, the acoustic radiation pressure deforms the drop shape so that its averaged, visually observed geometry is far from the sphere as illustrated in Figure 1 (c).

The acoustically deformed drop shapes and their stability were investigated, experimentally and theoretically, e.g., in [1, 15, 22, 23]. The applied mathematical analysis normally involves the well-known free boundary problem on the weightless drop motions in which the dynamic boundary condition describes not only the pressure jump caused by the surface tension (as, e.g., in [6]) but also the acoustic radiation pressure created by the external standing acoustic wave. The acoustic radiation pressure plays the role of an averaged vibrational force normally appearing in the vibrational mechanics problems [4]. The force also appears in [2, 3, 9, 10, 12] where the time-averaged shape of a contained liquid (called the vibroequilibrium) was analyzed provided by the high-frequency vibrational loads to the tank. A principal difference is that the papers [1, 15, 22, 23] *postulate* the acoustic radiation pressure be a given function, but the aforementioned "vibroequilibria" problems deal with the vibrational force which is a function of the averaged liquid shape.

The present paper generalizes analytical results from [2, 3, 12] to the problem on acoustically-levitating drops. Following [2, 3, 12], the starting point is the "ulage gas–liquid drop" interface problem considered within the framework of ideal compressible fluids with irrotational flows. Along with the corresponding differential formulation, four different Lagrange-type variational formulations are presented. The multi-timing technique is applied to separate quick and slow time scales in both differential and variational statements. The quick-time averaged interface problem yields a new free-surface problem on a slowly-oscillating drop in which the Langevin acoustic radiation pressure naturally appears in the dynamic interface condition. The pressure parametrically depends on the quick-time averaged drop shape whose free surface is a reflector. When there are no slow drop oscillations, we arrive at a static free-surface problem whose solution describes the visually observed, acoustically deformed drop shapes. These shapes are called the *drop vibroequilibria*. The main variational result consists of deriving a functional which can be interpreted as a quasi-potential energy of the drop vibroeqilibrium whose local minima correspond to the stable acoustically-deformed drops.

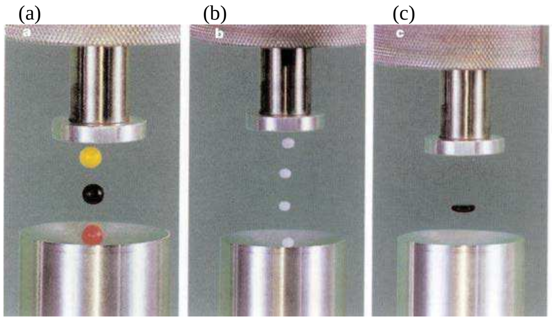

Figure 1: The acoustic levitator used in experiments [21]. An almost planar standing acoustic wave is created by the acoustic vibrator at the top and the spherical reflector at the bottom. The photos illustrate levitating (a) – three solid polymer spheres, (b) – four “liquid crystal” drops, and (c) – a liquid drop which is flattened by the nearly-planar standing acoustic wave.

2 Statement of the problem

Figure 2 schematically shows the “ullage gas–liquid drop” mechanical system situated in a closed box $Q = \{x \in \mathbb{R}^3 \mid W(x) < 0\}$, where $W(x)$ determines the piece-smooth box boundary and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the Cartesian coordinate system. The box domain Q does not depend on time t and consists of the gas $Q_1(t)$ and liquid $Q_2(t)$ domains ($Q = Q_1(t) \cup Q_2(t)$) separated by the drop surface $\Sigma(t) = \partial Q_2(t) = \{x \in Q_2 \mid \xi(x,t) = 0\}$ defined by the equality $\xi(x_1, x_2, x_3, t) = 0$ so that $\nabla\xi/|\nabla\xi|$ is the exterior normal vector with respect to $Q_2(t)$. The gas and liquid are assumed being *compressible ideal and barotropic* with *irrotational* flows. The box boundary $S = \partial Q$ includes the acoustic vibrator $S_0 \subset S$ and the reflecting surface $S_1 \subset S$, $S = S_0 \cup S_1$. For brevity, we exclude the gravity component assuming the zero-gravity condition.

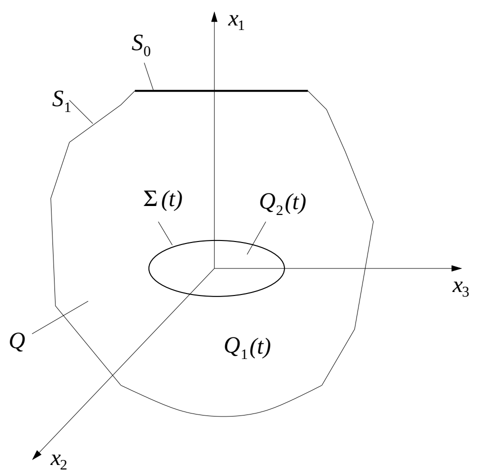

Figure 2: The "ullage gas–liquid drop" hydromechanic system situated in a rigid box Q. The acoustic vibrator is located on S_0, S_1 is the reflecting wall of the box.

After introducing the velocity potentials $\varphi_i = \varphi_i(x,t)$, the pressure $p_i = p_i(x,t)$ and the density $\rho_i = \rho_i(x,t)$ fields defined in $Q_i(t),\ i = 1,2$, one can write down the governing equations for the ideal barotropic fluids [2] as follows

$$\dot{\rho}_i + \operatorname{div}(\rho_i \nabla \varphi_i) = 0, \tag{1a}$$

$$\rho_i \nabla \left(\dot{\varphi}_i + \tfrac{1}{2} |\nabla \varphi_i|^2 \right) = -\nabla p_i, \tag{1b}$$

$$\rho_i = \rho_{0i} \left(\frac{p_i}{p_{0i}} \right)^{1/\gamma_i} \quad \text{in} \quad Q_i(t). \tag{1c}$$

Here, the time derivative is denoted by the dot and $\gamma_i,\ \ i = 1,2$ are the adiabatic indexes for the ullage gas and the liquid, respectively. The fluid domains should also satisfy the mass conservation condition

$$\int_{Q_i(t)} \rho_i \, \mathrm{d}Q = m_i = const_i, \quad i = 1,2. \tag{2}$$

Appropriate kinematic boundary conditions for (1) read as

$$\frac{\partial \varphi_i}{\partial n} = -\frac{\dot{\xi}}{|\nabla \xi|}, \quad i = 1,2 \ \text{ on } \ \Sigma(t), \tag{3a}$$

$$\rho_1 \frac{\partial \varphi_1}{\partial n} = \rho_{01} V_0(x) \sin \nu t \quad \text{on} \quad S_0, \tag{3b}$$

$$\frac{\partial \varphi_1}{\partial n} = 0 \quad \text{on} \quad S_1. \tag{3c}$$

These imply that fluid particles remain on the interface $\Sigma(t)$, define the normal velocity on the acoustic vibrator S_0, and suggest that the remaining box walls S_1 are a reflecting surface, respectively. Here, ν is the acoustic frequency and $V_0(x) \neq 0$ determines the vibrator shape (on S_0).

Finally, the compressible fluid interface problem requires the dynamic boundary condition

$$-p_2 + T_s(k_1 + k_2) = -p_1 \quad \text{on} \quad \Sigma(t) \tag{4}$$

giving the pressure balance between the drop and the ullage gas. It accounts for the surface tension associated with the $T_s(k_1 + k_2)$-term, where k_i, $i = 1, 2$ are the principal curvatures of $\Sigma(t)$ and T_s is the surface tension coefficient.

The problem (1)–(4) needs the following initial conditions

$$\begin{aligned} &\xi(x,0) = \tilde{\xi}_0(x); \quad \dot{\xi}(x,0) = \tilde{\xi}_1(x), \\ &\varphi_i(x,0) = \upsilon_i(x); \quad \dot{\varphi}(x,0) = \upsilon_{1i}(x), \quad i = 1, 2. \end{aligned} \tag{5}$$

3 The drop vibroequilibrium

3.1 Nondimensional statement

Henceforth, the free-interface problem (1)–(4) is considered in nondimensional statement assuming the characteristic size $l = 2R_0$ and the characteristic time ν^{-1}. The normalization suggests

$$x_{new} = l^{-1}x; \quad \xi_{new} = l^{-1}\xi; \quad \varphi_{inew} = l^{-2}\nu^{-1}\varphi_i; \quad p_{inew} = \rho_{0i} l^{-2}\nu^{-2} p_i,$$
$$p_{0inew} = \rho_{0i} l^{-2}\nu^{-2} p_{0i}, \quad m_{inew} = m_i l^3/\rho_{0i}, \quad i = 1, 2, \tag{6}$$

and introduces the following nondimensional parameters

$$\delta = \frac{\rho_{01}}{\rho_{02}}, \quad \nu_* = \frac{l^3 \rho_{02} \nu^2}{T_s}, \quad k^2 = \frac{\nu^2 l^2}{c_g^2}, \text{ and } k_*^2 = \frac{\nu^2 l^2}{c_l^2}, \tag{7}$$

where δ is the "gas–liquid" mean densities ratio, ν_* is the nondimensional acoustic frequency, and k and k_* are the wave numbers of acoustic (compressible) wave motions in gas and liquid, respectively; c_g and c_l are speeds of sound in the corresponding media.

After omitting the subscript *new*, the original free-interface problem (1)-(4) transforms to the nondimensional form

$$\dot{\rho}_i + \operatorname{div}(\rho_i \nabla \varphi_i) = 0, \tag{8a}$$

$$\rho_i \nabla \left(\dot{\varphi}_i + \tfrac{1}{2} |\nabla \varphi_i|^2 \right) = -\nabla p_i, \tag{8b}$$

$$\rho_i = \left(\frac{p_i}{p_{0i}} \right)^{1/\gamma_i} \quad \text{in} \quad Q_i(t), \tag{8c}$$

$$\int_{Q_i(t)} \rho_i \, \mathrm{d}Q = m_i, \quad i = 1, 2, \tag{8d}$$

$$\frac{\partial \varphi_1}{\partial n} = 0 \quad \text{on} \quad S_1, \tag{8e}$$

$$\rho_1 \frac{\partial \varphi_1}{\partial n} = \underbrace{\frac{\sup |V_0|}{c_g}}_{\epsilon} \underbrace{\frac{V_0(x)}{\sup |V_0|}}_{V(x) = O(1)} \frac{1}{k} \sin t \quad \text{on} \quad S_0, \tag{8f}$$

$$\frac{\partial \varphi_i}{\partial n} = -\frac{\dot{\xi}}{|\nabla \xi|}, \quad i = 1, 2, \tag{8g}$$

$$-p_2 + \underbrace{\nu_*^{-2}}_{\mu \mu_1 \epsilon^3} (k_1 + k_2) = -p_1 \underbrace{\delta}_{\mu_1 \epsilon} \quad \text{on} \quad \Sigma(t). \tag{8h}$$

An extra set of nondimensional parameters (and relationships between them) are introduced that is marked by the underbraces.

First, the small parameter

$$\epsilon = \frac{\sup |V_0|}{c_g} \ll 1 \tag{9}$$

implies the ratio between the maximum acoustic vibrator velocity and the sound speed in the ullage gas. *Secondly,* the ratio

$$\frac{\rho_{01}}{\rho_{02}} = \delta = \mu_1 \epsilon, \quad |\mu_1| \sim 1 \tag{10}$$

is assumed to of the same order than ϵ ($\mu_1 = O(1)$ is the proportionality coefficient). *Thirdly,* the nondimensional acoustic frequency is chosen as high as to provide the asymptotic relation

$$\nu_*^{-2} = \mu \mu_1 \epsilon^3, \quad \mu = O(1). \tag{11}$$

Fourthly, the wave numbers are

$$O(\epsilon) = k_*^2 \ll k^2 = O(1) \tag{12}$$

that means, from the physical point of view, that the forcing frequency may be close to lower acoustic resonant frequencies in the ullage gas, $k = O(1)$, but, because speed of sound in the liquid domain is much higher of that in the ullage gas (k_* is smaller of k on the $O(\epsilon^{1/2})$-scale), compressible liquid motions are far from a resonant condition and, in the first approximation, the drop is an incompressible liquid.

3.2 Multi-timing asymptotic technique

In this section, we apply the multi-timing technique assuming:

P1: *The nondimensional initial perturbations (5) provide that the free-interface problem (8) has the smooth solution in the closed domains of definition (up to the interface $\Sigma(t)$) such that $\forall t_1 < t_2 \;\; \varphi_i, p_i, \rho_i \in C^2(\hat{Q}_i), \;\; \hat{Q}_i = \{(t,x) : \forall t \in [t_1, t_2] \;\; x \in Q_i(t) \cup \Sigma(t)\}$. The interface $\Sigma(t)$ also is smooth, i.e. there exists a smooth homeomorphism $\mathcal{F}(\mathcal{F}^{-1})$ of a fixed single-connected domain Q_0 with a smooth boundary and the time-depending domain $\hat{Q}_2$ ($[t_1, t_2] \times \bar{Q}_0 \xrightarrow{\mathcal{F}} \bar{\hat{Q}}_2 \xrightarrow{\mathcal{F}^{-1}} [t_1, t_2] \times \bar{Q}_0$).*

P2: *ξ, φ_i, ρ_i, and $\mathcal{F}(\mathcal{F}^{-1})$ are analytical functions of the nondimensional parameter ϵ in a neighborhood of zero.*

As it is usually accepted in the vibrational mechanics, the quick time is associated with the nondimensional time t, but introducing the slow time τ suggests that it should be proportional to the square-root of the nondimensional potential forces. These forces are, in part, contributed by the surface tension which appears in the dynamic interface condition (8h) within the $O(\epsilon^3)$-multiplier. This means that the slow time can be defined as $\tau = \epsilon^{3/2} t$ and the nondimensional solution of (8) takes the form

$$\varphi_i = \varphi_i(x, t, \tau), \;\; p_i = p_i(x, t, \tau), \;\; \rho_i = \rho_i(x, t, \tau), \;\; \text{and } \xi = \xi(x, t, \tau).$$

Furthermore, based on assumption P2, this solution admits the asymptotic expansion

$$\begin{aligned}
\varphi_i &= \sum_{k=0}^{\infty} \epsilon^{k/2} \varphi_i^{(k/2)}(x,t,\tau); \quad p_i = \sum_{k=0}^{\infty} \epsilon^{k/2} p_i^{(k/2)}(x,t,\tau), \\
\rho_i &= \sum_{k=0}^{\infty} \epsilon^{k/2} \rho^{(k/2)}(x,t,\tau); \quad \xi = \sum_{k=0}^{\infty} \epsilon^{k/2} \xi_{k/2}(x,t,\tau)
\end{aligned} \tag{13}$$

and substituting (13) into (8) leads to a consequence of boundary value problems with respect to $\varphi_i^{(k/2)}, p_i^{(k/2)}, \rho_i^{(k/2)}, \xi_{k/2}, i = 1,2, k = 0,1,2....$ The problem (8) contains three small input parameters ϵ, $\epsilon^{3/2}$ and ϵ^3, but there is no the $O(\epsilon^{1/2})$-order input. This means that the $O(\epsilon^{1/2})$-order approximation is zero.

The $O(1)$*-order approximation* comes from the homogeneous problem

$$\begin{aligned}
&\dot\rho_i^{(0)} + \mathrm{div}(\rho_i^{(0)} \nabla \varphi_i^{(0)}) = 0; \quad \rho_i^{(0)} \nabla(\dot\varphi_i^{(0)} + \tfrac{1}{2}(\nabla \varphi_i^{(0)})^2) = -\nabla p_i^{(0)} \ \text{ in } \ Q_i^{(0)}, \\
&\rho_i^{(0)} = \left(\frac{p_i^{(0)}}{p_{0i}} \right)^{1/\gamma_i} \ \text{ in } \ Q_i^{(0)}; \quad \frac{\partial \varphi_1^{(0)}}{\partial n} = 0 \ \text{ on } \ S_1; \quad \frac{\partial \varphi_1^{(0)}}{\partial n} = 0 \ \text{ on } \ S_0, \\
&\frac{\partial \varphi_i^{(0)}}{\partial n} = -\frac{\dot\xi_0}{|\nabla \xi_0|}, \quad i = 1,2; \quad -p_2^{(0)} = 0 \ \text{ on } \ \Sigma^{(0)},
\end{aligned}$$

where the last pressure condition on $\Sigma^{(0)}$ shows that the drop motions are uncoupled with the compressible gas flow and, moreover, they do not depend on the surface tension. This means that, in the $O(1)$-approximation, the drop can slowly deform on the τ-scale and the zero-order solution is

$$\begin{aligned}
&\xi_0 = \xi_0(x,\tau); \quad \int_{Q_2^{(0)}(\tau)} \mathrm{d}Q = m_2; \quad \nabla \varphi_i^{(0)} = 0, \ i = 1,2, \\
&p_1^{(0)} = p_{01}; \ \rho_1^{(0)} = 1; \ \rho_2^{(0)} = p_2^{(0)} = 0
\end{aligned}$$

where $\xi_0(x,\tau) = 0$ defines the $O(1)$-order interface motions $\Sigma^{(0)} = \Sigma^{(0)}(\tau)$ which, in turn, defines the slowly-deforming domains $Q_i^{(0)}(\tau), \ i = 1,2$.

Henceforth, the $O(1)$-order drop deformations are associated with the quick-time averaged drop shape, i.e.

$$\Sigma_0(\tau) = \langle \Sigma(t,\tau) \rangle_t = \Sigma^{(0)}(\tau); \quad Q_i^{(0)}(\tau) = \langle Q_i(t,\tau) \rangle_t, \ i = 1,2, \tag{14}$$

and the higher-order asymptotic problems with respect to $\varphi_i^{(k/2)}, p_i^{(k/2)}, \rho_i^{(k/2)}, \xi_{k/2}, k \geq 2$ should be formulated in the averaged domains $Q_1^{(0)}(\tau)$ and $Q_2^{(0)}(\tau)$ as well as on the quick-time averaged interface $\Sigma_0(\tau)$.

The $O(\epsilon)$-*order approximation* comes from the problem

$$
\begin{aligned}
&k^2\ddot{\varphi}_1^{(1)} - \nabla^2\varphi_1^{(1)} = 0 \ \text{ in } \ Q_1^{(0)}(\tau); \quad \frac{\partial\varphi_1^{(1)}}{\partial n} = -\frac{\dot{\xi}_1}{|\nabla\xi_0|} \ \text{ on } \ \Sigma_0(\tau),\\
&\frac{\partial\varphi_1^{(1)}}{\partial n} = 0 \ \text{ on } \ S_1; \ \ \frac{\partial\varphi_1^{(1)}}{\partial n} = \frac{V(x)\sin t}{k} \ \text{ on } \ S_0,\\
&\frac{\partial\varphi_2^{(1)}}{\partial n} = -\frac{\dot{\xi}_1}{|\nabla\xi_0|}; \ \ p_2^{(1)} = \mu_1 p_{01} \ \text{ on } \ \Sigma_0(\tau),\\
&\dot{\rho}_2^{(1)} + \nabla^2\varphi_2^{(1)} = 0; \ \ \rho_2^{(1)} = 0 \ \text{ in } \ Q_2^{(0)}(\tau),
\end{aligned}
$$

where the last condition is due to $\rho_2^{(1)} = k_*^2 p_2^{(1)}$ and (12).

As it happened in the zero-order approximation, the dynamic interface condition (here, $p_2^{(1)} = \mu_1 p_{01} = const$) on the quick-time averaged interface $\Sigma_0(\tau)$ decouples the problem into two independent boundary value problems in $Q_2^{(0)}(\tau)$ and $Q_1^{(0)}(\tau)$.

Analyzing the boundary problem in $Q_2^{(0)}(\tau)$ shows that this approximation can only contribute into the slow-time drop deformations which, due to definition (14), are fully accounted for by the $O(1)$-order component. As a consequence,

$$\xi_1 = 0, \quad \nabla\varphi_2^{(1)} = 0, \quad \rho_2^{(1)} = 0, \quad p_2^{(1)} = \mu_1 p_{01}.$$

The boundary value problem in $Q_1^{(0)}(\tau)$ has the solution

$$\varphi_1^{(1)} = \epsilon\Phi_1(x,\tau)\sin t; \quad p_1^{(1)} = \Phi_1(x,\tau)\cos t, \tag{15}$$

where $\Phi_1(x)$ is the so-called *wave function* of the linear acoustic field in the ullage gas governed by the Neumann boundary value problem

$$\nabla^2\Phi_1 + k^2\Phi_1 = 0 \ \text{ in } \ Q_1^{(0)}(\tau); \quad \frac{\partial\Phi_1}{\partial n} = 0 \ \text{ on } \ S_1 \cup \Sigma_0(\tau);$$
$$\frac{\partial\Phi_1}{\partial n} = \frac{V(x)}{k} \ \text{ on } \ S_0 \tag{16}$$

and stated in the slowly-deforming gas domain where the quick-time averaged drop surface plays the role of a reflector.

The interface problem remains decoupled in the $O(\epsilon^{3/2})$-order approximation. For the gas domain $Q_1^{(0)}(\tau)$, the homogeneous τ-dependent boundary problem takes the form

$$\nabla^2\varphi_1^{(3/2)} = 0; \;\; p_1^{(3/2)} = \dot{\varphi}_1^{(3/2)} \;\text{ in }\; Q_1^{(0)}(\tau),$$
$$\frac{\partial\varphi_1^{(3/2)}}{\partial n} = 0 \;\text{ on }\; S_0 \cup S_1; \;\; \frac{\partial\varphi_1^{(3/2)}}{\partial n} = -\frac{\xi_{0\tau} + \dot{\xi}_{3/2}}{|\nabla\xi_0|} \;\text{ on }\; \Sigma_0(\tau),$$

but

$$\begin{aligned} &\nabla^2\varphi_2^{(3/2)} = 0; \;\; p_2^{(3/2)} = \dot{\varphi}_2^{(3/2)}; \;\; \rho_2^{(3/2)} = 0 \;\text{ in }\; Q_2^{(0)}(\tau),\\ &\frac{\partial\varphi_2^{(3/2)}}{\partial n} = -\frac{\xi_{0\tau} + \dot{\xi}_{3/2}}{|\nabla\xi_0|}; \;\; p_2^{(3/2)} = 0 \;\text{ on }\; \Sigma_0(\tau) \end{aligned} \tag{17}$$

describes the $O(\epsilon^{3/2})$-contribution to the drop motions which also is τ-dependent. This means that $\varphi_i^{(3/2)} = \varphi_i^{(3/2)}(x,\tau), i = 1, 2$.

Summarizing all asymptotic terms following from the constructed approximations gives the asymptotic solution

$$\varphi_2(x,t,\tau) = \epsilon^{3/2}\underbrace{\varphi_2^{(3/2)}(x,\tau)}_{\varphi(x,\tau)} + o(\epsilon^{3/2}), \tag{18a}$$

$$\xi(x,t,\tau) = \underbrace{\xi_0(x,\tau)}_{\zeta(x,\tau)} + o(\epsilon^{3/2}), \tag{18b}$$

$$\varphi_1(x,t,\tau) = \epsilon\,\underbrace{\Phi_1(x,\tau)}_{\Phi(x,\tau)}\sin t + \epsilon^{3/2}\varphi_1^{(3/2)}(x,\tau) + o(\epsilon^{3/2}). \tag{18c}$$

It shows that the lowest-order velocity field in the drop domain is of the order $O(\epsilon^{3/2})$ and this velocity field does not depend on the quick time. In the contrast, the lowest-order velocity field in the gas domain describes the linear acoustic standing wave for which the slowly-varying drop surface is a reflector. Finally, possible quick-time oscillations of the drop surface is of the $o(\epsilon^{3/2})$-order.

Because the dynamic interface condition (8h) contains, in the right hand side, the $O(\epsilon)$-multiplier, the drop may oscillate on the quick-time scale caused by the linear acoustic field (15) so that $\varphi_2^{(2)} =$

$\sin t\, F_1(x,\tau)$, but the velocity potential in $Q_1^{(0)}(\tau)$ takes the form $\varphi_1^{(2)} = \sin(2t)\, F_2(x,\tau) + \cos(2t)\, F_3(x,\tau)$. However, due to quadratic terms, the second-order pressure component in $Q_1^{(0)}(\tau)$ contains the quick-time averaged quantity

$$\langle p_1^{(2)} \rangle_t(x,\tau) = \tfrac{1}{4}(k^2(\Phi_1)^2 - (\nabla\Phi_1)^2) + const \tag{19}$$

associated with the so-called *Langevin acoustic radiation pressure*.

The $O(\epsilon^{5/2})$-order component has much more complicated structure, but it does not matter for the $O(\epsilon^3)$-order approximation which yields the *quick-time averaged* dynamic boundary condition

$$\begin{aligned}\varphi_{2\tau}^{(3/2)} + \tfrac{1}{2}\left(\nabla\varphi_2^{(3/2)}\right)^2 - \mu\mu_1(k_1+k_2) + {}& \\ + \tfrac{1}{4}\mu\mu_1(k^2(\Phi_1)^2 - (\nabla\Phi_1)^2) &= const \quad \text{on} \quad \Sigma_0(\tau),\end{aligned} \tag{20}$$

Accounting for the asymptotic solution (18), the quick-time averaged dynamic condition (20) as well as the governing boundary value problems for the lowest-order quantities in (18), we arrive, finally, at the following free-interface problem with respect to $\zeta(x,\tau) = \xi_0(x,\tau)$, $\varphi(x,\tau) = \varphi_2^{(3/2)}(x,\tau)$ and $\Phi(x,\tau) = \Phi_1(x,\tau)$

$$\begin{aligned}&\nabla^2\varphi = 0 \ \text{in} \ \Omega_2(\tau); \quad \frac{\partial\varphi}{\partial n} = -\frac{\zeta_\tau}{|\nabla\zeta|} \ \text{on} \ \Gamma(\tau); \quad \int_{\Omega_2(\tau)} \mathrm{d}\Omega = m_2,\\ &\varphi_\tau + \tfrac{1}{2}(\nabla\varphi)^2 - \mu\mu_1(k_1+k_2) + \\ &\qquad + \tfrac{1}{4}\mu\mu_1\left(k^2(\Phi)^2 - (\nabla\Phi)^2\right) = const \ \text{on} \ \Gamma(\tau),\end{aligned} \tag{21a}$$

$$\begin{aligned}&\nabla^2\Phi + k^2\Phi = 0 \ \text{in} \ \Omega_1(\tau); \quad \frac{\partial\Phi}{\partial n} = 0 \ \text{on} \ S_1 \cup \Gamma(\tau),\\ &\qquad\qquad \frac{\partial\Phi}{\partial n} = \frac{V(x)}{k} \ \text{on} \ S_0,\end{aligned} \tag{21b}$$

where $\Omega_1(\tau) = Q_1^{(0)}(\tau), \Omega_2(\tau) = Q_2^{(0)}(\tau)$, and $\Gamma(\tau) = \Sigma_0(\tau)$. This problem does not depend on t and describes a slow-time evolution of the acoustically levitating drop.

As matter of the fact, we proved the following theorem.

Theorem 3.1 *Under assumptions* **P1** *and* **P2**, *the quick-time averaged drop shape $\Omega_2(\tau)$ slowly oscillates governed by the free-surface problem (21).*

The free-interface problem (21) schematically takes the same form as that for a weightless drop. A difference consists of an extra term in the dynamic interface condition on $\Gamma(\tau)$ associated with the Langevin radiation pressure created by the acoustic vibrator in the quick-time averaged gas domain. This radiation pressure parametrically depends on the τ-instant drop shape due to the zero-Neumann boundary condition (21b) on $\Gamma(\tau)$. The latter implies that, in the lowest-order approximation, the slowly-oscillating drop surface plays the role of a reflector for the linear acoustic field in the ullage gas.

When assuming that the averaged drop shape does *not* oscillate, even on the slow-time scale, we come to the static free-interface problem

$$-\mu(k_1+k_2)+\frac{1}{4}\left(k^2(\Phi)^2-(\nabla\Phi)^2\right)=const \text{ on } \Gamma_0, \quad \int_{\Omega_{20}} \mathrm{d}\Omega = m_2, \quad (22a)$$

$$\nabla^2\Phi+k^2\Phi=0 \text{ in } \Omega_{10}; \quad \frac{\partial\Phi}{\partial n}=0 \text{ on } S_1\cup\Gamma_0,$$
$$\frac{\partial\Phi}{\partial n}=\frac{V(x)}{k} \text{ on } S_0. \quad (22b)$$

The drop shape Γ_0 is called the *drop vibroequilibria*.

4 Lagrangian formalism for (1)–(4)

Following [2], one can prove the following theorem providing equivalence of (1)–(4) and the Lagrangian variantional formulations.

Theorem 4.1 *Under assumption P1, the free-interface problem (1)-(4) is the necessary condition of the extremal points of the action*

$$G(\xi,\varphi_i,\rho_i)=\int_{t_1}^{t_2}[T-U-\Pi]\,\mathrm{d}t$$
$$=\int_{t_1}^{t_2}\left\{\sum_{i=1}^{2}\int_{Q_i(t)}\rho_i\left[\frac{(\nabla\varphi_i)^2}{2}-U_i(\rho_i)\right]\mathrm{d}Q-T_s|\Sigma|\right\}\mathrm{d}t \quad (23)$$

subject to the kinematic constraint (1) and assuming smooth isochronous variations

$$\delta\xi|_{t_1,t_2}=0; \quad \delta\rho_i|_{t_1,t_2}=0. \quad (24)$$

Here, $U_i(\rho_i)$ is the internal energy of the gas and liquid, respectively, i.e.

$$p_i \stackrel{def}{=} \rho_i^2 \frac{dU_i}{d\rho_i}. \tag{25}$$

Remark 4.1 *Because of constraint (1), the action G is a function of ξ and ρ_i.*

Proof. Let us employ the formula

$$\begin{aligned}\int_{Q(t)} [\dot\rho + (\nabla\varphi, \nabla\psi)]\,\mathrm{d}Q + \frac{d}{dt}\int_{Q(t)} \rho\varphi\,\mathrm{d}Q - \int_{S_0\subset S} \rho_{01} V_0 \varphi \sin(\nu t)\,\mathrm{d}S \\ = -\int_{Q(t)} [\dot\rho + \mathrm{div}(\rho\nabla\psi)]\varphi\,\mathrm{d}Q + \int_{S\setminus S_0} \rho\frac{\partial\psi}{\partial n}\varphi\,\mathrm{d}S + \int_{\Sigma(t)} \rho\frac{\partial\psi}{\partial n}\varphi\,\mathrm{d}S \\ + \int_{\Sigma(t)} \rho\frac{\dot\xi}{|\nabla\xi|}\varphi\,\mathrm{d}S + \int_{S_0}\left[\rho\frac{\partial\psi}{\partial n} - \rho_{01}V_0\sin(\nu t)\right]\varphi\,\mathrm{d}S\end{aligned} \tag{26}$$

for arbitrary domain Q, $\partial Q = \Sigma(t)\cup S\cup S_0$, so that $\Sigma(t)$ $(\xi(x,t)=0)$ is the time-depending interface ∂Q but $\varphi(x,t)$ and $\psi(x,t)$ are smooth functions. Assuming (1) and $\varphi=\varphi_1,\ \psi=\psi_1$, the right-hand side of (26) equals to zero. Analogously, since $\varphi=\varphi_2$ and $\psi=\psi_2$ $(S_0\cap\partial Q_2=\emptyset)$, the right-hand-side is also zero. Difference of the left-hand sides of the action gives

$$\begin{aligned}G(\xi,\varphi_i,\rho_i) = \int_{t_1}^{t_2}\Bigg\{\sum_{i=1}^{2}\int_{Q_i(t)} \rho_i\left[-\varphi_{it} - \frac{(\nabla\varphi_i)^2}{2} - U_i(\rho_i)\right]\mathrm{d}Q \\ -T_s|\Sigma| + \int_{S_0}\rho_{01}V_0\varphi_1\sin(\nu t)\,\mathrm{d}S\Bigg\}\,\mathrm{d}t - \sum_{i=1}^{2}(\rho_i\varphi_i)|_{t_1}^{t_2}.\end{aligned} \tag{27}$$

Assuming (1) satisfied, let us compute variations of G by ρ_i and ξ:

$$\begin{aligned}\delta_{\rho_i} G = \int_{t_1}^{t_2}\Bigg[\int_{Q_j(t)} \delta\rho_j\left[-\varphi_{jt} - \frac{(\nabla\varphi_j)^2}{2} - U_j(\rho_j) - \rho_j\frac{dU_j}{d\rho_j}\right]\mathrm{d}Q \\ - \int_{Q_j(t)} \rho_j[\delta\varphi_{jt} + (\nabla\varphi_j, \nabla\delta\varphi_j)]\,\mathrm{d}Q + \int_{S_0}\rho_{01}V_0\delta\varphi_1\sin(\nu t)\,\mathrm{d}S\Bigg]\,\mathrm{d}t \\ - [\delta\rho_j\varphi_j + \rho_j\delta\varphi_j]|_{t_1}^{t_2} = 0,\ j=1,2.\end{aligned} \tag{28}$$

Accounting for (26), the second line (for domain Q_2, $S_0 = \emptyset$) equals to zero, but (24) and constraint (1) lead to

$$-\varphi_{jt} - \frac{(\nabla\varphi_j)^2}{2} - gx_1 - U_j(\rho_j) - \rho_j \frac{dU_j}{d\rho_j} = 0. \tag{29}$$

Applying the grad-action to (29) and transforming the result based on the pressure definition (25) give (1b).

Computing the ξ-variation of (27), accounting for (24) and using the formulas for the area variation of $|\Sigma|$ give

$$\begin{aligned}\delta_\xi G = \int_{t_1}^{t_2} \Bigg[&\sum_{i=1}^{2} \int_{Q_i(t)} \rho_i[-\delta\varphi_{it} - (\nabla\varphi_i, \nabla\delta\varphi_i)]\,\mathrm{d}Q \\ &+ \int_{\Sigma(t)} \sum_{i=1}^{2} (-1)^i \frac{\delta\xi}{|\nabla\xi|}\rho_i \left[-\varphi_{it} - \frac{(\nabla\varphi_i)^2}{2} - U_i(\rho_i)\right] \mathrm{d}S \\ &- T_s \int_{\Sigma(t)} [-k_1 - k_2] \frac{\delta\xi}{|\nabla\xi|}\,\mathrm{d}S + \int_{S_0} \rho\delta\varphi_1 V_0 \sin(\nu t)\,\mathrm{d}S \Bigg] \mathrm{d}t \\ &- \sum_{i=1}^{2} (\rho_i\delta\varphi_i)|_{t_1}^{t_2} = 0. \end{aligned} \tag{30}$$

Accounting for (26) with φ and $\delta\varphi$ leads to

$$\begin{aligned}\delta_\xi G = \int_{t_1}^{t_2} \Bigg[\Bigg\{ \int_{\Sigma(t)} &\sum_{i=1}^{2} (-1)^i \rho_i \left[-\varphi_{it} - \frac{(\nabla\varphi_i)^2}{2} - U_i(\rho_i)\right] \\ &+ T_s[k_1 + k_2] \Bigg\} \frac{\delta\xi}{|\nabla\xi|}\,\mathrm{d}S \Bigg] \mathrm{d}t = 0, \end{aligned} \tag{31}$$

in what follows, the dynamic condition (4) is satisfied if and only if (29) ($\delta_{\rho_j} G = 0,\ j = 1, 2$) is satisfied.

The next formulation was called in [11] the *Bateman variational principle for a compressible fluid*. It is based on the functional which is not constrained to (1a), (3c), (3a), (3b) and the extremal point condition of the functional naturally deduces (1)–(3). According to [11], the functional should take the form

$$B(\xi,\varphi_i,\rho_i)=\int_{t_1}^{t_2}\left\{\sum_{i=1}^{2}\int_{Q_i(t)}\rho_i\left[-\dot\varphi_i-\frac{(\nabla\varphi_i)^2}{2}-U_i(\rho_i)\right]\mathrm{d}Q\right.$$
$$\left.-T_s|\Sigma|+\int_{S_0}\rho_{01}V_0\varphi_1\sin(\nu t)\,\mathrm{d}S\right\}\mathrm{d}t. \quad (32)$$

It differs from the action (27) due to the absence of the last summand.

Theorem 4.2 *Under assumption P1, the solution of (1)-(4) coincides with stationary points of the Bateman action (32) subject to isochronous smooth variations*

$$\delta\xi|_{t_1,t_2}=0;\ \delta\varphi_i|_{t_1,t_2}=0;\ \delta\rho_i|_{t_1,t_2}=0. \quad (33)$$

Proof. The proposition follows from the formulas for variations of (27) by ρ_j, ξ and the formula expressing the variation by φ_j

$$\delta_{\varphi_j}B=\int_{t_1}^{t_2}\left[-\int_{Q_j(t)}\rho_j[\delta\dot\varphi_j+(\nabla\varphi_j,\nabla\delta\varphi_j)]\,\mathrm{d}Q\right.$$
$$\left.+\int_{S_0}\rho_{01}\delta\varphi_jV_0\sin(\nu t)\,\mathrm{d}S\right]\mathrm{d}t=\int_{t_1}^{t_2}\left[\int_{Q_j(t)}[\dot\rho_j+div(\rho_j\nabla\varphi_j)]\delta\varphi_j\,\mathrm{d}Q\right.$$
$$-\int_{S\setminus S_0}\rho_j\frac{\partial\varphi_j}{\partial n}\delta\varphi_j\,\mathrm{d}S-\int_\Sigma\rho_j\left[\frac{\partial\varphi_j}{\partial n}+\frac{\dot\xi}{|\nabla\xi|}\right]\delta\varphi_j\,\mathrm{d}S$$
$$\left.-\delta_{ij}\int_{S_0}\left(\rho_1\frac{\partial\varphi_1}{\partial n}-\rho_{01}V_0\sin(\nu t)\right)\delta\varphi_1\,\mathrm{d}S\right]\mathrm{d}t+\rho_j\delta\varphi_j|_{t_1}^{t_2}=0, \quad (34)$$

where δ_{ij} is the Kronecker delta.

If (29) is fulfilled, but φ_j are the velocity potentials, the integrand in (32) coincides with the pressure defined by (25). Therefore, the Lagrangian in the Bateman action really coincides with the pressure, but *only on the solution of (1)–(4)*, namely, only on the extrema points of the action.

According to the *Berdichevskii idea*, we introduce the functional

$$B_b(\xi,\varphi_i)=\int_{t_1}^{t_2}\left\{\sum_{i=1}^{2}\int_{Q_i(t)}\{\sup_{\rho_i}\left(\rho_i\left[-\dot\varphi_i-\frac{(\nabla\varphi_i)^2}{2}-U_i(\rho_i)\right]\right)\right\}\mathrm{d}Q$$

$$- T_s|\Sigma| + \int_{S_0} \rho_{01} V_0 \varphi_1 \sin(\nu t)\, \mathrm{d}S\} \,\mathrm{d}t. \quad (35)$$

The following theorem gives the necessary generalization of the *Bateman-Berdichevskii* principle. The proof immediately follows from Theorem 4.2.

Theorem 4.3 *Under assumptions P1, the solution of (1)-(4) coincides with the extremal points of the action (35) subject to the smooth isochronous variations ξ and φ_i:*

$$\delta\xi|_{t_1,t_2} = 0;\ \delta\varphi_i|_{t_1,t_2} = 0. \quad (36)$$

The Bateman–Berdichevskii variational formulation explicitly employs the pressure (φ_i are the velocity potentials) so that varying the action by ρ_i is replaced by the sup-operation. Assuming the Euler equation (1b) is fulfilled due to the sup-operation, one can express p via φ: $p = P^{-1}(-\dot{\varphi} - \frac{1}{2}(\nabla\varphi)^2 - gx_1)$, where p is the pressure. Thereby, we arrive at the action

$$B_l(\xi, \varphi_i) = \int_{t_1}^{t_2} \left\{ \sum_{i=1}^{2} \int_{Q_i(t)} \left\{ P^{-1}\rho_i \left[-\dot{\varphi}_i - \frac{(\nabla\varphi_i)^2}{2} - U_i(\rho_i) \right] \right\} \mathrm{d}Q \right.$$
$$\left. -T_s|\Sigma| + \int_{S_0} \rho_{01} V_0 \varphi_1 \sin(\nu t)\, \mathrm{d}S \right\} \mathrm{d}t. \quad (37)$$

The proof of the following theorem automatically follows from the proof of Theorem 4.2.

Theorem 4.4 *Assuming P1 makes the (1)–(4) solution coinciding with the extremal points of the action (37) with respect to independent isochronous smooth variations of ξ and φ_i: $\delta\xi|_{t_1,t_2} = 0$; $\delta\varphi_i|_{t_1,t_2} = 0$.*

5 Quasi-potential energy of the drop vibroequilibria

We showed that the nondimensional problem (8) has an asymptotic solution (18) in which the lowest-order term responsible for the drop motions is uniquely a function of spatial variables and the slow time $\tau = \epsilon^{3/2} t$. The lowest-order term are governed by (21). We also proved theorems on the Lagrange-type variational formulations of the original free-interface problem (1)–(2). In this section, these variational formulations will be an object of the multi-timing technique to derive the quick-time averaged variational formulations and a quasi-potential energy of the drop vibroequilibria governed by (22).

Theorem 5.1 *Under assumptions P1 and P2, finding the quick-time averaged solution from the Lagrange variational formulation by Theorem 4.1 is equivalent to finding the stationary points of the nondimensional functional (23) (denoted by G^*)*

$$\langle G^*(\xi,\varphi_i,\rho_i)\rangle_t = const + \epsilon^{3/2}\mathcal{G}(\zeta,\varphi) + O(\epsilon^2),$$

within

$$\mathcal{G}(\zeta,\varphi,\Phi) = \int_{\tau_1}^{\tau_2}\left\{\int_{\Omega_2(\tau)} \tfrac{1}{2}(\nabla\varphi)^2\,\mathrm{d}Q - \mu\mu_1|\Gamma(\tau)|\right.$$
$$\left. + \frac{\mu_1}{4}\int_{\Omega_1(\tau)}\left[k^2\Phi^2 - (\nabla\Phi)^2\right]\mathrm{d}Q - \frac{\mu_1}{2k}\int_{S_0}\Phi V(x)\,\mathrm{d}S\right\}\mathrm{d}\tau \quad (38)$$

subject to the kinematic constraint

$$\nabla^2\varphi = 0 \ \textit{in}\ \Omega_2(\tau); \quad \frac{\partial\varphi}{\partial n} = -\frac{\zeta_\tau}{|\nabla\zeta|}\ \textit{on}\ \Gamma(\tau), \quad (39a)$$

$$\nabla^2\Phi + k^2\Phi = 0\ \textit{in}\ \Omega_1(\tau);\ \frac{\partial\Phi}{\partial n} = 0\ \textit{on}\ S_1\cup\Gamma(\tau),$$
$$\frac{\partial\Phi}{\partial n} = V(x)/k\ \textit{on}\ S_0 \quad (39b)$$

for isochronous smooth variations of $\delta\zeta|_{\tau_1,\tau_2} = 0$, where $\zeta(x,\tau) = 0$ governs the quick-time averaged deformations of the drop surface $\Gamma(\tau)$ ($\Omega_2(\tau)$ and $\Omega_1(\tau)$ are the liquid and gas domains separated by $\Gamma(\tau)$) on the slow-time $\tau = \epsilon^{3/2}t$ scale.

Proof. According to Theorem 4.1, finding the solution of (1)-(4) (nondimensional statement (8)) is equivalent to description of the extrema points of the action (23). Adopting the nondimensional variational statement, substituting the asymptotic solution (18) into variational and differential expressions of Theorem 4.1 and choosing $|t_2 - t_1| > \epsilon^{-3/2}$, we get $\langle G(\xi,\varphi_i,\rho_i)\rangle_t = const + \epsilon^{3/2}\mathcal{G}(\zeta,\varphi) + O(\epsilon^2)$ and the kinematic constraint (39).

Let ζ,φ be a local extrema point of the action (38) subject to (39). Obviously, ζ and φ satisfy (21). Taking (18) in the nondimensional formulation of Theorem 4.1 gives, within to higher-order components, a stationary point of G_*.

Theorem 5.2 *Under assumptions P1 and P2, finding the quick-time averaged solution from the Bateman variational formulation by Theorem 4.2 is equivalent to finding the stationary points of the time-averaged nondimensional Bateman-type action (32) (denoted by B^*)*

$$\langle B^*(\xi,\varphi_i,\rho_i)\rangle_t = const + \epsilon^{3/2}\mathcal{B}(\zeta,\varphi,\Phi) + O(\epsilon^2),$$

where

$$\mathcal{B}(\zeta,\varphi,\Phi) = \int_{\tau_1}^{\tau_2}\Bigg\{\int_{\Omega_2(\tau)}\left[-\varphi_\tau - \tfrac{1}{2}(\nabla\varphi)^2\right]\,\mathrm{d}Q - \mu\mu_1|\Gamma(\tau)| + \frac{\mu_1}{4}\int_{\Omega_1(\tau)}\left[k^2\Phi^2 - (\nabla\Phi)^2\right]\,\mathrm{d}Q - \frac{\mu_1}{2k}\int_{S_0}\Phi\, V(x)\,\mathrm{d}S\Bigg\}\,\mathrm{d}\tau, \quad (40)$$

subject to isochronous smooth variations

$$\delta\zeta|_{\tau_1,\tau_2} = 0;\;\; \delta\varphi|_{\tau_1,\tau_2} = 0;\;\; \delta\Phi|_{\tau_1,\tau_2} = 0.$$

Proof. The proof scheme is the same as in the previous theorem.

Remark 5.1 *The quick-time averaged variational formulations of the Bateman and Bateman–Berdichevskii types lead to Theorem 5.2 which can be treated as the Bateman–Luke variational principle for the weightless drop dynamics levitating in the zero-gravity and affected by the surface tension as well as the Langevin radiation pressure.*

Remark 5.2 *Assuming the τ-independent solutions in Theorems 5.1 and 5.2 leads to the quasi-potential energy of the mechanical system. This gives the following theorem.*

Theorem 5.3 *Finding the stable drop vibroequilibria from (22) is equivalent to minimization of the quasi-potential energy functional*

$$U = \mu|\Gamma_0| - \frac{1}{4}\int_{\Omega_{10}}\left(k^2\Phi^2 - (\nabla\Phi)^2\right)\,\mathrm{d}Q + \frac{1}{2k}\int_{S_0} V(x)\,\Phi\,\mathrm{d}S \quad (41)$$

subject to

$$\int_{\Omega_{20}}\mathrm{d}Q = m_2 = const \quad (42)$$

and

$$\nabla^2\Phi + k^2\Phi = 0 \text{ in } \Omega_{10};\;\; \frac{\partial\Phi_1}{\partial n} = 0 \text{ on } S_1\cup\Gamma_0, \quad \frac{\partial\Phi_1}{\partial n} = V(x)/k \text{ on } S_0. \quad (43)$$

6 Conclusions

Based on differential and variational formulations, we study the quick-time averaged motions of an acoustically levitated drop. An emphasis is on the visually-observed quasi-static drop shapes called, in the present paper, the drop vibroequilibria. Along with the differential formulation of the problem on the drop vibroequilibria, a functional responsible for the quasi-potential energy of the system is derived.

The forthcoming studies should focus on small drop oscillations with respect to the vibroequlibria. This implies the corresponding spectral theorems which can be considered as a generalization of the famous Rayleigh results.

Another open problem may consists of appropriate numerical methods which should solve the problem on the drop equilibria. The variational formulation of Theorem 5.3 should facilitate the numerical method.

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