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A complete weakly-nonlinear multimodal system for modeling the liquid sloshing dynamics in an upright annular cylindrical tank *

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Employing the Lukovsky–Miles equations, a weakly–nonlinear modal system is derived for studying the resonant liquid sloshing in an upright [multi-] annular cylindrical tank. The system contains all up to the third order polynomial quantities in terms of the hydrodynamic generalized coordinates and can be adopted for getting the Narimanov-

Використовуючи рівняння Луковського–Майлса, виводиться слабо-нелінійна модальна система для дослідження резонансних коливань рідини в вертикальному баці [мульти-] кільцевого перерізу. Система утримує в собі всі аж до третього порядку поліноміальні члени в термінах узагальнених гідродинамічних координат та може бути базою для отриманння адаптивних модальних рівнянь, а також рівнянь типу Наріманова–Моісєєва. Останні рівняння показали свою ефективність для моделювання резонансних коливань рідини.

Используя уравнения Луковского–Майлса, выводится слабо-нелинейная модальная система для исследования резонансных колебаний жидкости в вертикальном баке [мульти-] кольцевого сечения. Система включает все до третьего порядка полиномиальные члены в терминах обобщенных гидродинамических координат и может быть базой для получения адаптивных модальных уравнений, а также уравнений типа Нариманова–Моисеева. Последние уравнения показали свою эффетивность для моделирования резонансных колебаний жидкости.

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A Introduction

The nonlinear multimodal method is an efficient analytical tool for studying the sloshing dynamics of an inviscid incompressible liquid with irrotational flow. The method makes it possible to derive compact systems of nonlinear ordinary differential equations which could be employed in classification and quantification of nonlinear liquid motions in resonantly excited tanks. The most general, fully-nonlinear modal equations were independently derived by Lukovsky and Miles in 1976 (see, details in the books [3, 4]). These equations could be a base for getting approximate weakly-nonlinear modal equations which are exemplified in [1,3]. Derivation of these approximate modal equations involves nontrivial analytical algorithms which should, in some cases, use the computer algebra [5, 6]. An alternative algorithm was recently proposed in [2] for the spherical tank shape. The present paper extends this algorithm to the case of an annular upright cylindrical tank.

B Modal solution and the Lukovsky–Miles equations

The absolute velocity potential $\Phi(x, y, z, t)$, and the free surface $\Sigma(t)$ are the two unknowns of the nonlinear free-boundary problem which describes the liquid sloshing dynamics under assumption that the liquid is ideal incompressible with irrotational flow. Normally, the liquid motions are considered in the tank-fixed coordinate system $Oxyz$ and, for brevity, we assume that the tank is exposed to translatory motions whose velocity is defined by the time-dependent vector $\boldsymbol{v}_0(t) = (v_{O1}(t), v_{02}(t), v_{03}(t))$ in projections on the axes of this coordinate system.

In accordance with the multimodal method, the velocity potential is postulated in the form

$$\Phi(x, y, z, t) = \boldsymbol{v}_0 \cdot \boldsymbol{r} + \varphi(x, y, z, t), \quad \boldsymbol{r} = (x, y, z), \tag{1}$$

where, after introducing the cylindrical coordinate system whose Ox axis coincides with the symmetry axis,

$$\varphi(x, r, \theta, t) = \sum_{Mi} P_{Mi}(t)\, \psi_{Mi}(x, r) \cos(M\theta) + \sum_{mi} R_{mi}(t)\, \psi_{mi}(x, r) \sin(m\theta). \tag{2}$$

Here, the capital indexes imply summation from zero to infinity but small indexes mean the summation from 1 to infinity, namely, $M = 0, 1, 2\ldots$, $m = 1, 2\ldots$, $i = 1, 2\ldots$. The functions $\psi_{Mi}(x, r)$ are the meridional projections of the natural sloshing modes which take, for the considered tank shape, the form

$$\psi_{Mi} = \frac{\cosh(k_{Mi}(x+h))}{\cosh(k_{Mi}h)} f_{Mi}(k_{Mi}r) \tag{3}$$

so that k_{Mi} and $f_{Mi}(k_{Mi}r)$ can be found in an analytical form (the solution for the single-ring cross-sectional shape is given in [1, 3]). The functions $P_{Mi}(t)$ and $R_{Mi}(t)$ can be treated as the generalized velocities.

The free surface equation admits in the considered case the single-valued modal solution

$$x = f(r, \theta, t) = \sum_{Mi} p_{Mi}(t) f_{Mi}(r) \cos(M\theta) + \sum_{mi} r_{mi}(t) f_{mi}(r) \sin(m\eta), \tag{4}$$

where $p_{Mi}(t)$ and $r_{Mi}(t)$ play the role of the generalized coordinates.

The paper [2] showed that the fully-nonlinear modal equations by Lukovsky & Miles for the axisymmetric tanks take the form

$$\sum_{Mn} \frac{\partial A^p_{Ab}}{\partial p_{Mn}} \dot{p}_{Mn} + \sum_{mn} \frac{\partial A^p_{Ab}}{\partial r_{mn}} \dot{r}_{mn} = \sum_{Mn} A^{pp}_{(Ab)(Mn)} P_{Mn} + \sum_{mn} A^{pr}_{(Ab),(mn)} R_{mn}, \tag{5a}$$

$$\sum_{Mn} \frac{\partial A^r_{ab}}{\partial p_{Mn}} \dot{p}_{Mn} + \sum_{mn} \frac{\partial A^r_{ab}}{\partial r_{mn}} \dot{r}_{mn} = \sum_{Mn} A^{pr}_{(Mn),(ab)} P_{Mn} + \sum_{mn} A^{rr}_{(ab),(mn)} R_{mn} \tag{5b}$$

implying the *kinematic subsystem* but the *dynamic modal equations* read as

$$\sum_{Mn} \frac{\partial A^p_{Mn}}{\partial p_{Ab}} \dot{P}_{Mn} + \sum_{mn} \frac{\partial A^r_{mn}}{\partial p_{Ab}} \dot{R}_{mn} + \frac{1}{2} \sum_{MnLk} \frac{\partial A^{pp}_{(Mn)(Lk)}}{\partial p_{Ab}} \dot{P}_{Mn} \dot{P}_{Lk}$$
$$+ \frac{1}{2} \sum_{mnlk} \frac{\partial A^{rr}_{(mn)(lk)}}{\partial p_{Ab}} \dot{R}_{mn} \dot{R}_{lk} + \frac{1}{2} \sum_{Mnlk} \frac{\partial A^{pr}_{(Mn)(lk)}}{\partial p_{Ab}} \dot{P}_{Mn} \dot{R}_{lk} + \frac{\partial l}{\partial p_{Ab}} = 0, \tag{6a}$$

$$\sum_{Mn} \frac{\partial A^p_{Mn}}{\partial r_{ab}} \dot{P}_{Mn} + \sum_{mn} \frac{\partial A^r_{mn}}{\partial r_{ab}} \dot{R}_{mn} + \frac{1}{2} \sum_{MnLk} \frac{\partial A^{pp}_{(Mn)(Lk)}}{\partial r_{ab}} \dot{P}_{Mn} \dot{P}_{Lk}$$

$$+\frac{1}{2}\sum_{mnlk}\frac{\partial A^{rr}_{(mn)(lk)}}{\partial r_{ab}}\dot R_{mn}\dot R_{lk}+\frac{1}{2}\sum_{Mnlk}\frac{\partial A^{pr}_{(Mn)(lk)}}{\partial r_{ab}}\dot P_{Mn}\dot R_{lk}+\frac{\partial l}{\partial r_{ab}}=0. \tag{6b}$$

Here, $A = 0,1,2\ldots$, $a,b = 1,2\ldots$ and the following coefficients are function of the generalized coordinates

$$A^{p}_{Ab}=\rho\int\limits_{r_0}^{r}\int\limits_{0}^{2\pi}\cos(A\theta)\,\Xi^{(0)}_{Ab}\left(r;\theta;p_{Ij}(t);r_{ij}(t)\right)\,d\theta\,dr, \tag{7a}$$

$$A^{r}_{Ab}=\rho\int\limits_{r_0}^{r}\int\limits_{0}^{2\pi}\sin(a\theta)\,\Xi^{(0)}_{Ab}\left(r;\theta;p_{Ij}(t);r_{ij}(t)\right)\,d\theta\,dr, \tag{7b}$$

$$A^{pp}_{(Ab)(Mn)}=\int\limits_{r_0}^{r}\int\limits_{0}^{2\pi}\Big[\cos(A\theta)\cos(M\theta)\cdot\Xi^{(1)}_{(Ab)(Mn)}\left(r;\theta;p_{Ij}(t);r_{ij}(t)\right)$$
$$+\sin(A\theta)\sin(M\theta)\cdot\Xi^{2}_{(Ab)(Mn)}\left(r;\theta;p_{Ij}(t)r_{ij}(t)\right)\Big]\,d\theta\,dr, \tag{8a}$$

$$A^{rr}_{(ab)(mn)}=\int\limits_{r_0}^{r}\int\limits_{0}^{2\pi}\Big[\sin(a\theta)\sin(m\theta)\cdot\Xi^{(1)}_{(ab)(mn)}\left(r;\theta;p_{Ij}(t);r_{ij}(t)\right)$$
$$+\cos(a\theta)\cos(m\theta)\cdot\Xi^{2}_{(ab)(mn)}\left(r;\theta;p_{Ij}(t)r_{ij}(t)\right)\Big]\,d\theta\,dr, \tag{8b}$$

$$A^{pr}_{(Ab)(mn)}=\int\limits_{r_0}^{r}\int\limits_{0}^{2\pi}\Big[\cos(A\theta)\sin(m\theta)\cdot\Xi^{(1)}_{(Ab)(mn)}\left(r;\theta;p_{Ij}(t);r_{ij}(t)\right)$$
$$-\sin(A\theta)\cos(m\theta)\cdot\Xi^{(2)}_{Am}\left(r;\theta;p_{Ij}(t);r_{ij}(t)\right)\Big]\,d\theta\,dr. \tag{8c}$$

C Weakly-nonlinear modal equations

Our task consists of deriving, in an explicit form, a Taylor–type approximation of (7) and (8), resolving the kinematic subsystem with respect to P_{Mn} and R_{mn}, and deriving the weakly-nonlinear modal equations with respect to the generalized coordinates only. Denoting

$$f = \sum_{Mi} p_{Mi} f_{Mi} \cos(M\theta) + \sum_{mi} r_{mi} f_{mi} \sin(m\theta) \tag{9a}$$

deduces that

$$\begin{aligned} f^2 = & \sum_{Mi}\sum_{Nj} p_{Mi} p_{Nj} f_{Mi} f_{Nj} \cos(M\theta)\cos(N\theta) \\ & + \sum_{Mi}\sum_{mj} p_{Mi} r_{mj} f_{Mi} f_{mj} \cos(M\theta)\sin(m\theta) \\ & + \sum_{mi}\sum_{nj} r_{mi} r_{nj} f_{mi} f_{nj} \sin(m\theta)\sin(n\theta), \end{aligned} \tag{9b}$$

$$\begin{aligned} f^3 = & \sum_{Mi}\sum_{Nj}\sum_{Kl} p_{Mi} p_{Nj} p_{Kl} f_{Mi} f_{Nj} f_{Kl} \cos(M\theta)\cos(N\theta)\cos(K\theta) \\ & + \sum_{Mi}\sum_{Nj}\sum_{ml} p_{Mi} p_{Nj} r_{ml} f_{Mi} f_{Nj} f_{ml} \cos(M\theta)\cos(N\theta)\sin(m\theta) \\ & + \sum_{Mi}\sum_{nj}\sum_{ml} p_{Mi} r_{nj} r_{ml} f_{Mi} f_{nj} f_{ml} \cos(M\theta)\sin(n\theta)\sin(m\theta) \\ & + \sum_{mi}\sum_{nj}\sum_{kl} r_{mi} r_{nj} r_{kl} f_{mi} f_{nj} f_{kl} \sin(m\theta)\sin(n\theta)\sin(k\theta). \end{aligned} \tag{9c}$$

If the following notations are adopted

$$\Delta_{\underbrace{i\dots j}_{N_1},\underbrace{n\dots k}_{N_2}} = \int_0^{2\pi} \underbrace{\cos(i\theta)\dots\cos(j\theta)}_{N_1} \cdot \underbrace{\sin(i\theta)\dots\sin(j\theta)}_{N_2}\, d\theta, \tag{10a}$$

$$\lambda_{\underbrace{(mi)\dots(nj)}_{N_1},\underbrace{(kl)\dots(sw)}_{N_2}} = \int_{r_0}^{r} \underbrace{f_{mi}\dots f_{nj}}_{N_1} \cdot \underbrace{f'_{kl}\dots f'_{sw}}_{N_2}\, dr, \tag{10b}$$

then

$$\int_{r_0}^{r}\int_0^{2\pi} f^2\, dr\, d\theta = \sum_{MiNj} p_{Mi} p_{Nj} \Delta_{MN} \lambda_{(Mi)(Nj)}$$

$$+\sum_{Mimj} p_{Mi} r_{mj} \Delta_{M,m} \lambda_{(Mi)(mj)} + \sum_{minj} r_{mi} r_{nj} \Delta_{,mn} \lambda_{(mi)(nj)}. \quad (11)$$

Furthermore, expanding A^p_{Mi} and A^r_{mi} up to the third polynomial terms in p_{Mi} and r_{mi} gives

$$A^p_{Ab} = \rho \int_{r_0}^{r} \int_{0}^{2\pi} \cos(A\theta) \cdot \underbrace{\frac{\sinh k_{Ab}(x+h)}{k_{Ab}\cosh(k_{Ab}h)} f_{Ab}(k_{Ab}r)\Big|_0^{x=f(r,\theta,t)}}_{\Xi^{(0)}_{Ab}(r;\theta;p_{Ij}(t);r_{ij}(t))} d\theta dr, \quad (12a)$$

$$A^r_{ab} = \rho \int_{r_0}^{r} \int_{0}^{2\pi} \sin(A\theta) \cdot \underbrace{\frac{\sinh k_{ab}(x+h)}{k_{ab}\cosh(k_{ab}h)} f_{ab}(k_{ab}r)\Big|_0^{x=f(r,\theta,t)}}_{\Xi^{(0)}_{ab}(r;\theta;p_{Ij}(t);r_{ij}(t))} d\theta dr, \quad (12b)$$

where

$$\begin{aligned}\Xi^{(0)}_{Ab} &= \frac{f_{Ab}}{k_{Ab}\cosh(k_{Ab}h)} [\sinh(k_{Ab}x)\cosh(k_{Ab}h) + \cosh(k_{Ab}x)\sinh(k_{Ab}h)]\Big|_0^x \\ &= \frac{f_{Ab}}{k_{Ab}} [\sinh(k_{Ab}x) + \tanh(k_{Ab})\cosh(k_{Ab}x)]\Big|_0^x \\ &= f_{Ab}\left[x + \frac{k_{Ab}\tanh(k_{Ab}h)}{2}x^2 + \frac{k^2_{Ab}}{6}x^3\right]. \quad (13)\end{aligned}$$

Tedious derivations lead to the expressions

$$\begin{aligned}A^p_{Ab} &= \sum_{Mi} p_{Mi} \Delta_{AM} \lambda_{(Ab)(Mi)} \\ &+ \frac{k_{Ab}\tanh(k_{Ab}h)}{2} \Bigg[\sum_{MiNj} p_{Mi} p_{Nj} \Delta_{AMN} \lambda_{(Ab)(Mi)(Nj)} \\ &\qquad + \sum_{minj} r_{mi} r_{nj} \Delta_{A,mn} \lambda_{(Ab)(mi)(nj)}\Bigg] \\ &+ \frac{k^2_{Ab}}{6} \Bigg\{\sum_{MiNjKl} p_{Mi} p_{Nj} p_{Kl} \Delta_{AMNK} \lambda_{(Ab)(Mi)(Nj)(Kl)}\end{aligned}$$

$$+ \sum_{Minjml} p_{Mi} r_{nj} r_{ml} \Delta_{AM,nm} \lambda_{(Ab)(Mi)(nj)(ml)} \Bigg\}, \quad (14a)$$

$$A^r_{ab} = \sum_{mi} r_{mi} \Delta_{,am} \lambda_{(ab)(mi)} + \frac{k_{ab}\tanh(k_{ab}h)}{2} \sum_{Mimj} p_{Mi} r_{mj} \Delta_{M,am} \lambda_{(ab)(Mi)(mj)} + \frac{k^2_{ab}}{6} \Bigg\{ \sum_{MiNjml} p_{Mi} p_{Nj} r_{ml} \Delta_{MN,am} \lambda_{(ab)(Mi)(Nj)(ml)} + \sum_{minjkl} r_{mi} r_{nj} r_{kl} \Delta_{,amnk} \lambda_{(ab)(mi)(nj)(kl)} \Bigg\}, \quad (14b)$$

and

$$\frac{\partial A^p_{Ab}}{\partial p_{Mn}} = \Delta_{AM} \lambda_{(Ab)(Mn)} + k_{Ab} \tanh(k_{Ab}h) \sum_{Nj} p_{Nj} \Delta_{AMN} \lambda_{(Ab)(Mn)(Nj)} + \frac{k^2_{Ab}}{2} \sum_{NjKl} p_{Nj} p_{Kl} \Delta_{AMNK} \lambda_{(Ab)(Mn)(Kl)(Nj)} + \frac{k^2_{Ab}}{6} \sum_{kjml} r_{kj} r_{ml} \Delta_{AM,km} \lambda_{(Ab)(Mn)(kj)(ml)}, \quad (15a)$$

$$\frac{\partial A^p_{Ab}}{\partial r_{mn}} = k_{Ab} \tanh(k_{Ab}h) \sum_{kj} r_{kj} \Delta_{A,mk} \lambda_{(Ab)(mn)(kj)} + \frac{k^2_{Ab}}{3} \sum_{Mikj} p_{Mi} r_{kj} \Delta_{AM,mk} \lambda_{(Ab)(Mi)(mn)(kj)}, \quad (15b)$$

$$\frac{\partial A^r_{ab}}{\partial p_{Mn}} = \frac{1}{2} k_{ab} \tanh(k_{ab}h) \sum_{mj} r_{mj} \Delta_{M,am} \lambda_{(ab)(Mn)(mj)} + \frac{k^2_{ab}}{3} \sum_{Njml} p_{Nj} r_{ml} \Delta_{MN,am} \lambda_{(ab)(Mn)(Nj)(ml)}, \quad (15c)$$

$$\frac{\partial A^r_{ab}}{\partial r_{mn}} = \Delta_{,am}\lambda_{(ab)(mn)} + \frac{k_{ab}\tanh(k_{ab}h)}{2}\sum_{Mi} p_{Mi}\Delta_{M,am}\lambda_{(ab)(Mi)(mn)} + \frac{k_{ab}^2}{6}\sum_{MiNj} p_{Mi}p_{Nj}\Delta_{MN,am}\lambda_{(ab)(Mi)(Nj)(mn)} + \frac{k_{ab}^2}{2}\sum_{ijkl} r_{ij}r_{kl}\Delta_{,amik}\lambda_{(ab)(mn)(ij)(kl)}. \tag{15d}$$

Expanding $A^{pp}_{(Mi)(Nj)}$, $A^{rr}_{(mi)(nj)}$ and $A^{pr}_{(Mi),(nj)}$ into the Taylor series gives

$$A^{pp}_{(Ab)(Mn)} = \rho\int_Q \nabla\left(\psi_{Ab}\cos(A\theta)\right)\cdot\nabla\left(\psi_{Mn}\cos(M\theta)\right)dQ, \tag{16}$$

where

$$\nabla\left(\psi_{Ab}(x,r)\cos(A\theta)\right) = \frac{\partial\psi_{Ab}\cos(A\theta)}{\partial x} + \frac{\partial\psi_{Ab}\cos(A\theta)}{\partial\theta} + \frac{1}{r}\frac{\partial\psi_{Ab}\cos(A\theta)}{\partial r}$$

$$\frac{\partial\psi_{Ab}\cos(A\theta)}{\partial x} = \frac{k_{Ab}\sinh k_{Ab}(x+h)}{\cosh(k_{Ab}h)}\cdot f_{Ab}(k_{Ab}r)\sin(A\theta)$$

$$\frac{\partial\psi_{Ab}\cos(A\theta)}{\partial\theta} = -\frac{A\cosh k_{Ab}(x+h)}{\cosh(k_{Ab}h)}\cdot f_{Ab}(k_{Ab}r)\cos(A\theta)$$

$$\frac{\partial\psi_{Ab}\cos(A\theta)}{\partial r} = \frac{k_{Ab}\cosh k_{Ab}(x+h)}{\cosh(k_{Ab}h)}\cdot f'_{Ab}(k_{Ab}r)\cos(A\theta)$$

$$(\nabla\psi_N\cdot\nabla\psi_K) = \frac{k_{Ab}k_{Mn}\sinh\left[k_{Ab}(x+h)\right]\sinh\left[k_{Mn}(x+h)\right]}{\cosh(k_{Ab}h)\cosh(k_{Mn}h)}f_{Ab}f_{Mn}\cos(A\theta)\cos(M\theta) + \frac{AM\cosh\left[k_{Ab}(x+h)\right]\cosh\left[k_{Mn}(x+h)\right]}{\cosh(k_{Ab}h)\cosh(k_{Mn}h)}f_{Ab}f_{Mn}\sin(A\theta)\sin(M\theta) + \frac{k_{An}k_{Mn}\cosh\left[k_{Ab}(x+h)\right]\cosh\left[k_{Mn}(x+h)\right]}{r^2\cosh(k_{Ab}h)\cosh(k_{Mn}h)}f'_{Ab}f'_{Mn}\cos(A\theta)\cos(M\theta). \tag{17}$$

After simplification for (8)

$$\Xi^{(1)}_{(Ab)(Mn)} = \int\limits_0^{f(r,\theta,t)} \frac{k_{Ab}k_{Mn}}{\cosh(k_{Ab}h)\cosh(k_{Mn}h)} \cdot \left(\sinh\left[k_{Ab}(x+h)\right]\sinh\left[k_{Mn}(x+h)\right] f_{Ab}f_{Mn} + \frac{1}{r^2}\cosh\left[k_{Ab}(x+h)\right]\cosh\left[k_{Mn}(x+h)\right] f'_{Ab}f'_{Mn}\right) dx, \quad (18a)$$

$$\Xi^{(2)}_{(Ab)(Mn)} = \int\limits_0^{f} \frac{AM\cosh\left[k_{Ab}(x+h)\right]\cosh\left[k_{Mn}(x+h)\right]}{\cosh\left(k_{Ab}h\right)\cosh\left(k_{Mn}h\right)} f_{Ab}f_{Mn}dx, \quad (18b)$$

$$A^{rr}_{(ab)(mn)} = \rho\int\limits_Q \left(\nabla\varphi_n\nabla\varphi_k\right) = \rho\int\limits_Q \nabla\left(\psi_{ab}\sin\left(a\theta\right)\right)\cdot\nabla\left(\psi_{mn}\sin(m\theta)\right)dQ, \quad (19)$$

$$A^{pr}_{(Ab)(mn)} = \rho\int\limits_Q \left(\nabla\varphi_N\nabla\varphi_k\right) = \rho\int\limits_Q \nabla\left(\psi_{Ab}\cos(A\theta)\right)\cdot\left(\psi_{mn}\sin(m\theta)\right)dQ. \quad (20)$$

Expanding $\Xi^{(1)}_{(Ab)(Mn)}$ and $\Xi^{(2)}_{(Ab)(Mn)}$ up to the second polynomial order in p_{Mi}, r_{mi} gives

$$\Xi^{(1)}_{(Ab)(Mn)} = \frac{K_{(Ab)(Mn)}}{K^{(0)}_{(Ab)(Mn)}}\left\{K^{(2)}_{(Ab)(Mn)}f_{Ab}f_{Mn} + K^{(1)}_{(Ab)(Mn)}\frac{f'_{Ab}f'_{Mn}}{r^2}\right\} + K_{(Ab)(Mn)}\left\{K^{(5)}_{(Ab)(Mn)}f_{Ab}f_{Mn} + \frac{f'_{Ab}f'_{Mn}}{r^2}\right\}\cdot x + \frac{K_{(Ab)(Mn)}}{2}\left\{K^{(4)}_{(Ab)(Mn)}f_{Ab}f_{Mn} + K^{(3)}_{(Ab)(Mn)}\frac{f'_{Ab}f'_{Mn}}{r^2}\right\}\cdot x^2, \quad (21a)$$

$$\Xi^{2}_{(Ab)(Mn)} = AMf_{Ab}f_{Mn}\left\{\frac{K^{(1)}_{(Ab)(Mn)}}{K^{(0)}_{(Ab)(Mn)}} + x + \frac{K^{(3)}_{(Ab)(Mn)}}{2}\cdot x^2\right\}, \quad (21b)$$

where

$$K_{(Ab)(Mn)} = k_{Ab}k_{Mn}; \quad K^{(0)}_{(Ab)(Mn)} = \frac{1}{k^2_{Ab} - k^2_{Mn}};$$

$$K^{(1)}_{(Ab)(Mn)} = k_{Ab}\tanh(k_{Ab}h) - k_{Mn}\tanh(k_{Mn}h);$$
$$K^{(2)}_{(Ab)(Mn)} = k_{Ab}\tanh(k_{Mn}h) - k_{Mn}\tanh(k_{Ab}h);$$
$$K^{(3)}_{(Ab)(Mn)} = k_{Ab}\tanh(k_{Ab}h) + k_{Mn}\tanh(k_{Mn}h);$$
$$K^{(4)}_{(Ab)(Mn)} = k_{Ab}\tanh(k_{Mn}h) + k_{Mn}\tanh(k_{Ab}h);$$
$$K^{(5)}_{(Ab)(Mn)} = \tanh(k_{Ab}h)\tanh(k_{Mn}h); \quad (22)$$

thus

$$\begin{aligned} A^{pp}_{(Ab)(Mn)} &= \mu^{(1)}_{(Ab)(Mn)}\Delta_{AM} + \mu^{(2)}_{(Ab)(Mn)}\Delta_{,AM} \\ &+ \Big(K_{(Ab)(Mn)}K^{(5)}_{(Ab)(Mn)} + AM\Big)\sum_{Ni} p_{Ni}\lambda_{(Ab)(Mn)(Ni)}\Delta_{AMN} \\ &+ K_{(Ab)(Mn)}\sum_{Ni} p_{Ni}\frac{\lambda_{(Ni),(Ab)(Mn)}}{r^2}\Delta_{AMN} \end{aligned}$$

$$\begin{aligned} &+ \mu^{(4)}_{(Ab)(Mn)}\sum_{NiZj} p_{Ni}p_{Zj}\lambda_{(Ab)(Mn)(Ni)(Zj)}\Delta_{AMNZ} \\ &+ \mu^{(4)}_{(Ab)(Mn)}\sum_{lizj} r_{li}r_{zj}\lambda_{(Ab)(Mn)(li)(zj)}\Delta_{AM,lz} \\ &+ \mu^{(3)}_{(Ab)(Mn)}\sum_{NiZj} p_{Ni}p_{Zj}\frac{\lambda_{(Ni)(Zj),(Ab)(Mn)}}{r^2}\Delta_{AMNZ} \end{aligned}$$

$$\begin{aligned} &+ \mu^{(3)}_{(Ab)(Mn)}\sum_{lizj} r_{li}r_{zj}\frac{\lambda_{(li)(zj),(Ab)(Mn)}}{r^2}\Delta_{AM,lz} \\ &+ \mu^{(5)}_{(Ab)(Mn)}\sum_{NiZj} p_{Ni}p_{Zj}\lambda_{(Ab)(Mn)(Ni)(Zj)}\Delta_{NZ,AM} \\ &+ \mu^{(5)}_{(Ab)(Mn)}\sum_{lizj} r_{li}r_{zj}\lambda_{(Ab)(Mn)(li)(zj)}\Delta_{,AMlz}; \quad (23a) \end{aligned}$$

$$\begin{aligned} A^{rr}_{(ab)(mn)} &= \mu^{(1)}_{(ab)(mn)}\Delta_{,am} + \mu^{(2)}_{(ab)(mn)}\Delta_{am} \\ &+ K_{(ab)(mn)}K^{(5)}_{(ab)(mn)}\sum_{Ni} p_{Ni}\lambda_{(ab)(mn)(Ni)}\Delta_{N,am} \end{aligned}$$

$$+K_{(ab)(mn)}\sum_{Ni}p_{Ni}\frac{\lambda_{(Ni),(ab)(mn)}}{r^2}\Delta_{N,am}+am\sum_{Ni}p_{Ni}\lambda_{(ab)(mn)(Ni)}\Delta_{amN}$$
$$+\mu^{(4)}_{(ab)(mn)}\sum_{NiZj}p_{Ni}p_{Zj}\lambda_{(ab)(mn)(Ni)(Zj)}\Delta_{NZ,am}$$
$$+\mu^{(4)}_{(ab)(mn)}\sum_{lizj}r_{li}r_{zj}\lambda_{(am)(mn)(li)(zj)}\Delta_{,amlz}$$
$$+\mu^{(3)}_{(ab)(mn)}\sum_{NiZj}p_{Ni}p_{Zj}\frac{\lambda_{(Ni)(Zj),(ab)(mn)}}{r^2}\Delta_{NZ,am}$$
$$+\mu^{(3)}_{(ab)(mn)}\sum_{lizj}r_{li}r_{zj}\frac{\lambda_{(li)(zj),(ab)(mn)}}{r^2}\Delta_{,amlz}$$
$$+\mu^{(5)}_{(ab)(mn)}\sum_{NiZj}p_{Ni}p_{Zj}\lambda_{(ab)(mn)(Ni)(Zj)}\Delta_{NZ,am}$$
$$+\mu^{(5)}_{(ab)(mn)}\sum_{lizj}r_{li}r_{zj}\lambda_{(ab)(mn)(li)(zj)}\Delta_{,amlz};\qquad(24a)$$

$$A^{pr}_{(Ab)(mn)}=\mu^{(1)}_{(Ab)(mn)}\Delta_{Am}-\mu^{(2)}_{(Ab)(mn)}\Delta_{,Am}$$
$$+K_{(Ab)(mn)}K^{(5)}_{(Ab)(mn)}\sum_{zj}r_{zj}\lambda_{(Ab)(mn)(zj)}\Delta_{A,mz}$$
$$+K_{(Ab)(mn)}\sum_{zj}r_{zj}\frac{\lambda_{(zj),(Ab)(mn)}}{r^2}\Delta_{A,mz}-Am\sum_{zj}r_{zj}\lambda_{(Ab)(mn)(zj)}\Delta_{m,Az}$$
$$+\mu^{(4)}_{(Ab)(mn)}\sum_{Nizj}p_{Ni}r_{zj}\lambda_{(Ab)(mn)(Ni)(zj)}\Delta_{AN,mz}$$
$$+\mu^{(3)}_{(Ab)(mn)}\sum_{Nizj}p_{Ni}r_{zj}\frac{\lambda_{(Ni)(zj),(Ab)(mn)}}{r^2}\Delta_{AN,mz}$$
$$-\mu^{(5)}_{(Ab)(mn)}\sum_{Nizj}p_{Ni}r_{zj}\lambda_{(Ab)(mn)(Ni)(zj)}\Delta_{AN,mz}\qquad(25a)$$

and

$$\mu^{(1)}_{(Ab)(Mn)}=\frac{K_{(Ab)(Mn)}}{K^{(0)}_{(Ab)(Mn)}}\left\{K^{(2)}_{(Ab)(Mn)}\lambda_{(Ab)(Mn)}+K^{(1)}_{(Ab)(Mn)}\frac{\lambda_{,(Ab)(Mn)}}{r^2}\right\};$$

$$\mu^{(2)}_{(Ab)(Mn)}=\frac{K^{(1)}_{(Ab)(Mn)}AM}{K^{(0)}_{(Ab)(Mn)}}\lambda_{(Ab)(Mn)};\ \mu^{(3)}_{(Ab)(Mn)}=\frac{K_{(Ab)(Mn)}K^{(3)}_{(Ab)(Mn)}}{2};$$

$$\mu^{(4)}_{(Ab)(Mn)} = \frac{K_{(Ab)(Mn)}K^{(4)}_{(Ab)(Mn)}}{2}; \; \mu^{(5)}_{(Ab)(Mn)} = \frac{K^{(3)}_{(Ab)(Mn)}AM}{2}. \quad (26a)$$

Using these expressions and the formulas from Appendix in [2] leads to the desirable weakly-nonlinear modal equations in which all the zero hydrodynamic coefficients are excluded.

D Concluding remarks

The present paper extends the derivation algorithm from [2] to the case of an upright multi-annular cylindrical tank. The analogous expressions for the Taylor approximation of the nonlinear quantities of the Lukovsky–Miles modal equations were obtained but the final form adaptive and Narimanov–Moiseev type equations can easily follow from the Taylor approximations and the formulas presented in Appendix of [2].

The forthcoming studies should focus on investigation of the secondary resonance phenomena as well as on examining the resonant liquid sloshing and comparison with recent experimental data.

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