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A variational method for solutions to the Beltrami equation

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Dedicated to memory of Professor Promarz M. Tamrazov

We give a brief overview of our results in the theory of variations for classes of regular solutions to the degenerate Beltrami equation with constraints of the set-theoretic and integral types for the coefficient. The variational maximum principles and other necessary extremum conditions are formulated and applications to one of the main equation of the mathematical physics are obtained.

1. Introduction. Let D be a domain in the complex plane $\mathbb{C}$. It is well known that a K-quasiconformal mapping $f : D \to \mathbb{C}$, $K \geq 1$, is an orientation-preserving homeomorphic $W^{1,2}_{\mathrm{loc}}(D)$ solution to the Beltrami differential equation

$$f_{\overline{z}} = \mu(z) \cdot f_z \tag{1.1}$$

when the measurable coefficient μ satisfies the *uniform ellipticity condition* $|\mu(z)| \leq (K-1)/(K+1)$ almost everywhere in D. That is why the Beltrami equation turned out to be instrumental, in particular, in the study of Riemann surfaces, Teichmüller spaces, Kleinian groups, meromorphic functions, low dimensional topology, holomorphic motion, complex dynamics, Clifford analysis and control theory.

A special place in the theory of quasiconformal mappings assigned to the development and application of the variational method. The calculus of variations can be used to solve extremal problems for functionals over given family of mappings and to determine the extremal functions or the extremal value of the functional. Another main direction in the variational theory concerns the problem of minimization of the dilatation. It is intrinsically connected with the theory of Teichmüller spaces, measurable foliations on surfaces, holomorphic motions, etc., see, for example, [29] and the survey papers [27, 28]. The variational theory for quasiconformal maps and univalent functions with quasiconformal extensions was substantially developed by Belinskii, Krushkal, Kühnau, Lavrent'ev, Lehto, Renelt, Schiffer and Schober et al., see, e.g., [4, 5, 26, 30, 32, 37, 38, 46, 47] and the references therein. On the other hand, one can apply the variational theory to obtain existence and representation theorems for solutions of some partial differential equations by constructing appropriate functionals over a compact class of quasiconformal mappings. These equations arise from variational procedure as a necessary condition for the extremum, see, e.g., [45, 46] and [17, 18]. Other important approaches to the variational theory belong to Kühnau. He developed powerful methods for solving the variational problems in geometric function theory and its applications to electrostatics, fluid mechanics etc., see, e.g., [27, 28, 30].

In this paper we replace the condition of uniform ellipticity $|\mu(z)| \leq \leq (K-1)/(K+1)$ by a weaker condition $|\mu(z)| < 1$ almost everywhere in D and study more general classes of homeomorphic solutions to the Beltrami equation than those of quasiconformal. The degeneracy of the ellipticity for the Beltrami equations (1.1) will be controlled by the *dilatation coefficient*

$$K_\mu(z) := \frac{1+|\mu(z)|}{1-|\mu(z)|} \in L^1_{\mathrm{loc}}\,. \tag{1.2}$$

Recall that the problem on existence of homeomorphic solutions for the equation (1.1) was resolved for the uniformly elliptic case long ago, see e.g. [1, 33]. The existence problem for the degenerate Beltrami equations (1.1) when $K_\mu \notin L^\infty$ is currently an active area of research, see e.g. the monographs [3, 23, 36] and the surveys [22, 48], and further references therein. A series of criteria on the existence of regular solutions for the Beltrami equation (1.1) were given in the recent papers [6, 7]. There we called a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(D)$ by a *regular solution* of (1.1) if f satisfies (1.1) a.e. in D and the Jacobian $J_f(z) = |f_z|^2 - |f_{\bar z}|^2 \neq 0$ a.e. in D. We see that every quasiconformal mapping is a regular solution to the Beltrami equation. Throughout this paper we will deal with regular solutions only.

The main goal of this paper is to give a brief overview of our recent results, devoted to the theory of variations for compact classes of regular solutions to the degenerate Beltrami equation, see [14, 34, 35]. While constructing the variations, we follow the approach proposed in [10 — 13, 16, 19, 41]. The mentioned approach uses the convexity of the set of complex dilatations which turns out to be the common property of compact classes of regular solutions of the Beltrami equation. The compactness theory is also a significant part of the variational method. The basic theorems of existence and compactness for different classes of regular solutions to the Betrami equation can be found in [6 — 8, 19, 22, 34 — 36, 39 — 41, 43], see also the references therein.

This paper is organized as follows. After the introduction, we recall the definition of the class FMO of functions with finite mean oscillation that, in particular, contains the well-known class BMO of functions with bounded mean oscillation introduced by John and Nirenberg [25]. New criteria of compactness for some classes of regular solutions to the Beltrami equation will be given in Section 5.1 just in terms of FMO functions. In Section 3 we collect some facts from the theory of Sobolev spaces and the composition operators that will be utilized for the construction of admissible variations. The admissible variations for a class H of regular normalized solutions $f(z)$ to the Beltrami equation (1.1) are given in Section 4. The variational formula is applied to deduce the corresponding variational maximum principles for some compact classes of regular solutions to the Beltrami equation with constraints on μ of the set-theoretic type, see Section 5, as well as with constraints on μ of the integral type, see Section 6. Some geometric properties of regular solutions to the Beltrami equation are given in Section 7. We complete the paper with an application of the variational method to the study of the generalized degenerate Caushy–Riemann system.

2. Bounded and finite mean oscillation. Later on, $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$, $D(r) = D(0, r)$, $\mathbb{D} = D(0, 1)$, $\operatorname{dist}(E, F) = \sup_{x\in E,\, y\in F} |x - y|$ is the Euclidean distance between the sets E and F in $\mathbb{C}$, $\operatorname{mes} E$ is the Lebesgue measure of the set $E \subset \mathbb{C}$, $dm(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$, and $dS(z) = \left(1 + |z|^2\right)^{-2} dm(z)$ stands for the *element of a spherical area* in $\overline{\mathbb{C}}$.

Following [24], we say that a function $\varphi : D \to \mathbb{R}$ has *finite mean oscillation at a point* $z_0 \in D$ if

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{|D(z_0,\varepsilon)|} \int\limits_{D(z_0,\varepsilon)} |\varphi(z) - \widetilde{\varphi}_\varepsilon(z_0)|\ dm(z) < \infty\,, \tag{2.1}$$

where $|D(z_0,\varepsilon)|$ is the area of the disk $D(z_0,\varepsilon)$ and

$$\widetilde{\varphi}_\varepsilon(z_0) = \frac{1}{|D(z_0,\varepsilon)|} \int\limits_{D(z_0,\varepsilon)} \varphi(z)\ dm(z) \ < \ \infty$$

is the mean value of the function $\varphi(z)$ over the disk $D(z_0,\varepsilon)$. We say also that a function $\varphi : D \to \mathbb{R}$ is of *finite mean oscillation in the domain* D, abbreviated as $\varphi \in \mathrm{FMO}(D)$ or simply $\varphi \in \mathrm{FMO}$, if φ has a finite mean oscillation at every point $z_0 \in D$.

From the definition it follows that the well-known class BMO of functions with bounded mean oscillation, introduced by John and Nirenberg [25] is a subset of FMO. There exist examples showing that FMO is not $\mathrm{BMO}_{\mathrm{loc}}$, see e.g. [23]. Although $\mathrm{FMO} \subset L^1_{\mathrm{loc}}$ but FMO is not a subset of L^p_{loc} for any $p > 1$ in comparison with $\mathrm{BMO}_{\mathrm{loc}} \subset L^p_{\mathrm{loc}}$ for all $p \in [1,\infty)$.

The concept of finite mean oscillation can also be extended to infinity in the standard way. Namely, given a domain $D \subseteq \overline{\mathbb{C}}$, $\infty \in D$, and a function $\varphi : D \to \mathbb{R}$. We say that φ has *finite mean oscillation at* ∞ if the function $\varphi^*(z) = \varphi\,(1/\overline{z})$ has finite mean oscillation at 0. Clearly, by the inverse change of variables $z \to 1/\overline{z}$, the latter is equivalent to the condition

$$\int\limits_{|z|\geq R} |\varphi(z) - \widetilde{\varphi}_R| \frac{dm(z)}{|z|^4} = O\left(\frac{1}{R^2}\right) \quad \text{as } R \to \infty\,, \tag{2.2}$$

where

$$\widetilde{\varphi}_R = \frac{R^2}{\pi} \int\limits_{|z|\geq R} \varphi(z)\, \frac{dm(z)}{|z|^4}\,.$$

In terms of the spherical area, the condition (2.2) can be written in the form

$$\overline{\lim_{r\to 0}} \frac{1}{S(B_r)} \int\limits_{B_r} |\varphi(z) - \varphi^*_r|\ dS(z) \ < \ \infty\,,$$

where B_r is the circle with the center at ∞ and the radius r in the spherical metric, $S(B_r)$ is the spherical area of the circle B_r and

$$\varphi^*_r = \frac{1}{S(B_r)} \int\limits_{B_r} \varphi(z)\ dS(z) \ < \ \infty$$

is the average of the function φ over the circle B_r in the spherical area.

Recall also that a point $z_0 \in D \subseteq \mathbb{C}$ is called a *Lebesgue point* of a function $\varphi : D \to \mathbb{R}$ if

$$\lim_{\varepsilon \to 0} \frac{1}{|D(z_0,\varepsilon)|} \int\limits_{D(z_0,\varepsilon)} |\varphi(z) - \varphi(z_0)| \; dm(z) = 0 \,. \tag{2.3}$$

It is well-known that, for every locally integrable function $\varphi : D \to \mathbb{R}$, a.e. point in D is its Lebesgue point. Similarly, a point $z_0 = \infty \in D$ is called a Lebesgue point of a function $\varphi : D \to \mathbb{R}$, $D \subseteq \overline{\mathbb{C}}$, if

$$\overline{\lim_{R \to \infty}} \int\limits_{|z| \geq R} |\varphi(z) - \varphi(\infty)| \, \frac{dm(z)}{|z|^4} = O\left(\frac{1}{R^2}\right) \quad \text{as} \quad R \to \infty \,. \tag{2.4}$$

In other words, the condition (2.4) is equivalent to the convergence $\varphi^*_r \to \varphi(\infty)$ as $r \to 0$ of the averages in the spherical area.

3. Sobolev spaces and composition operators. Let D be a domain in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$. We recall that the Sobolev space $L^1_p(D)$, $p \geq 1$, consists of locally integrable functions $\varphi : D \to \mathbb{R}$ with generalized derivatives and the seminorm

$$\|\varphi\|_{L^1_p(D)} = \| \nabla \varphi\|_{L_p(D)} = \left(\int\limits_D | \nabla \varphi|^p \, dm \right)^{1/p} < \infty, \tag{3.1}$$

where m is the Lebesgue measure in $\mathbb{R}^n$, $\nabla\varphi$ is the *generalized gradient* of the function φ, $\nabla\varphi = \left(\frac{\partial\varphi}{\partial x_1}, \dots, \frac{\partial\varphi}{\partial x_n}\right)$, $x = (x_1, \dots, x_n)$, defined by the conditions

$$\int\limits_D \varphi \, \frac{\partial \eta}{\partial x_i} \, dm = -\int\limits_D \frac{\partial \varphi}{\partial x_i} \, \eta \, dm \quad \forall \, \eta \in C^\infty_0(D), \; i = 1, 2, \dots, n. \tag{3.2}$$

We denote by $C^\infty_0(D)$ the space of all infinitely smooth functions with a compact support in D. Similarly, a vector-function is said to belong to the Sobolev class $L^1_p(D)$ if its coordinate functions belong to $L^1_p(D)$. The classes $W^{1,p}(D) = L^1_p(D) \cap L_p(D)$ differ from the classes $L^1_p(D)$ only by the norm $\|\varphi\|_{W^{1,p}(D)} = \|\varphi\|_{L_p(D)} + \| \nabla \varphi\|_{L_p(D)}$. The following proposition holds (see [49, 50]).

Lemma 3.1. *Let $f : D \to D'$, $D' \subset \mathbb{R}^n$, be a homeomorphism. Then the following conditions are equivalent:*

1) *the composition $f^*\varphi = \varphi \circ f$ generates the bounded operator*

$$f^* : L^1_p(D') \to L^1_q(D), \quad 1 \le q \le p < \infty, \tag{3.3}$$

2) *the mapping f belongs to the class $W^{1,1}_{\rm loc}(D)$, and the function*

$$K_p(x,\, f) := \inf \left\{ k(x) : \|Df\|(x) \le k(x) |J_f(x)|^{\frac{1}{p}} \right\} \tag{3.4}$$

belongs to $L_r(D)$, where r is determined from the equality $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

There $\|Df\|(x)$ denotes the operator norm of the Jacobi matrix Df of the mapping f at the point x, $\|Df\|(x) := \sup\limits_{h\in\mathbb{R}^n,\, |h|=1} Df \cdot h$.

Whence, in particular at $n = 2$, $p = 2$, and $q = 1$, we have

Proposition 3.1. *Let f be an sense-preserving homeomorphism of the class $W^{1,1}_{\rm loc}$ with $K_{\mu_f} \in L^1_{\rm loc}$ between domains D and D' in $\mathbb{C}$. Then $g \circ f \in W^{1,1}_{\rm loc}$ for any mapping $g : D' \to \mathbb{C}$ of the class $W^{1,2}_{\rm loc}$.*

As is well known, any quasiconformal mapping g in $\mathbb{C}$ belongs to the class $W^{1,2}_{\rm loc}$ (see, e.g., Theorem IV.1.2 in [33]).

Corollary 3.1. *For any sense-preserving homeomorphism $f : D \to D'$ of the class $W^{1,1}_{\rm loc}$ with $K_{\mu_f} \in L^1_{\rm loc}$ and a quasiconformal mapping $g : \mathbb{C} \to \mathbb{C}$, the composition $g \circ f$ belongs to $W^{1,1}_{\rm loc}$.*

Quite similarly to Theorem 5.4.6 in [9], we obtain

Lemma 3.2. *Let f be a homeomorphism between domains D and D' in $\mathbb{R}^n$, let the composition operator $f^* : L^1_p(D') \to L^1_q(D)$, $1 \le q \le p < \infty$, be bounded, and let f has the N^{-1}-property. Then, for $g \in L^1_p(D')$ a.e.,*

$$\frac{\partial (g \circ f)}{\partial x_i}(x) = \sum_{k=1}^{n} \frac{\partial g}{\partial y_k}\,(f(x))\, \frac{\partial f_k}{\partial x_i}(x), \quad i = 1, \ldots, n. \tag{3.5}$$

Combining Lemmas 3.1 and 3.2 similarly to IC(1) in [1], we have

Proposition 3.2. *Let f be a regular homeomorphism between domains D and D' in $\mathbb{C}$ with $K_{\mu_f} \in L^1_{\rm loc}$. Then, for $g \in W^{1,2}_{\rm loc}$, a.e.*

$$(g\circ f)_z = (g_w \circ f) f_z + (g_{\overline{w}} \circ f)\overline{f_{\overline{z}}}, \quad (g\circ f)_{\overline{z}} = (g_w \circ f) f_{\overline{z}} + (g_{\overline{w}} \circ f)\overline{f_z}. \tag{3.6}$$

Corollary 3.2. *In particular, formulas* (3.6) *hold for quasiconformal mappings* g.

4. Construction of variations. For the illustration of general variational procedure, we consider first a class H of regular solutions $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ of the Beltrami equation (1.1), normaized by $f(0)=0,\ f(1)=1,\ f(\infty)=$ $=\infty$, when the Beltrami coefficients μ vary over a convex set $\mathfrak{M}$ of measurable functions with $K_\mu\in L^1_{\rm loc}$. If $f\in H$, then the *complex dilatation* $\mu_f(z)=f_{\bar z}(z)/f_z(z)$ for the mapping f exists a.e. in $\mathbb{C}$ and this dilatation coincides a.e. with the corresponding Beltrami coefficient $\mu(z)$.

Theorem 4.1. *Let* $f\in H$ *and have complex dilatation* $\mu\in\mathfrak{M}$. *Then for every* $\nu\in\mathfrak{M}$ *the mappings*

$$f_\varepsilon(\zeta)=f(\zeta)-\frac{\varepsilon}{\pi}\int\limits_{\mathbb{C}}(\nu(z)-\mu(z))\,\varphi(f(z),\,f(\zeta))\,\left(f_z/\overline{f_z}\right)\,J_f(z)\,dm(z)+$$

$$+o(\varepsilon,\,\zeta)\in H\ , \tag{4.1}$$

for each $\varepsilon\in[0,1/2)$. *Here*

$$\varphi(w,\,w')=\frac{1}{w-w'}\cdot\frac{w'}{w}\cdot\frac{w'-1}{w-1}\,, \tag{4.2}$$

and $o(\varepsilon,\zeta)/\varepsilon\to 0$ *locally uniformly with respect to* $\zeta\in\mathbb{C}$.

The proof of Theorem 4.1 is based on the theory of composition operators, given in Section 3, see in details [20] and [21], and the known theorem on the differentiability of families of quasiconformal mappings with respect to a parameter [1, Chapter 5].

Kernel (4.2) from the variational formula (4.1) is commonly called the *variational derivative* in the class of homeomorphisms $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ with the normalizations $f(0)=0,\ f(1)=1,\ f(\infty)=\infty$ (see [46]).

Remark 4.1. Variational derivatives for other normalizations can be found, for example, in [4] and [26]. In particular, the normalization $f(z)=$ $=z+o(1)$, where $o(1)\to 0$ as $z\to\infty$, implies variational derivative for mappings whose characteristic is equal to zero in a neighborhood of the infinity (see, e.g., [2]):

$$\varphi(w,\,w')=\frac{1}{(w-w')}\ . \tag{4.3}$$

5. On classes of mappings with constraints on μ of the set-theoretic type. Recall that the function $f : D \to \mathbb{C}$ is called *absolutely continuous on lines*, which is written as $f \in \mathrm{ACL}$, if, for any closed rectangle R in D whose sides are parallel to the coordinate axes, $f|_R$ is absolutely continuous on almost all linear segments in R, which are parallel to the sides of R (see, e.g., [1]). Let $Q : D \to [1, \infty]$ be a measurable function. A sense-preserving homeomorphism $f : D \to \mathbb{C}$ of the class ACL is called $Q(z)$-*quasiconformal* ($Q(z)$-q.c.) if $K_{\mu_f}(z) \le Q(z)$ a.e.

Andreian-Cazacu, Volkovyskii, Gutlyanskiĭ, Ioffe, Krushkal, Kühnau, Lehtinen, Renelt, Teichmüller, Schiffer, Schober, and others studied the classes of $Q(z)$-q.c. mappings for which $\mu(z) \in \Delta_{q(z)}$ a.e. where

$$\Delta_{q(z)} = \{\nu \in \mathbb{C} : |\nu| \le q(z)\} , \qquad q(z) = \frac{Q(z) - 1}{Q(z) + 1} ,$$

as well as the classes with additional constraints of the form $\mathcal{F}(\mu(z), z) \le 0$ a.e. where $\mathcal{F}(\mu, z) : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ satisfies the Caratheodory conditions, i.e. $\mathcal{F}(\mu, \cdot)$ is measurable on $\mathbb{C}$ for all $t \in [1,\, Q]$ and $\mathcal{F}(\cdot,\, z)$ is continuous on $[1,\, Q]$ for a.e. z. Finally, one of the Schiffer–Schober statements [46] led to the consideration of classes with constraints of the general set-theoretic form:

$$\mu(z) \in M(z) \subseteq \Delta_{q(z)} \quad \text{a.e.} \tag{5.1}$$

However, this development occurred for a long time in the frame of Q-q.c. mappings, since it was assumed that $Q \in L^\infty$.

Denote by $\mathfrak{M}_M$ the class of all measurable functions $\mu(z)$ satisfying (5.1) where we do not assume that $Q \in L^\infty$. We also denote by H^*_M the class of all regular solutions $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Beltrami equation with coefficients μ in $\mathfrak{M}_M$, normalized by $f(0) = 0,\ f(1) = 1,\ f(\infty) = \infty$.

A measurable function $Q : \mathbb{C} \to [1, \infty]$ is said to be *exponentially bounded in measure* if there exist constants $T \ge 1,\ \gamma > 0,\ c > 0$ such that for all $t \ge T$

$$\mathrm{mes}\{z \in \mathbb{C} : Q(z) > t\} \le ce^{-\gamma t} . \tag{5.2}$$

It is known that if Q is exponentially bounded in measure, then the corresponding class H^*_M is compact with respect to the topology of the locally uniform convergence, see e.g. [19], Theorem 12.2.

Recall that a family of compact sets $M(z) \subseteq \mathbb{D}$, $z \in \mathbb{C}$, is called *measurable in the parameter z* if the set $E_0 = \{z \in \mathbb{C} : M(z) \subseteq M_0\}$ is measurable by Lebesgue for any closed set $M_0 \subseteq \mathbb{C}$. In the sequel we will use the following notations

$$Q_M(z) := \frac{1+q_M(z)}{1-q_M(z)}\,, \quad q_M(z) := \max_{\nu\in M(z)} |\nu|\,. \tag{5.3}$$

5.1. Compactness criteria for the class H^*_M. Let $\mathcal{G}$ be the group of Möbius transformations of $\mathbb{D}$ onto itself. A set M in $\mathbb{D}$ is called *invariant–convex* if all sets $g(M)$, $g \in \mathcal{G}$, are convex. In particular, such sets are convex.

The following compactness theorems for the class H^*_M have been proven in [35].

Theorem 5.1. *Let $M(z)$, $z \in \mathbb{C}$, be a family of invariant–convex compact sets in $\mathbb{D}$ measurable in the parameter z. If $Q_M \in \mathrm{FMO}(\overline{\mathbb{C}})$, then the class H^*_M is compact.*

Corollary 5.1. *Let $M(z)$, $z \in \mathbb{C}$, be a family of invariant–convex compact sets in $\mathbb{D}$ measurable in z. If every point $z_0 \in \overline{\mathbb{C}}$ is a Lebesgue point of Q_M, then the class H^*_M is compact.*

Corollary 5.2. *Let $M(z)$, $z \in \mathbb{C}$, be a family of invariant–convex compact sets in $\mathbb{D}$ measurable in the parameter. If*

$$\overline{\lim_{\varepsilon\to 0}}\frac{1}{|D(z_0,\varepsilon)|}\int\limits_{D(z_0,\varepsilon)} Q_M(z)\; dm(z) \;<\; \infty \qquad \forall\, z_0\in\mathbb{C}$$

and

$$\overline{\lim_{r\to 0}}\,\frac{1}{S(B_r)}\int\limits_{B_r} Q_M(z)\,dS(z) < \infty \quad at\ \ z_0=\infty\,,$$

then the class H^*_M is compact.

Theorem 5.2. *Let $M(z)$, $z \in \mathbb{C}$, be a family of invariant-convex compact sets in $\mathbb{D}$ measurable in the parameter. If*

$$\int\limits_0^{\varepsilon}\frac{dr}{rq_{z_0}(r)} = \infty \qquad\qquad \forall\, z_0\in\mathbb{C}\,, \tag{5.4}$$

where $\varepsilon > 0$, $q_{z_0}(r)$ is the average of the function $Q_M(z)$ over the circle $|z-z_0| = r$, and

$$\int\limits_{\delta}^{\infty}\frac{dR}{Rq_{\infty}(R)} = \infty \quad at\ \ z_0=\infty\,, \tag{5.5}$$

where $\delta > 0$, $q_\infty(R)$ is the average of the function $Q_M(z)$ over the circle $|z| = R$. Then the class H^*_M is compact.

Corollary 5.3. *Let $M(z)$, $z \in \mathbb{C}$, be a family of invariant-convex compact sets in $\mathbb{D}$ measurable in the parameter. If*

$$q_{z_0}(r) = O\left(\log \frac{1}{r}\right) \quad as \ r \to 0 \tag{5.6}$$

at every point $z_0 \in \mathbb{C}$ and

$$q_{z_0}(R) = O\left(\log R\right) \quad as \ R \to \infty \tag{5.7}$$

at the point $z_0 = \infty$. Then the class H_M^* is compact.

For every nondecreasing function $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$, the *inverse function* $\Phi^{-1} : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ can be well defined by setting

$$\Phi^{-1}(\tau) = \inf_{\Phi(t) \geq \tau} t\,.$$

Here, inf is equal to ∞ if the set of $t \in [0, \infty]$ such that $\Phi(t) \geq \tau$ is empty.

Theorem 5.3. *Let $M(z)$, $z \in \mathbb{C}$, be a family of invariant-convex compact sets in $\mathbb{D}$ measurable in the parameter. If*

$$\int\limits_{\mathbb{C}} \Phi\left(Q_M(z)\right) dS(z) < \infty\,, \tag{5.8}$$

where $\Phi : [0, \infty] \to [0, \infty]$ is a nondecreasing convex function such that

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau\left[\Phi^{-1}(\tau)\right]} \;=\; \infty \tag{5.9}$$

for some $\delta > \Phi(0)$, then the class H_M^* is compact.

Note that the condition (5.9) is not only sufficient but also necessary for the compactness of the class H_M^* with restrictions of integral type (5.8), see Theorem 5.1 in [42].

5.2. A variational maximum principle. The functional $\Omega : H \to \mathbb{R}$ is called *Gâteaux differentiable* if

$$\Omega(f_\varepsilon) = \Omega(f) + \varepsilon \mathrm{Re} \int\limits_{\mathbb{C}} g\, d\varkappa + o(\varepsilon) \tag{5.10}$$

for every variation $f_\varepsilon = f + \varepsilon g + o(\varepsilon)$ in the class H where $\varkappa = \varkappa_f$ is a finite complex Borel measure (Radon measure) with a compact support and $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$ locally uniformly in $\mathbb{C}$ (see [47]). In other words, there exists the functional $L(g; f)$ that is continuous and linear in the first variable and such that

$$\Omega(f_\varepsilon) = \Omega(f) + \varepsilon \operatorname{Re} L(g; f) + o(\varepsilon)\,. \tag{5.11}$$

We say that Ω is *Gâteaux differentiable without degeneration* on the class H, if the kernel $\varphi(w, f(\zeta))$ is locally integrable for any $f \in H$ with respect to the product of measures $dm(w) \otimes d\varkappa(\zeta)$ and

$$\mathcal{A}(w) := \frac{1}{\pi} \int\limits_{\mathbb{C}} \varphi(w, f(\zeta))\, d\varkappa(\zeta) \neq 0 \quad \text{for a.e. } w \in \mathbb{C}. \tag{5.12}$$

In what follows we will assume that the functional Ω possesses the above property. We also say that the function $f \in H^*_M$ is *extremal* if

$$\max_{\varphi \in H^*_M} \Omega(\varphi) = \Omega(f).$$

Theorem 4.1 allows us to prove a maximum principle which states, that under the above conditions, the complex dilatation μ of an extreme mapping f belongs to the set of the extreme points of the convex set M. For the illustration of this principle we specify the set M and start with the case when M are subordinated to some set-theoretic constrains.

Theorem 5.4. *Let $M(z)$, $z \in \mathbb{C}$, be the family of compact convex sets in $\mathbb{D}$ measurable in the parameter z and such that $Q_M \in L^1_{\text{loc}}$. If $f \in H^*_M$ is extremal, then its complex characteristic satisfies the inclusion $\mu(z) \in \partial M(z)$ for almost all $z \in \mathbb{C}$.*

5.3. Other necessary conditions of the extremum. Given $\mu \in \mathfrak{M}_M$, we denote by $\omega_\mu(z)$ a *cone of admissible directions* for the set $M(z)$ at the point $\mu(z)$, i.e., the set of all $\omega \in \mathbb{C}$, $\omega \neq 0$, such that $\mu(z) + \varepsilon\omega \in M(z)$ for all $\varepsilon \in [0, \varepsilon_0]$ and some $\varepsilon_0 > 0$ (see, e.g., [31]).

Theorem 5.5. *Under the conditions of Theorem* 5.4, *the extremal function f satisfies the inequalities $\operatorname{Re} \omega\, \mathcal{B}(z) \geq 0$ for a.e. $z \in \mathbb{C}$ at all ω in the cone of admissible directions $\omega_\mu(z)$ where $\mathcal{B}(z) = \mathcal{A}(f(z))\, f_z^2$ and $\mathcal{A}(w)$ is defined by relation* (5.12).

Corollary 5.4. *If additionally there exists the tangent at every point of* $\partial M(z)$ *for a.e.* $z \in \mathbb{C}$*, then* $n(z)\,\mathcal{B}(z) \geq 0$ *a.e. Here* $n(z)$ *stands for the unit vector of the internal normal to* $\partial M(z)$ *at the point* $\mu(z)$*.*

Let us consider the important partial case, when

$$M(z) = \{\varkappa \in \mathbb{C} : |\varkappa - c(z)| \leq k(z)\} \subseteq \mathbb{D}$$

where the functions $c(z)$ and $k(z)$ are measurable. By the variational maximum principle, $n(z) = (c(z) - \mu(z))/k(z)$. Moreover, the relation from Corollary 5.4 is equivalent a.e. to the equality $(c(z) - \mu(z))/k(z) = \overline{\mathcal{B}(z)}/|\mathcal{B}(z)|$, i.e., $\mu(z) = c(z) - k(z)\,\overline{\mathcal{B}(z)}/|\mathcal{B}(z)|$. Thus, we arrive at the following statement.

Corollary 5.5. *Let* $M(z),\ z \in \mathbb{C}$*, be the family of disks given above. If*

$$Q(z) := \frac{1 + k(z) + |c(z)|}{1 - k(z) - |c(z)|} \in L^1_{\mathrm{loc}}\ ,$$

then the extremal function $f \in H^*_M$ *satisfies the equation*

$$f_{\overline{z}} = c(z) f_z - k(z)\,\frac{\overline{\mathcal{A}(f(z))}}{|\mathcal{A}(f(z))|}\,\overline{f_z}\,. \tag{5.13}$$

If $Q(z) \in L^\infty$, we recognize the well-known necessary conditions of the extremum for quasiconformal mappings.

6. On the classes of mappings with constraints of the integral type. Here we consider mappings of the Sobolev class $W^{1,1}_{\mathrm{loc}}$ with constraints of the integral type on the dilatation coefficient. Let us remark that similar classes of quasiconformal mappings in the mean were studied by Ahlfors, Biluta, Boyarskii, Gol'berg, Gutlyanskiĭ, Kruglikov, Krushkal, Kud'yavin, Kühnau, Perovich, Pesin, Ryazanov and others (see, e.g., references to [36, Chapt. 12] and [19, Chapt. 21]).

Let $\Phi : [0,\infty] \to [0,\infty]$ be a nondecreasing convex function. Then we denote by $\mathfrak{F}^\Phi$ the class of all regular solutions $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Beltrami equation normalized by $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$, with coefficients μ such that

$$\int\limits_{\mathbb{C}} \Phi\left(K_\mu(z)\right) dS(z) \leq 1\,. \tag{6.1}$$

We also denote by $\mathfrak{M}^\Phi$ the corresponding class of complex dilatations.

Note that, under the given conditions on Φ, the function

$$\mathcal{D}(\tau) := \Phi\left(\frac{1+\tau}{1-\tau}\right), \quad \tau \in [0,\, 1], \tag{6.2}$$

is convex. Therefore, the class of complex dilatations $\mathfrak{M}^\Phi$ is also convex.

Finally, recall that the function $\Phi : [0, \infty] \to [0, \infty]$ is called *strictly convex*, if it is convex, nondecreasing, and $\lim\limits_{t\to\infty} \Phi(t)/t = \infty$.

6.1. The compactness theorem for classes $\mathfrak{F}^\Phi$. The following compactness theorem for classes $\mathfrak{F}^\Phi$ was obtained in [34].

Theorem 6.1. *Let $\Phi : [0, \infty] \to [0, \infty]$ be a continuous strictly convex function such that*

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{6.3}$$

for some $\delta > \Phi(0)$. Then the class $\mathfrak{F}^\Phi$ is compact.

Note that the condition (6.3) is not only sufficient but also necessary for the compactness of class $\mathfrak{F}^\Phi$ with restrictions of integral type (6.1), see Theorem 5.1 in [42].

6.2. A variational maximum principle.

Theorem 6.2. *Let $\Phi : [0, \infty] \to [0, \infty]$ be a nondecreasing convex function with $\Phi(Q) \neq 0$ where $Q = \sup_{\Phi(t)<\infty} t$ and let the functional $\Omega : \mathfrak{F}^\Phi \to \mathbb{R}$ be Gâteaux differentiable without degeneration. Suppose that $\max \Omega$ over the class $\mathfrak{F}^\Phi$ is attained for a mapping $f \in \mathfrak{F}^\Phi$. Then the dilatation coefficient K_μ, $\mu = \mu_f$, satisfies the equality*

$$\int\limits_{\mathbb{C}} \Phi(K_\mu(z))\, dS(z) = 1. \tag{6.4}$$

6.3. Other necessary conditions of the extremum.

Theorem 6.3. *Under conditions of Theorem* 6.2,

$$f_{\overline{z}} = -k(z)\, \frac{\overline{\mathcal{A}(f(z))}}{|\mathcal{A}(f(z))|}\, \overline{f_z} \tag{6.5}$$

where $k(z) = (K_\mu(z) - 1)/(K_\mu(z) + 1)$ and

$$\int\limits_{\mathbb{C}} \mathcal{D}(k(z))\, dS(z) = 1\,. \tag{6.6}$$

Here $\mathcal{A}$ and $\mathcal{D}$ are given above by relations (5.12) *and* (6.2), *respectively.*

7. On Belinskii's conformality. Recall that a mapping f is called conformal at a point z_0 if f is differentiable at the point z_0 by Darboux–Stolz, i.e.,

$$f(z) - f(z_0) = f_z(z_0)(z - z_0) + f_{\overline{z}}(z_0)\overline{(z - z_0)} + o(|z - z_0|) \ , \qquad (7.1)$$

and if $f_{\overline{z}}(z_0) = 0$, whereas $f_z(z_0) \neq 0$, where $o(|z - z_0|)/|z - z_0| \to 0$ as $z \to z_0$.

The example $w = z(1 - \ln|z|)$ by Shabat (see [4]) shows that, for a continuous complex dilatation $\mu(z)$, the mapping $w = f(z)$ can be nondifferentiable in the sense of Darboux–Stolz.

If the complex dilatation $\mu(z)$ is continuous at a point z_0, then, as was first established by Belinskii (see [4]), the mapping $w = f(z)$ is differentiable at z_0 in the following meaning:

$$\Delta w = A(\rho) \left[\Delta z + \mu_0 \overline{\Delta z} + o(\rho)\right] , \qquad (7.2)$$

where $\mu_0 = \mu(z_0)$, $\rho = |\Delta z + \mu_0 \overline{\Delta z}|$, $A(\rho)$ depends only on ρ and $o(\rho)/\rho \to 0$ as $\rho \to 0$. Here $A(\rho)$ can have no limit as $\rho \to 0$. However,

$$\lim_{\rho \to 0} \frac{A(t\rho)}{A(\rho)} = 1 \quad \forall\, t > 0. \qquad (7.3)$$

A mapping f is said to be *differentiable by Belinskii* at a point z_0 if conditions (7.2)–(7.3) are satisfied with some $\mu_0 \in \mathbb{D}$. In this definition, for a discontinuous $\mu(z)$, the equality $\mu_0 = \mu(z_0)$ in relation (7.2) does not obligatorily hold. If $\mu_0 = 0$, one says that f is *conformal by Belinskii* at the point z_0 (see [15]).

The function $\mu(z)$ is called *approximately continuous at a point* $z_0 \in \mathbb{C}$, if there exists a measurable set E on which $\mu(z) \to \mu(z_0)$ as $z \to z_0$, and z_0 is a density point of E, i.e.,

$$\lim_{\varepsilon \to 0} \frac{\mathrm{mes}\{E \cap D(z_0, \varepsilon)\}}{\mathrm{mes}\{D(z_0, \varepsilon)\}} = 1.$$

It will be shown below that the approximate continuity of μ remains the sufficient condition for the differentiability of f by Belinskii with $\mu_0 = \mu(z_0)$.

An analog of the following theorem for quasiconformal mappings can be found in [15].

Theorem 7.1. *Let D be a domain in $\mathbb{C}$, $0 \in D$, and let $f : D \to \mathbb{C}$ be a regular solution of the Beltrami equation* (1.1), $f(0) = 0$, *and let*

$$\limsup_{r\to 0} \frac{1}{|D(r)|} \int\limits_{D(r)} \Phi\left(K_\mu(z)\right)\, dm(z) < \infty \tag{7.4}$$

for a nondecreasing convex function $\Phi : [0, \infty] \to [0, \infty]$ such that

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{7.5}$$

for some $\delta > \Phi(0)$. Then the following assertions are equivalent:

1) *f is conformal by Belinskii at zero;*

2) *for any $\zeta \in \mathbb{D}$*

$$\lim_{\substack{\tau\to 0,\\ \tau>0}} \frac{f(\tau\zeta)}{f(\tau)} = \zeta; \tag{7.6}$$

3) *for any $\delta \in (0,\, 1)$ at $|z'| < \delta|z|$ and $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$,*

$$\lim_{z\to 0} \left\{ \frac{f(z')}{f(z)} - \frac{z'}{z} \right\} = 0; \tag{7.7}$$

4) *for any $\zeta \in \mathbb{D}$*

$$\lim_{\substack{z\to 0,\\ z\in\mathbb{C}^*}} \frac{f(z\zeta)}{f(z)} = \zeta. \tag{7.8}$$

In this case, the limit in (7.8) *is locally uniform in ζ in $\mathbb{D}$.*

Corollary 7.1. *In particular, the conclusions of the lemma are proper, if*

$$\int\limits_{D} \Phi\left(K_{\mu_n}(z)\right) dS(z) \le M < \infty \tag{7.9}$$

for some strictly convex function $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ with condition (7.5).

We now give the most interesting consequence of Theorem 7.1.

Theorem 7.2. *Let D be a domain in $\mathbb{C}$, $z_0 \in D$, $f : D \to \mathbb{C}$ be a regular solution of the Beltrami equation* (1.1) *and*

$$\limsup_{r\to 0} \frac{1}{|D(z_0,\, r)|} \int\limits_{D(z_0,\, r)} \Phi\left(K_\mu(z)\right)\, dm(z) < \infty \tag{7.10}$$

for a strictly convex function $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ with condition (7.5). *If $\mu(z)$ is approximately continuous at the point z_0, then the mapping f is differentiable by Belinskii at this point with $\mu_0 = \mu(z_0)$.*

Equality (7.7) and the triangle inequality yield

Corollary 7.2. *Under the hypothesis and one of conditions* 1)–4) *of Theorem* 7.1 *at $|z'| \le \delta|z|$ for any $\delta > 0$, there exists*

$$\lim_{\substack{z\to 0\\ z\,\in\,\mathbb{C}^*}} \left\{ \frac{|f(z')|}{|f(z)|} - \frac{|z'|}{|z|} \right\} = 0. \tag{7.11}$$

Using this result and the Stolz theorem, we have proved the following

Theorem 7.3. *Under the hypothesis and condition* 1) *of Theorem* 7.1,

$$\lim_{z\to 0} \frac{\ln|f(z)|}{\ln|z|} = 1. \tag{7.12}$$

Theorems 7.2 and 7.3 yield immediately

Corollary 7.3. *Under hypothesis of Theorem* 7.2 *with $\mu(z_0) = 0$,*

$$\lim_{z\to z_0} \frac{\ln|f(z) - f(z_0)|}{\ln|z - z_0|} = 1. \tag{7.13}$$

8. On some applications. The theory of the Belinskii conformality and differentiability developed in the last section makes possible to apply our theory of variational method to one of the main equations of mathematical physics, see [14].

Namely, let us consider in a domain $D \subset \overline{\mathbb{C}}$ the equation

$$\operatorname{div}(K \operatorname{grad} u) = 0 \tag{8.1}$$

which is the basic equation in the theory of stationary flows, hydrodynamics, and magneto- and electrostatics of inhomogeneous media. It is convenient to interpret the coefficient K as a function of the complex variable $z = x + iy$. In this case, we assume that the coefficient K is positive and is uniformly separated from 0 that is natural from the physical viewpoint. In addition, we can always attain that $\operatorname{ess\,inf} K(z) \geq 1$ by an additional normalization.

Here, we do not suppose that the coefficient K is differentiable or at least continuous or bounded. In this case, as a (weak) solution of Eq. (8.1), we understand a function U that possesses a locally conjugate function V such that the couple (U, V) has the first generalized derivatives, except isolated singularities, and satisfies a.e. the generalized Cauchy–Riemann system

$$V_x = -KU_y, \quad V_y = KU_x \tag{8.2}$$

in a relevant neighborhood of every point of the domain D.

It is easy to see that the system of equations (8.2) is equivalent to a single complex Beltrami equation of the second kind

$$F_{\overline{z}} = -k(z)\overline{F_z}\,, \tag{8.3}$$

where $F = U + iV,\ z = x + iy$ and

$$k(z) = \frac{K(z) - 1}{K(z) + 1}\,. \tag{8.4}$$

The analog of the below presented theorem was first announced for $K \in L^\infty$ under the Belinskii–Wittich–Teichmüller conditions in [16], and then it was proved under the weaker conditions of approximate continuity in the dissertation [41], see also the monograph [19].

Theorem 8.1. *Let D be a domain in $\overline{\mathbb{C}}$, and let $K : D \to [1, \infty]$ be a measurable function such that*

$$\int\limits_D \Phi\left(K(z)\right)\, dS(z) < \infty \tag{8.5}$$

for the strictly convex function $\Phi : [1, \infty] \to [0, \infty]$ with condition (7.5). *Then, for any $z_0 \in D\backslash\{\infty\}$, there exists a solution of Eq.* (8.1) *that is representable in the form*

$$U(z, z_0) = \ln|f(z) - f(z_0)|^{-1}\,, \tag{8.6}$$

where f is a regular solution of the equation

$$f_{\overline{z}} = -k(z)\,\frac{f(z)-f(z_0)}{\overline{f(z)-f(z_0)}}\;\overline{f_z} \tag{8.7}$$

in $\overline{\mathbb{C}}$ *with the normalizations* $f(0)=0$, $f(1)=1$, $f(\infty)=\infty$. *Here* $k(z)$ *is given by* (8.4) *in* D, $k(z)\equiv 0$ *outside of* D. *In this case, the solution* $U(z,\,z_0)$ *itself is extended by continuity in* $\overline{\mathbb{C}}$ *and is a harmonic function in the additional domain* $\mathbb{C}\setminus\overline{D}$ *where we set* $K(z)\equiv 1$.

In addition, if the function $K(z)$ *is approximately continuous at the point* z_0 *and satisfies the condition*

$$\limsup_{r\to 0}\frac{1}{|D(z_0,\,r)|}\int\limits_{D(z_0,\,r)}\Phi\left(K(z)\right)\,dm(z)<\infty\,, \tag{8.8}$$

then

$$\lim_{z\to z_0}\frac{U(z,\,z_0)}{\ln|z-z_0|^{-1}}=\frac{1}{K(z_0)}\,. \tag{8.9}$$

Remark 8.1. Analogously, if $K(1/z)$ is approximately continuous at 0 and satisfies (8.8) there, then

$$\lim_{z\to\infty}\frac{U(z,\,\infty)}{\ln|z|}=\frac{1}{K(\infty)}\,. \tag{8.10}$$

Remark 8.2. We note that, by the Lebesgue theorem, condition (8.5) yields condition (8.8) for almost all $z_0\in D$ (see, e.g., Theorem IV(5.4) in [44]). Moreover, by the Denjoy theorem, any almost everywhere finite measurable function is almost everywhere approximately continuous (see, e.g., Theorem IV(10.6) in [44]).

Corollary 8.1. *Under condition* (8.5)*, the solution of* (8.1) *with the property* (8.9) *exists for almost all* $z_0\in D$.

On the history of the problem for quasiconformal mappings with restrictions of the integral type on the Lavrent'ev characteristic, see the papers [20, 21].

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