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Fractal operator in weighted L_p space

Dedicated to memory of Professor Promarz M. Tamrazov

Fractal operators are the data representatives in fractal data compaction methods. With fractal compression, encoding is extremely computationally expensive. Decoding, however, is quite fast [1]. In order to increase encoding efficiency the prerequisites for fractal operator could be eased. The aim is to find new spaces and conditions under which iterations of the operator converge to some fixed point.

1. Introduction. Iterations convergence of fractal operator is being studied in different metric spaces. It is already studied in the space of compact subsets of real plane with Hausdorff metric [2]; p-summable functions for $1 \leqslant p < \infty$ with standard distance; space of continuous functions on real interval [3] and bounded functions on real plane with uniform metric.

This article describes fractal operator in weighted $L_p\big[(a,\, b)\big]$ space for $1 \leqslant p < \infty$. There are conditions found for the operator to be eventually contractive on some closed subset of the space. It guarantees that the operator has unique fixed point and its iterations converge to this point.

2. Fractal operator concept and properties. Let $I = [a,\, b] \subset \mathbb{R}$ be a closed interval,

$$L_p^{w(\cdot)}(I) = \Big\{ f\colon I \longrightarrow \mathbb{R} \;\Big|\; \|f\|_{L_p^{w(\cdot)}(I)} = \Big(\int_I w(x)|f(x)|^p\,dx \Big)^{\frac{1}{p}} < +\infty \Big\} \text{ —}$$

weighted $L_p(I)$ space with summable function $w\colon L_p(I) \longrightarrow \mathbb{R}$. As well as $L_p(I)$, $L_p^{w(\cdot)}(I)$ is considered as a set of equivalence classes of functions equal nearly everywhere.

The given space could be considered as a generalization of $L_p(I)$ space, having function $w(x) \equiv 1$, $L_p^{w(\cdot)}(I) = L_p(I)$.

Hereunder for space $L_p^{w(\cdot)}(I)$ we consider $1 \leqslant p < \infty$ and a standard metric ρ.

Consider an approximation of function $f \in L_p^{w(\cdot)}(I)$ with an attractor of IFS $(T^{\circ k})_{k=0}^{\infty}$ which is initiated by the operator T in the case where the attractor is a fixed point f_T^* of operator T. Such an approximation is usually called fractal, because f_T^* could be a function with the graph that has a fractional Hausdorff dimension [2].

Eventually contractive operator is an operator that holds Lipschitz condition and there is a power of its iteration that is a contractive operator. The minimum of such powers is eventually contractive operator's index.

Contractive index of an eventually contractive operator of index k is

$$L_T^{(k)} = \sup_{\substack{\{f,g\} \subset L_p^{w(\cdot)}(I), \\ f \neq g}} \frac{\rho(T^{\circ k} f,\, T^{\circ k} g)}{\rho(f,\, g)} < +\infty, \quad k \in \mathbb{N} \cup \{0\}.$$

It is followed from the definition that the eventually contractive operator is an operator that holds Lipschitz condition and all its iterations beginning with a particular power are contractive operators.

Indeed, since $L_T^{(m)} < 1$, then, staring from k,

$$L_T^{mk+p} \leqslant (L_T^m)^k (L_T^1)^p < 1, \quad 0 \leqslant p \leqslant m-1$$

Introduce a fractal operator. Set $n \in \mathbb{N}$, $n \geqslant 2$. Choose:

- a partition $\lambda = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of interval I; denote $I_i = [x_{i-1},\, x_i), i = 1, 2, \ldots, n-1$; $I_n = [x_{n-1},\, x_n]$; it is clear that $I_i \subset I, i = 1, 2, \ldots, n$; $\bigcup_{i=1}^{n} I_i = I$; $I_{i_1} \cap I_{i_2} = \varnothing, 1 \leqslant i_1 < i_2 \leqslant n$;

- sets of points $\{\alpha_i\}_{i=1}^n$, $\{\beta_i\}_{i=1}^n$ from I such that $a \leqslant \alpha_i < \beta_i \leqslant b$, $i = 1, 2, \ldots, n$;

- a set of diffeomorphisms $\{\phi_i\}_{i=1}^n$, $\phi_i \colon [\alpha_i,\, \beta_i) \longrightarrow I_i, i = 1, 2, \ldots, n$, i.e. differentiable bijections with differentiable inverses;

- a set $\{\psi_i\}_{i=1}^n$ of mappings $\psi_i \in C(I \times \mathbb{R})$, $i = 1, 2, \ldots, n$, which are uniformly continuous as functions of the first argument and hold Lipschitz condition: $|\psi_i(x,\, y_1) - \psi_i(x,\, y_2)| \leqslant d_i |y_1 - y_2|$ for any $x \in I$, $\{y_1, y_2\} \subset \mathbb{R}$ and some $d_i > 0$.

Fractal operator of function $f \in L_p^{w(\cdot)}(I)$ is the mapping

$$(Tf)(x) = \sum_{i=1}^{n} \psi_i\big(\phi_i^{-1}(x),\, f(\phi_i^{-1}(x))\big)\mathbb{1}_{I_i}(x) \tag{1}$$

for $x \in I$, $f \in L_p^{w(\cdot)}(I)$.

Here $\mathbb{1}_A(x)$ is the indicator function of set A. The product of indicator's zero value and undefined expression is zero.

Thus, an operator T (which is non-linear, in general case) is defined by setting a partition of interval I into subintervals I_i and defining mappings $\Phi_i(x,\, y) = \big(\phi_i(x),\, \psi_i(x,\, y)\big)$. T maps any function f into partially-defined function that is a deformed copy of restriction of f on each its domain interval.

3. Eventually contractive operator in $L_p^{w(\cdot)}(I)$. Let T be a fractal operator, $f \in L_p^{w(\cdot)}(I)$. To verify that Tf is a function in $L_p^{w(\cdot)}(I)$, let us estimate the norm $\|Tf\|^p_{L_p^{w(\cdot)}(I)}$.

If a function w is finite and summable then define the signed measure $\nu(E) = \int_E w\, d\mu$, $E \subset \mathbb{R}$. Radon—Nikodym theorem states that for a signed measure ν the function w is defined uniquely accurate within 0-measure sets. Then represent w as Radon—Nikodym derivative $w = \frac{d\nu}{d\mu}$ and in $\int_I w(x)|Tf(x)|^p\, d\mu$ change the measure, we obtain $\int_I |Tf(x)|^p d\nu = \|Tf\|^p_{L_p(I)} < \infty$. Now, let w be infinite and summable, than

$$\int_I w(x)|(Tf)(x)|^p\, dx = \int_I w(x)\Big|\sum_{i=1}^{n} \psi_i\big(\phi_i^{-1}(x),\, f(\phi_i^{-1}(x))\big)\mathbb{1}_{I_i}(x)\Big|^p\, dx =$$

$$= \sum_{i=1}^{n} \int_{I_i} w(x)\big|\psi_i\big(\phi_i^{-1}(x),\, f(\phi_i^{-1}(x))\big)\big|^p\, dx =$$

$$= \sum_{i=1}^{n} \int_{\phi_i^{-1}(I_i)} w\big(\phi_i(x)\big)\big|\psi_i\big(x,\, f(x)\big)\big|^p \phi_i'(x)\, dx =$$

$$= \sum_{i=1}^{n} \int_{\phi_i^{-1}(I_i)} w\big(\phi_i(x)\big)\big|\psi_i\big(x,\, f(x_0)\big) +$$

$$+ \psi_i\big(x,\, f(x)\big) - \psi_i\big(x,\, f(x_0)\big)\big|^p \phi_i'(x)\, dx.$$

After using the inequality $(a+b)^n \leqslant 2^{n-1}(a^n + b^n)$, $\{a, b\} \subset \mathbb{R}$, $n > 1$, we obtain

$$\sum_{i=1}^{n}\int_{\phi_i^{-1}(I_i)} w\big(\phi_i(x)\big)\big|\psi_i\big(x,\, f(x_0)\big)+\psi_i\big(x,\, f(x)\big)-\psi_i\big(x,\, f(x_0)\big)\big|^p\phi_i'(x)dx \leqslant$$

$$\leqslant 2^{p-1}\sum_{i=1}^{n}\int_{\phi_i^{-1}(I_i)} w\big(\phi_i(x)\big)(|\psi_i(x,\, f(x_0))|^p+$$

$$+|\psi_i(x,\, f(x))-\psi_i(x,\, f(x_0))|^p)\phi_i'(x)\, dx \leqslant$$

$$\leqslant 2^{p-1}\sum_{i=1}^{n}\int_{\phi_i^{-1}(I_i)} w\big(\phi_i(x)\big)(|\psi_i(x,\, f(x_0))|^p+$$

$$+d_i^p|f(x)-f(x_0)|^p)\phi_i'(x)\, dx < \infty.$$

Verify that T is a continuous operator. It is follows from:

$$\rho(Tf,\, Tg)^p = \int_I w(x)|(Tf)(x)-(Tg)(x)|^p\, dx =$$

$$= \sum_{i=1}^{n}\int_{I_i} w(x)|\psi_i(\phi_i^{-1}(x),\, f(\phi_i^{-1}(x)))-\psi_i(\phi_i^{-1}(x),\, g(\phi_i^{-1}(x)))|^p\, dx \leqslant$$

$$\leqslant \sum_{i=1}^{n} d_i^p\int_{I_i} w(x)|f(\phi_i^{-1})-g(\phi_i^{-1})|\, dx =$$

$$= \sum_{i=1}^{n} d_i^p\int_{\phi_i^{-1}(I_i)} w\big(\phi_i(x)\big)|f(x)-g(x)|^p\phi_i'(x)\, dx \leqslant$$

$$\leqslant (\sum_{i=1}^{n} d_i^p\max_I \phi_i'(x))\int_{\phi_i^{-1}(I_i)} w\big(\phi_i(x)\big)|f(x)-g(x)|^p\ dx.$$

Thus, it is proved the following.

Statement 3.1. *Consider the interval* $I=[a,\, b]$*, the partition* $\lambda = \{x_0, x_1, \ldots, x_n\}$*, the set of points* $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n\}$*, the sets of bijective functions* $\{\phi_i\}_{i=1}^n$ *and mappings* $\{\psi_i\}_{n=1}^n$ *defined in the section 2. Then the operator* T *defined in* (1) *maps continuously* $L_p^{w(\cdot)}(I)$ *into itself for a summable function* w.

Further, estimate distances between fractal operator iteration's images of two different functions. First, estimate $\rho(T^{\circ 2}f,\, T^{\circ 2}g)$.

$$\rho(T^{\circ 2}f,\, T^{\circ 2}g)^p = \int_I w(x)|(T(Tf))(x) - (T(Tg))(x)|^p dx =$$

$$= \sum_{i=1}^{n} \int_{I_i} w(x)|\psi_i(\phi_i^{-1}(x),\, Tf(\phi_i^{-1}(x))) - \psi_i(\phi_i^{-1}(x),\, Tg(\phi_i^{-1}(x)))|^p dx \leqslant$$

$$\leqslant \sum_{i=1}^{n} d_i^p \int_{I_i} w(x)|Tf(\phi_i^{-1}(x)) - Tg(\phi_i^{-1}(x))|^p dx =$$

$$= \sum_{i=1}^{n} d_i^p \int_{\phi_i^{-1}(I_i)} w(x)|Tf(x),\, Tg(x)|^p \phi_i'(x) dx =$$

$$= \sum_{i=1}^{n} d_i^p \int_{\phi_i^{-1}(I_i)} w(x)| \sum_{j=1}^{n} \psi_j(\phi_j^{-1}(x),\, f(\phi_j^{-1}(x)))\mathbb{1}_{I_j}(x) -$$

$$- \sum_{j=1}^{n} \psi_j(\phi_j^{-1}(x),\, g(\phi_j^{-1}(x)))\mathbb{1}_{I_j}(x)|^p dx =$$

$$= \sum_{i=1}^{n} d_i^p \sum_{j=1}^{n} \int_{I_j \cap \phi_i^{-1}(I_i)} w(x)|\psi_j(\phi_j^{-1}(x),\, f(\phi_j^{-1}(x))) -$$

$$-\psi_j(\phi_j^{-1}(x),\, g(\phi_j^{-1}(x)))|^p dx \leqslant$$

$$\leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} d_i^p d_j^p \int_{I_j \cap \phi_i^{-1}(I_i)} w(x)|f(\phi_j^{-1}) - g(\phi_j^{-1})|^p dx =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} d_i^p d_j^p \int_{\phi_j^{-1}(I_j \cap \phi_i^{-1}(I_i))} w(x)|f(x) - g(x)|^p \phi_i'(\phi_j(x))\phi_j'(x)\, dx =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (d_i d_j)^p \int_{\phi_j^{-1}(I_j \cap \phi_i^{-1}(I_i))} w(x)|f(x) - g(x)|^p (\phi_i \circ \phi_j)'(x)\, dx.$$

With the help of mathematical induction show that the following holds

$$\rho(T^k f,\, T^k g)^p \leqslant$$

$$\leqslant \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} (d_{i_1} \cdots d_{i_k})^p \int_{\phi_{i_k}(I_{i_k} \cap \cdots \cap \phi_{i_2}^{-1}(I_{i_2} \cap \phi_{i_1}^{-1}(I_{i_1}))\cdots)} |f(x) - g(x)|^p \times$$

$$\times (\phi_{i_1} \circ \cdots \circ \phi_{i_k})'(x)\, dx.$$

Having induction base, show induction step.

$$\rho(T^{\circ(k+1)}f,\, T^{\circ(k+1)}g)^p = \int_I w(x)|(T(T^{\circ k}f))(x) - (T(T^{\circ k}g))(x)|^p dx =$$

$$= \sum_{i=1}^{n} \int_{I_i} w(x)|\psi_i(\phi_i^{-1}(x),\, T^{\circ k}f(\phi_i^{-1}(x))) -$$

$$-\psi_i(\phi_i^{-1}(x),\, T^{\circ k}g(\phi_i^{-1}(x)))|^p dx \leqslant$$

$$\leqslant \sum_{i=1}^{n} d_i^p \int_{I_i} w(x)|T^{\circ k}f(\phi_i^{-1}(x)) - T^{\circ k}g(\phi_i^{-1}(x))|^p dx =$$

$$= \sum_{i=1}^{n} d_i^p \int_{\phi_i^{-1}(I_i)} w(x)|T^{\circ k}f(x),\, T^{\circ k}g(x)|^p \phi_i'(x) dx \leqslant$$

$$\leqslant \sum_{i_1=1}^{n} \cdots \sum_{i_{k+1}=1}^{n} (d_{i_1} \cdots d_{i_{k+1}})^p \int_{\phi_{i_{k+1}}(I_{i_{k+1}} \cap \cdots \cap \phi_{i_2}^{-1}(I_{i_2} \cap \phi_{i_1}^{-1}(I_{i_1}))\cdots)} w(x) \times$$

$$\times |f(x) - g(x)|^p (\phi_{i_1} \circ \cdots \circ \phi_{i_{k+1}})'(x)\, dx.$$

Further, denote the following values:

$$v_{i_1 \ldots i_k,\, p} = \sup_{\substack{f \neq g, \\ \{f,\, g\} \subset F}} \Big[\int_{\phi_{i_k}^{-1}(I_{i_k} \cap \cdots \cap \phi_{i_2}^{-1}(I_{i_2} \cap \phi_{i_1}^{-1}(I_{i_1}))\cdots)} |f - g|^p \times$$

$$\times |(\phi_{i_1} \circ \cdots \circ \phi_{i_k})'|\, dx \Big(\int_I |f - g|^p\, dx \Big)^{-1} \Big],$$

$$w_{k,p} = \sum_{i_1, \ldots, i_k = 1}^{n} (d_{i_1} \cdots d_{i_k})^p v_{i_1 \ldots i_k, p}\,.$$

Consider a set of functions F such that $F \subset T(F)$. Then

$$\sup_{\substack{f \neq g, \\ \{f,\, g\} \subset F}} \frac{\rho(T^{\circ k}f,\, T^{\circ k}g)^p}{\rho(f,\, g)^p} \leqslant w_{k,p}\,.$$

The existence of finite limit $w_p = \lim_{k \to \infty}(w_{k,p})^{1/k} < \infty$ follows from the fact that the given sequence is submultiplicative [4]. Thus, it is proved the following

Statement 3.2. *For the operator T defined in* (1) *and a set of functions F such that $F \subset T(F)$, there is a finite limit $w_p < \infty$.*

Further, having the sequence of inequalities

$$\sup_{\substack{f\neq g,\\ \{f,g\}\subset F}} \frac{\rho(Tf,\,Tg)^p}{\rho(f,\,g)^p} \leqslant \sum_{i=1}^{n} d_i^p v_{i,\,p},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sup_{\substack{f\neq g,\\ \{f,g\}\subset F}} \frac{\rho(Tf,\,Tg)^p}{\rho(f,\,g)^p} \leqslant \sum_{i_1,\ldots,i_k=1}^{n} d_{i_1}^p \ldots d_{i_k}^p v_{i_1\ldots i_k,\,p}\,,$$

one can be convinced that the operator T is eventually contractive if there is such a number k that $w_{k,p} < 1$. Thus,

Statement 3.3. *For the operator T defined in (1) and a set of functions F such that $F \subset T(F)$, if $w_p < 1$, then operator T is eventually contractive in F.*

Applying the well-known Banach fixed-point theorem generalization [5], it is easy to obtain the following

Statement 3.4. *For the operator T defined in (1) and a set of functions F such that $F \subset T(F)$, F is closed, if $w_p < 1$, than operator T has the unique fixed point f_T^* in $L_p^{w(\cdot)}(I)$, and for any $f \in F$ we have $T^{\circ k} f \to f_T^*$, $k \to \infty$.*

Thus, it is shown that a fractal operator could be considered on a weighted L_p space and could be used for some data compaction.

References

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