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Boundary value problems with finite groups of Lipschitz shifts

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Dedicated to the memory of Professor Promarz M. Tamrazov

Applying the theory of Mellin pseudodifferential operators with non-regular symbols we establish Fredholm criteria and index formulas for singular integral operators with piecewise slowly oscillating coefficients and finite non-cyclic groups of Lipschitz shifts whose derivatives admit slowly oscillating discontinuities. Such operators studied on the Lebesgue spaces are related to boundary value problems with finite groups of shifts.

1. Introduction. Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting a Banach space X, and let $\mathcal{K}(X)$ be the closed two-sided ideal of all compact operators in $\mathcal{B}(X)$. An operator $A \in \mathcal{B}(X)$ is said to be *Fredholm*, if its image is closed and the spaces $\ker A$ and $\ker A^*$ are finite-dimensional. In that case the number $\operatorname{Ind} A = \dim\ker A - \dim\ker A^*$ is referred to as the *index* of A (see, e.g., [1, p. 9]).

Let $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ be the unit circle with counter-clockwise orientation and let G be a finite group of Lipschitz homeomorphisms of $\mathbf{T}$ onto itself that have slowly oscillating (see Section 2) derivatives. By [2], G has one of the two following forms:

$$G = \{e, \alpha, \dots, \alpha^{n-1}\}, \quad G = \{e, \alpha, \dots, \alpha^{n-1}, \beta, \alpha\beta, \dots, \alpha^{n-1}\beta\}, \qquad (1)$$

where $n \in \mathbf{N}$, e is the unit of G, the shift α preserves the orientation on $\mathbf{T}$, $\alpha^n = e$ and $\alpha^k \neq e$ if $k = 1, 2, \dots, n-1$, the shift β reverses the orientation on $\mathbf{T}$ and $\beta^2 = e$,

$$\alpha^k\beta = \beta\alpha^{n-k} \quad \text{for all} \quad k = 1, 2, \dots, n, \qquad (2)$$

and $(g_1 g_2)(t) = g_2[g_1(t)]$ for all $t \in \mathbf{T}$ and all $g_1, g_2 \in G$.

Let $1 < p < \infty$. Then the Cauchy singular integral operator $S_{\mathbf{T}}$ given for $f \in L^1(\mathbf{T})$ and almost all $t \in \Gamma$ by

$$(S_{\mathbf{T}} f)(t) := \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{\mathbf{T} \setminus \mathbf{T}(t,\varepsilon)} \frac{f(\tau)}{\tau - t}\, d\tau, \tag{3}$$

where $\mathbf{T}(t,\varepsilon) := \big\{\tau \in \mathbf{T} : |\tau - t| < \varepsilon\big\}$, is bounded on the Lebesgue space $L^p(\mathbf{T})$, where $\|f\|_{L^p(\mathbf{T})} := \Big(\int_{\mathbf{T}} |f(\tau)|^p |d\tau|\Big)^{1/p}$ (see, e.g., [1, Section 1.42]).

Our goal is the Fredholm study of the next boundary value problem: Find a function Φ analytic in $\mathbf{C} \setminus \mathbf{T}$, represented by the Cauchy type integral over $\mathbf{T}$ with a density $\varphi \in L^p(\mathbf{T})$ and satisfying the boundary condition

$$\sum_{g \in G} a_g^+(t) \Phi^+[g(t)] = \sum_{g \in G} a_g^-(t) \Phi^-[g(t)] + f(t) \quad \text{for} \quad t \in \mathbf{T}, \tag{4}$$

where $\Phi^{\pm}(t)$ are angular boundary values of Φ on $\mathbf{T}$, $a_g^{\pm}$ are piecewise slowly oscillating (see Section 2) functions in $L^\infty(\mathbf{T})$, and $f \in L^p(\mathbf{T})$. By the Sokhotski–Plemelj formulas $\Phi^{\pm} = \pm P_{\mathbf{T}}^{\pm} \varphi$, with boundary value problem (4) we can associate the equivalent singular integral operator with shifts

$$B = \sum_{g \in G} (a_g^+ V_g P_{\mathbf{T}}^+ + a_g^- V_g P_{\mathbf{T}}^-) \in \mathcal{B}(L^p(\mathbf{T})), \tag{5}$$

where V_g are the shift operators given by $V_g f = f \circ g$, $P_{\mathbf{T}}^{\pm} = 2^{-1}(I \pm S_{\mathbf{T}})$, I is the identity operator and $S_{\mathbf{T}}$ is the Cauchy singular integral operator given by (3).

The Fredholm theory for the operator (5) with continuous coefficients and cyclic groups of shifts was constructed by G. S. Litvinchuk (see [3]). The case of piecewise continuous coefficients and cyclic groups of shifts preserving or changing orientation was studied by I. Gohberg and N. Ya. Krupnik, N. K. Karapetiants and S. G. Samko, N. L. Vasilevski and M. V. Shapiro (see [3, 4] and the references therein).

In the case of continuous coefficients $a_g^{\pm}$ and finite non-cyclic groups (1) of shifts, a Fredholm criterion and an index formula for the operator (5) on the spaces $L^p(\Gamma)$ with $p \in (1, \infty)$ and a closed Liapunov curve Γ were obtained in [2]. The Fredholm theory for the Banach algebra generated by operators (5) with piecewise continuous coefficients on the spaces

$L^p(\Gamma,\rho)$ with power weights ρ was constructed in [5] (see also papers by G. Yu. Vinogradova for the case $\Gamma=\mathbf{R}$ in [4]).

In the present paper we establish a Fredholm criterion and an index formula for the operator (5) under the following conditions: the finite group G in (1) is non-cyclic, the coefficients $a_g^{\pm}$ are piecewise slowly oscillating and admit only finite sets of discontinuities on $\mathbf{T}$, the derivatives α' and β' are slowly oscillating on $\mathbf{T}$, and the fixed points of the shifts α and $\delta:=\alpha\circ\beta$ are isolated discontinuity points for the derivatives α' and δ', respectively. The study is based on the Fredholm theory for Mellin pseudodifferential operators with non-regular symbols, which was constructed in [6 — 8].

The paper is organized as follows. Section 2 contains preliminaries on slowly oscillating and piecewise slowly oscillating functions. Section 3 contains necessary results on Mellin pseudodifferential operators. In Section 4 the operator B with shifts given by (5) is reduced to an equivalent matrix operator B_Γ without shifts. In Section 5 the operator B_Γ is reduced to a finite family of Mellin pseudodifferential operators, which together describe the Fredholmness of B_Γ. Finally, applying the results of Section 3, we obtain a Fredholm criterion and an index formula for the operator B in Sections 6 and 7, respectively.

2. The C^*-algebra of $PSO(\mathbf{T})$ functions. Let $C(\mathbf{T})$, $PC(\mathbf{T})$ and $SO(\mathbf{T})$ denote the C^*-subalgebras of $L^\infty(\mathbf{T})$ consisting, respectively, of all continuous functions on $\mathbf{T}$, all piecewise continuous functions on $\mathbf{T}$, that is, the functions having one-sided limits at each point $t\in\mathbf{T}$, and all slowly oscillating functions on $\mathbf{T}$, that is, the functions f that are slowly oscillating at each point $\lambda\in\mathbf{T}$:

$$\lim_{\varepsilon\to 0}\operatorname{ess\,sup}\big\{|f(z_1)-f(z_2)|:\, z_1,z_2\in\mathbf{T}_\varepsilon(\lambda)\big\}=0,$$

where $\mathbf{T}_\varepsilon(\lambda):=\{z\in\mathbf{T}:\,\varepsilon/2\le|z-\lambda|\le\varepsilon\}$. Denoting by $SO_\lambda(\mathbf{T})$ the C^*-subalgebra of $L^\infty(\mathbf{T})$ consisting of the continuous functions on $\mathbf{T}\setminus\{\lambda\}$ that are slowly oscillating at $\lambda\in\mathbf{T}$, we deduce that $SO(\mathbf{T})$ is the smallest C^*-subalgebra of $L^\infty(\mathbf{T})$ containing all C^*-algebras $SO_\lambda(\mathbf{T})$ for $\lambda\in\mathbf{T}$.

Let $PSO(\mathbf{T}):=\operatorname{alg}(SO(\mathbf{T}),PC(\mathbf{T}))$ be the C^*-subalgebra of $L^\infty(\mathbf{T})$ generated by the C^*-algebras $SO(\mathbf{T})$ and $PC(\mathbf{T})$.

Since $C(\mathbf{T})\subset SO(\mathbf{T})\subset PSO(\mathbf{T})$, it follows from [9] that

$$M(SO(\mathbf{T}))=\bigcup_{t\in\mathbf{T}}M_t(SO(\mathbf{T})),\quad M(PSO(\mathbf{T}))=M(SO(\mathbf{T}))\times\{0,1\},$$

where the fibers of the maximal ideal space $M(SO(\mathbf{T}))$ are given for $t \in \mathbf{T}$ by

$$M_t(SO(\mathbf{T})) = \{\xi \in M(SO(\mathbf{T})) : \xi|_{C(\mathbf{T})} = t\}, \tag{6}$$

and $t(f) = f(t)$ for $f \in C(\mathbf{T})$.

Theorem 1 [9, Theorem 4.6]. *If $\xi \in M_t(SO(\mathbf{T}))$ with $t \in \mathbf{T}$ and $\mu \in \{0, 1\}$, then the characters $(\xi, \mu) \in M(PSO(\mathbf{T}))$ possess the following properties: $(\xi, \mu)|_{SO(\mathbf{T})} = \xi$, $(\xi, \mu)|_{C(\mathbf{T})} = t$, $(\xi, \mu)|_{PC(\mathbf{T})} = (t, \mu)$; and $(t, 0)a = a(t - 0)$ and $(t, 1)a = a(t + 0)$ are the left and right one-sided limits of a function $a \in PC(\mathbf{T})$ at the point $t \in \mathbf{T}$.*

Given $a \in PSO(\mathbf{T})$, we put $a(\xi^+) := a(\xi, 1)$ and $a(\xi^-) := a(\xi, 0)$ for all $\xi \in M(SO(\mathbf{T}))$.

By the proof of [9, Theorem 6.2], the C^*-algebra $SO(\mathbf{T})$ is contained in the C^*-algebra $QC(\mathbf{T})$ of quasicontinuous functions on $\mathbf{T}$, and by [10],

$$QC(\mathbf{T}) := (C(\mathbf{T}) + H^\infty) \cap (C(\mathbf{T}) + \overline{H^\infty}) = L^\infty(\mathbf{T}) \cap VMO(\mathbf{T}),$$

where the C^*-algebra H^∞ consists of all functions being non-tangential limits on $\mathbf{T}$ of the functions bounded and analytic on the open unit disc, and $VMO(\mathbf{T})$ is the Banach space of functions of vanishing mean oscillation. Hence, the compactness criteria in [11], the fact $SO(\mathbf{T}) \subset L^\infty(\mathbf{T}) \cap VMO(\mathbf{T})$, and [12, Theorem 4.1 and Proposition 4.5] together imply the following.

Theorem 2. *Let $1 < p < \infty$.*

(a) *If $a \in SO(\mathbf{T})$, then $aS_{\mathbf{T}} - S_{\mathbf{T}}aI \in \mathcal{K}(L^p(\mathbf{T}))$.*

(b) *If α is an orientation-preserving Lipschitz homeomorphism of $\mathbf{T}$ onto itself and $\alpha' \in SO(\mathbf{T})$, then $V_\alpha S_{\mathbf{T}} - S_{\mathbf{T}} V_\alpha \in \mathcal{K}(L^p(\mathbf{T}))$.*

(c) *If β is an orientation-reversing Lipschitz homeomorphism of $\mathbf{T}$ onto itself and $\beta' \in SO(\mathbf{T})$, then $V_\beta S_{\mathbf{T}} + S_{\mathbf{T}} V_\beta \in \mathcal{K}(L^p(\mathbf{T}))$.*

3. Mellin pseudodifferential operators. Let a be an absolutely continuous function of finite total variation $V(a) = \int_{\mathbf{R}} |a'(x)|dx$ on $\mathbf{R}$. The set $V(\mathbf{R})$ of all absolutely continuous functions of finite total variation on $\mathbf{R}$ becomes a Banach algebra with the norm $\|a\|_V := \|a\|_{L^\infty(\mathbf{R})} + V(a)$.

Following [6, 7], let $C_b(\mathbf{R}_+, V(\mathbf{R}))$ denote the Banach algebra of all bounded continuous $V(\mathbf{R})$-valued functions on $\mathbf{R}_+ = (0, \infty)$ with the norm

$$\|\mathfrak{a}(\cdot, \cdot)\|_{C_b(\mathbf{R}_+, V(\mathbf{R}))} = \sup_{t \in \mathbf{R}_+} \|\mathfrak{a}(t, \cdot)\|_V.$$

As usual, let $C_0^\infty(\mathbf{R}_+)$ be the set of all infinitely differentiable functions of compact support on $\mathbf{R}_+$.

Take the Lebesgue space $L^p(\mathbf{R}_+, d\mu)$ with invariant measure $d\mu(t) = dt/t$ on $\mathbf{R}_+$. The following boundedness result for Mellin pseudodifferential operators follows from [7, Theorem 6.1] (see also [6, Theorem 3.1]).

Theorem 3. *If* $\mathfrak{a} \in C_b(\mathbf{R}_+, V(\mathbf{R}))$*, then the Mellin pseudodifferential operator* $\mathrm{Op}(\mathfrak{a})$*, defined for functions* $f \in C_0^\infty(\mathbf{R}_+)$ *by the iterated integral*

$$[\mathrm{Op}(\mathfrak{a})f](t) = \frac{1}{2\pi}\int\limits_{\mathbf{R}} dx \int\limits_{\mathbf{R}_+} \mathfrak{a}(t,x)\left(\frac{t}{\tau}\right)^{ix} f(\tau)\frac{d\tau}{\tau} \quad \textit{for} \quad t \in \mathbf{R}_+,$$

extends to a bounded linear operator on the space $L^p(\mathbf{R}_+, d\mu)$ *and there is a number* $C_p \in (0,\infty)$ *depending only on* p *such that*

$$\|\,\mathrm{Op}(\mathfrak{a})\|_{\mathcal{B}(L^p(\mathbf{R}_+,d\mu))} \le C_p\|\mathfrak{a}\|_{C_b(\mathbf{R}_+,V(\mathbf{R}))}.$$

Let $SO(\mathbf{R}_+, V(\mathbf{R}))$ denote the Banach subalgebra of $C_b(\mathbf{R}_+, V(\mathbf{R}))$ consisting of all $V(\mathbf{R})$-valued functions $\mathfrak{a}$ on $\mathbf{R}_+$ that slowly oscillate at 0 and ∞, that is,

$$\lim_{r\to 0} \mathrm{cm}_r^C(\mathfrak{a}) = \lim_{r\to\infty} \mathrm{cm}_r^C(\mathfrak{a}) = 0,$$

where

$$\mathrm{cm}_r^C(\mathfrak{a}) = \max\big\{\big\|\mathfrak{a}(t,\cdot) - \mathfrak{a}(\tau,\cdot)\big\|_{L^\infty(\mathbf{R})} : t,\tau \in [r,2r]\big\}.$$

Let $\mathcal{E}(\mathbf{R}_+, V(\mathbf{R}))$ be the Banach algebra of all $V(\mathbf{R})$-valued functions $\mathfrak{a} \in SO(\mathbf{R}_+, V(\mathbf{R}))$ such that

$$\lim_{|h|\to 0}\sup_{t\in\mathbf{R}_+}\big\|\mathfrak{a}(t,\cdot) - \mathfrak{a}^h(t,\cdot)\big\|_V = 0\,,$$

where $\mathfrak{a}^h(t,x) := \mathfrak{a}(t, x+h)$ for all $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$.

To study the Fredholmness of Mellin pseudodifferential operators $\mathrm{Op}(\mathfrak{a})$, we also need the Banach algebra $\widetilde{\mathcal{E}}(\mathbf{R}_+, V(\mathbf{R}))$ consisting of all functions $\mathfrak{a} \in \mathcal{E}(\mathbf{R}_+, V(\mathbf{R}))$ such that

$$\lim_{M\to\infty}\sup_{t\in\mathbf{R}_+}\int\limits_{\mathbf{R}\setminus[-M,M]}\left|\frac{\partial\mathfrak{a}(t,x)}{\partial x}\right| dx = 0.$$

Thus, $\mathcal{E}(\mathbf{R}_+, V(\mathbf{R}))$ and $\widetilde{\mathcal{E}}(\mathbf{R}_+, V(\mathbf{R}))$ are Banach subalgebras of the algebras $SO(\mathbf{R}_+, V(\mathbf{R})) \subset C_b(\mathbf{R}_+, V(\mathbf{R}))$.

Below we need the following Fredholm criterion and index formula for Mellin pseudodifferential operators $\mathrm{Op}(\mathfrak{a})$ with symbols $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbf{R}_+, V(\mathbf{R}))$, which were obtained in [8, Theorem 4.3] on the base of [6, Theorems 12.2 and 12.5]. Let $M_t(SO(\mathbf{R}_+))$ denote the fibers the maximal ideal space $M(SO(\mathbf{R}_+))$ defined similarly to (6).

Theorem 4. *If* $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbf{R}_+, V(\mathbf{R}))$, *then the Mellin pseudodifferential operator* $\mathrm{Op}(\mathfrak{a})$ *is Fredholm on the space* $L^p(\mathbf{R}_+, d\mu)$ *if and only if*

$$\mathfrak{a}(t, \pm\infty) \neq 0 \quad \textit{for all} \quad t \in \mathbf{R}_+, \qquad \mathfrak{a}(\xi, x) \neq 0 \quad \textit{for all} \quad (\xi, x) \in \Delta \times \overline{\mathbf{R}}, \tag{7}$$

where $\Delta = M_0(SO(\mathbf{R}_+)) \cup M_\infty(SO(\mathbf{R}_+))$. *In the case of Fredholmness*

$$\mathrm{Ind}\,\mathrm{Op}(\mathfrak{a}) = \lim_{\tau \to +\infty} \frac{1}{2\pi} \big\{ \arg \mathfrak{a}(t, x) \big\}_{(t,x) \in \partial\Pi_\tau},$$

where $\Pi_\tau = [\tau^{-1}, \tau] \times \overline{\mathbf{R}}$ *and* $\big\{\arg \mathfrak{a}(t, x)\big\}_{(t,x) \in \partial\Pi_\tau}$ *denotes the increment of* $\arg \mathfrak{a}(t, x)$ *when the point* (t, x) *traces the boundary* $\partial\Pi_\tau$ *of* Π_τ *counter-clockwise.*

4. Reduction to an operator B_Γ without shifts. Let the group G be of the second form in (1). Then the operator (5) takes the form

$$B = A_+ P_{\mathbf{T}}^+ + A_- P_{\mathbf{T}}^-, \tag{8}$$

where $A_\pm$ are functional operators given by

$$A_\pm := \sum_{k=0}^{n-1} a_k^\pm V_\alpha^k + \sum_{k=0}^{n-1} a_{n+k}^\pm V_\alpha^k V_\beta \tag{9}$$

and $a_k^\pm, a_{n+k}^\pm \in PSO(\mathbf{T})$ for all $k = 0, 1, \ldots, n-1$.

To study the operator (8), we first apply a reduction to an operator without shifts.

Fix $t_0 \in \mathbf{T}$ such that $\beta(t_0) = t_0$. Then the points $t_0, \alpha(t_0), \ldots, \alpha^{n-1}(t_0)$ are pairwise distinct. In what follows we assume without loss of generality that $t_0 \prec \alpha(t_0) \prec \ldots \prec \alpha^{n-1}(t_0) \prec t_0$. The shift $\delta = \alpha \circ \beta$ reverses the orientation on $\mathbf{T}$ and maps the endpoints of the arc segment $[t_0, \alpha(t_0)] \subset \mathbf{T}$ to each other: $\delta(t_0) = (\alpha \circ \beta)(t_0) = \alpha(t_0)$, $\delta[\alpha(t_0)] = (\alpha \circ \beta)[\alpha(t_0)] = t_0$. Hence the open arc $(t_0, \alpha(t_0)) \subset \mathbf{T}$ contains exactly one fixed point of the shift $\delta = \alpha \circ \beta$. We denote this point by t_1.

Let Γ be the arc segment $[t_0, t_1] \subset \mathbf{T}$,

$$\mathbf{T} = \bigcup_{k=0}^{n-1} \big(\alpha^k(\Gamma) \cup (\beta \circ \alpha^k)(\Gamma)\big)$$

and the arcs Γ, $\alpha(\Gamma), \dots, \alpha^{n-1}(\Gamma)$, $\beta(\Gamma)$, $(\beta \circ \alpha)(\Gamma), \dots, (\beta \circ \alpha^{n-1})(\Gamma)$ can admit pairwise intersections only at the points of the set $\bigcup_{k=0}^{n-1} \{\alpha^k(t_0), \alpha^k(t_1)\}$.

Let $L^p_{2n}(\Gamma)$ denote the Banach space of vector functions $\psi = \{\psi_k\}_{k=0}^{2n-1}$ where $\psi_k \in L^p(\Gamma)$ and $\|\psi\|_{L^p_{2n}(\Gamma)} = \Big(\sum_{k=0}^{2n-1} \|\psi_k\|^p_{L^p(\Gamma)}\Big)^{1/p}$. Consider the isomorphism

$$\Upsilon : L^p(\mathbf{T}) \to L^p_{2n}(\Gamma), \quad (\Upsilon\varphi)(t) = \{\varphi_k(t)\}_{k=0}^{2n-1} \ \text{ for } \ t \in \Gamma, \tag{10}$$

where $\varphi_k(t) = \varphi[\alpha^k(t)]$ if $k = 0, 1, \dots, n-1$ and $\varphi_k(t) = \varphi[(\beta \circ \alpha^k)(t)]$ if $k = n, n+1, \dots, 2n-1$. Then for every $k = 0, 1, \dots, n-1$ it follows that

$$\begin{gathered} \Upsilon V_\beta \Upsilon^{-1} = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} I, \qquad \Upsilon V_\alpha^k \Upsilon^{-1} = \begin{bmatrix} Y_k & 0 \\ 0 & Y_{n-k} \end{bmatrix} I, \\ Y_k = \begin{bmatrix} 0 & I_{n-k} \\ I_k & 0 \end{bmatrix} I, \end{gathered} \tag{11}$$

where I_k is the $k \times k$ identity matrix. Making use of (11) we immediately obtain the following lemma.

Lemma 5. *If $A_\pm$ are functional operators* (9) *and Υ is given by* (10)*, then $\Upsilon A_\pm \Upsilon^{-1} = \mathcal{A}_\pm I \in \mathcal{B}(L^p_{2n}(\Gamma))$, where*

$$\mathcal{A}_\pm(t) = \begin{bmatrix} \mathcal{A}_1^\pm(t) & \mathcal{A}_2^\pm(t) \\ \mathcal{A}_3^\pm(t) & \mathcal{A}_4^\pm(t) \end{bmatrix} \quad \textit{for} \ \ t \in \Gamma,$$

$$\mathcal{A}_1^\pm(t) = \begin{bmatrix} a_0^\pm(t) & a_1^\pm(t) & \dots & a_{n-1}^\pm(t) \\ a_{n-1}^\pm[\alpha(t) & a_0^\pm[\alpha(t)] & \dots & a_{n-2}^\pm[\alpha(t)] \\ \vdots & \vdots & \ddots & \vdots \\ a_1^\pm[\alpha^{n-1}(t)] & a_2^\pm[\alpha^{n-1}(t)] & \dots & a_0^\pm[\alpha^{n-1}(t)] \end{bmatrix},$$

$$\mathcal{A}_4^\pm(t) = \begin{bmatrix} a_0^\pm[\beta(t)] & a_{n-1}^\pm[\beta(t)] & \dots & a_1^\pm[\beta(t)] \\ a_1^\pm[\beta(\alpha(t))] & a_0^\pm[\beta(\alpha(t))] & \dots & a_2^\pm[\beta(\alpha(t))] \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1}^\pm[\beta(\alpha^{n-1}(t))] & a_{n-2}^\pm[\beta(\alpha^{n-1}(t))] & \dots & a_0^\pm[\beta(\alpha^{n-1}(t))] \end{bmatrix},$$

and the matrix functions $\mathcal{A}_2^{\pm}$ *and* $\mathcal{A}_3^{\pm}$ *are obtained, respectively, from* $\mathcal{A}_1^{\pm}$ *and* $\mathcal{A}_4^{\pm}$ *by replacing* $a_k^{\pm}$ *by* $a_{n+k}^{\pm}$ *for* $k = 0, 1, \dots, n-1$.

If the functions $\alpha', \beta' \in SO(\mathbf{T})$, then for every $k = 0, 1, \dots, n-1$ it follows from Theorem 2 that

$$V_\alpha^k S_{\mathbf{T}} V_\alpha^{-k} \simeq S_{\mathbf{T}}, \quad V_\alpha^k V_\beta S_{\mathbf{T}} V_\beta^{-1} V_\alpha^{-k} \simeq -S_{\mathbf{T}}, \tag{12}$$

where $A \simeq B$ means that the operator $A - B$ is compact.

Lemma 6. *If* $\alpha', \beta' \in SO(\mathbf{T})$ *and* $\delta = \alpha \circ \beta$, *then*

$$\Upsilon S_{\mathbf{T}} \Upsilon^{-1} \simeq \begin{bmatrix} S_\Gamma & -H \\ H & -S_\Gamma \end{bmatrix}, \quad H = \begin{bmatrix} R_0 & 0 & 0 & \dots & 0 & R_1 \\ 0 & 0 & 0 & \dots & R_1 & R_0 \\ 0 & 0 & 0 & \dots & R_0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & R_1 & R_0 & \dots & 0 & 0 \\ R_1 & R_0 & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{13}$$

where the operators $R_0, R_1 \in \mathcal{B}(L^p(\Gamma))$ *are given for* $t \in \Gamma$ *by*

$$\big(R_0\varphi\big)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - \beta(t)}\, d\tau, \quad \big(R_1\varphi\big)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - \delta(t)}\, d\tau. \tag{14}$$

Proof. The operator $\Upsilon S_{\mathbf{T}} \Upsilon^{-1} \in \mathcal{B}(L^p_{2n}(\Gamma))$ is the operator matrix

$$\Upsilon S_{\mathbf{T}} \Upsilon^{-1} = \begin{bmatrix} S_\Gamma^{(1)} & -S_\Gamma^{(2)} \\ S_\Gamma^{(3)} & -S_\Gamma^{(4)} \end{bmatrix}, \quad S_\Gamma^{(r)} = \Big[S_{k,j}^{(r)}\Big]_{k,j=0}^{n-1} \quad (r = 1, 2, 3, 4),$$

where for $t \in \Gamma$ and all $k, j = 0, 1, \dots, n-1$ we have

$$\begin{aligned}
\big[S_{k,j}^{(1)}\varphi\big](t) &= \frac{1}{\pi i} \int_\Gamma \frac{(\alpha^j)'(\tau)}{\alpha^j(\tau) - \alpha^k(t)}\, \varphi(\tau) d\tau, \\
\big[S_{k,j}^{(2)}\varphi\big](t) &= \frac{1}{\pi i} \int_\Gamma \frac{(\beta \circ \alpha^j)'(\tau)}{(\beta \circ \alpha^j)(\tau) - \alpha^k(t)}\, \varphi(\tau) d\tau, \\
\big[S_{k,j}^{(3)}\varphi\big](t) &= \frac{1}{\pi i} \int_\Gamma \frac{(\alpha^j)'(\tau)}{\alpha^j(\tau) - (\beta \circ \alpha^k)(t)}\, \varphi(\tau) d\tau, \\
\big[S_{k,j}^{(4)}\varphi\big](t) &= \frac{1}{\pi i} \int_\Gamma \frac{(\beta \circ \alpha^j)'(\tau)}{(\beta \circ \alpha^j)(\tau) - (\beta \circ \alpha^k)(t)}\, \varphi(\tau) d\tau.
\end{aligned}$$

The operators $S^{(1)}_{k,j}$ and $S^{(4)}_{k,j}$ are compact if $k \neq j$, because in that case $\alpha^k(\Gamma) \cap \alpha^j(\Gamma) = \emptyset$ and $(\beta \circ \alpha^k)(\Gamma) \cap (\beta \circ \alpha^j)(\Gamma) = \emptyset$. On the other hand, if $k = j$, then we infer from (12) that

$$S^{(1)}_{k,k} \simeq S_\Gamma, \quad S^{(4)}_{k,k} \simeq S_\Gamma.$$

Analogously, $(\beta \circ \alpha^j)(\Gamma)$ intersects $\alpha^k(\Gamma)$ only if either $j = n - k$ (here $j = 0$ if $k = 0$) or $j = n - k - 1$, and then

$$\begin{aligned}(\beta \circ \alpha^j)(\Gamma) \cap \alpha^k(\Gamma) &= \{\alpha^k(t_0)\} \quad \text{if} \;\; j = n - k,\\ (\beta \circ \alpha^j)(\Gamma) \cap \alpha^k(\Gamma) &= \{\alpha^k(t_1)\} \quad \text{if} \;\; j = n - k - 1.\end{aligned}$$

Hence the operators $S^{(2)}_{k,j}$ and $S^{(3)}_{k,j}$ are compact if $j \notin \{n-k, n-k-1\}$. On the other hand, for $j = n - k$ and $j = n - k - 1$ we have the relations

$$\begin{aligned}S^{(2)}_{k,n-k} \simeq R_0, \quad S^{(2)}_{k,n-k-1} \simeq R_1,\\ S^{(3)}_{k,n-k} \simeq R_0, \quad S^{(3)}_{k,n-k-1} \simeq R_1.\end{aligned} \tag{15}$$

Indeed, let $\varepsilon > 0$ be sufficiently small and let χ^+_ε, $\widetilde{\chi}^-_\varepsilon$, χ^-_ε, $\widetilde{\chi}^+_\varepsilon$ be the characteristic functions of the arc segments

$$\begin{aligned}\gamma^+_\varepsilon &:= [t_0, t_0 e^{i\varepsilon}] \subset \Gamma, \qquad \widetilde{\gamma}^-_\varepsilon := \beta(\gamma^+_\varepsilon) \subset \mathbf{T} \setminus \Gamma,\\ \gamma^-_\varepsilon &:= [t_1 e^{-i\varepsilon}, t_1] \subset \Gamma, \quad \widetilde{\gamma}^+_\varepsilon := \delta(\gamma^-_\varepsilon) \subset \mathbf{T} \setminus \Gamma,\end{aligned}$$

respectively. Clearly,

$$S^{(r)}_{k,n-k} \simeq \chi^+_\varepsilon S^{(r)}_{k,n-k} \chi^+_\varepsilon I, \quad S^{(r)}_{k,n-k-1} \simeq \chi^-_\varepsilon S^{(r)}_{k,n-k-1} \chi^-_\varepsilon I$$

for $r = 2, 3$ and any sufficiently small $\varepsilon > 0$. Then, taking into account (2) and (12), setting $\sigma := \beta(t)$ for $t \in \gamma^+_\varepsilon$, $\varsigma := \delta(t)$ for $t \in \gamma^-_\varepsilon$ and denoting by K compact operators in $\mathcal{B}(L^p(\mathbf{T}))$, for $t \in \mathbf{T}$ we get

$$\begin{aligned}\big[\chi^+_\varepsilon S^{(2)}_{k,n-k}(\chi^+_\varepsilon \varphi)\big](t) &= \frac{\chi^+_\varepsilon(t)}{\pi i} \int_\Gamma \frac{(\beta \circ \alpha^{n-k})'(\tau)\, \chi^+_\varepsilon(\tau)\varphi(\tau)}{(\beta \circ \alpha^{n-k})(\tau) - \alpha^k(t)}\, d\tau =\\ &= \frac{\widetilde{\chi}^-_\varepsilon(\sigma)}{\pi i} \int_\Gamma \frac{(\beta \circ \alpha^{n-k})'(\tau)\, \chi^+_\varepsilon(\tau)\varphi(\tau)\, d\tau}{(\beta \circ \alpha^{n-k})(\tau) - (\beta \circ \alpha^{n-k})(\sigma)} =\\ &= \frac{\widetilde{\chi}^-_\varepsilon(\sigma)}{\pi i} \int_\Gamma \frac{\chi^+_\varepsilon(\tau)\varphi(\tau)}{\tau - \sigma}\, d\tau + \big[\chi^+_\varepsilon K(\chi^+_\varepsilon \varphi)\big](\beta^{-1}(\sigma)) =\\ &= \frac{\chi^+_\varepsilon(t)}{\pi i} \int_\Gamma \frac{\chi^+_\varepsilon(\tau)\varphi(\tau)\, d\tau}{\tau - \beta(t)} + \big[\chi^+_\varepsilon K(\chi^+_\varepsilon \varphi)\big](t),\end{aligned}$$

$$\begin{aligned}\big[\chi_\varepsilon^- S^{(2)}_{k,n-k-1}(\chi_\varepsilon^-\varphi)\big](t) &= \frac{\chi_\varepsilon^-(t)}{\pi i}\int_\Gamma \frac{(\beta\circ\alpha^{n-k-1})'(\tau)\,\chi_\varepsilon^-(\tau)\varphi(\tau)}{(\beta\circ\alpha^{n-k-1})(\tau)-\alpha^k(t)}\,d\tau = \\ &= \frac{\widetilde{\chi}_\varepsilon^+(\varsigma)}{\pi i}\int_\Gamma \frac{(\beta\circ\alpha^{n-k-1})'(\tau)\,\chi_\varepsilon^-(\tau)\varphi(\tau)\,d\tau}{(\beta\circ\alpha^{n-k-1})(\tau)-(\beta\circ\alpha^{n-k-1})(\varsigma)} = \\ &= \frac{\widetilde{\chi}_\varepsilon^+(\varsigma)}{\pi i}\int_\Gamma \frac{\chi_\varepsilon^-(\tau)\varphi(\tau)}{\tau-\varsigma}\,d\tau + \big[\chi_\varepsilon^- K(\chi_\varepsilon^-\varphi)\big](\delta^{-1}(\varsigma)) = \\ &= \frac{\chi_\varepsilon^-(t)}{\pi i}\int_\Gamma \frac{\chi_\varepsilon^-(\tau)\varphi(\tau)\,d\tau}{\tau-\delta(t)} + \big[\chi_\varepsilon^- K(\chi_\varepsilon^-\varphi)\big](t),\end{aligned}$$

which proves the first two relations in (15) in view of (14). The second two relations in (15) are proved analogously, which completes the proof.

By (8), (9) and by Lemmas 5 and 6, we obtain

Corollary 7. *If B is the operator* (8)*, then*

$$\begin{aligned}B_\Gamma := \Upsilon B \Upsilon^{-1} &\simeq \mathcal{A}_+\begin{bmatrix} P_\Gamma^+ & -2^{-1}H \\ 2^{-1}H & P_\Gamma^- \end{bmatrix} + \mathcal{A}_-\begin{bmatrix} P_\Gamma^- & +2^{-1}H \\ -2^{-1}H & P_\Gamma^+ \end{bmatrix} = \\ &= \mathcal{C}_+P_\Gamma^+ + \mathcal{C}_-P_\Gamma^- + 2^{-1}\mathcal{A}\begin{bmatrix} 0 & -H \\ H & 0 \end{bmatrix},\end{aligned} \tag{16}$$

where H is given by (13)*, $\mathcal{A}=\mathcal{A}_+-\mathcal{A}_-$, and the matrix functions $\mathcal{C}_\pm$ are given by*

$$\mathcal{C}_+(t) = \begin{bmatrix} \mathcal{A}_1^+(t) & \mathcal{A}_2^-(t) \\ \mathcal{A}_3^+(t) & \mathcal{A}_4^-(t) \end{bmatrix}, \quad \mathcal{C}_-(t) = \begin{bmatrix} \mathcal{A}_1^-(t) & \mathcal{A}_2^+(t) \\ \mathcal{A}_3^-(t) & \mathcal{A}_4^+(t) \end{bmatrix}.$$

Let $\Gamma^0 := \Gamma\setminus\{t_0,t_1\}$. Passing from Γ to a segment $[0,\arg(t_1/t_0)]\subset\mathbf{R}$, one can prove that there exists an orientation-reversing Lipschitz homeomorphism $\widehat{\beta}$ of Γ onto $\beta(\Gamma)$ such that t_0 is its fixed point, $\widehat{\beta}'\in SO(\Gamma)\cap$ $\cap C(\Gamma^0)$ and the function $\beta'-\widehat{\beta}'$ is continuous at the points t_0 and t_1 and has zero values there. Let χ_Γ^+ and χ_Γ^- be the characteristic functions of Γ and $\beta(\Gamma)$. Then, setting $\sigma := \widehat{\beta}(t)$ for $t\in\Gamma$ and extending the orientation-preserving shift $\zeta = \beta\circ\widehat{\beta}^{-1} : \beta(\Gamma)\to\beta(\Gamma)$ to $\mathbf{T}\setminus\beta(\Gamma)$ as the identity shift, we infer that $\zeta'\in SO(\mathbf{T})$ and therefore, by Theorem 2,

$$\frac{\chi_\Gamma^+(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau-\beta(t)}\,d\tau - \frac{\chi_\Gamma^+(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau-\widehat{\beta}(t)}\,d\tau =$$

$$= \frac{\chi_\Gamma^-(\sigma)}{\pi i} \int_{\mathbf{T}} \frac{\chi_\Gamma^+(\tau)\varphi(\tau)}{\tau - (\beta \circ \widehat{\beta}^{-1})(\sigma)}\, d\tau - \frac{\chi_\Gamma^-(\sigma)}{\pi i} \int_{\mathbf{T}} \frac{\chi_\Gamma^+(\tau)\varphi(\tau)}{\tau - \sigma}\, d\tau =$$
$$= \frac{\chi_\Gamma^-(\sigma)}{\pi i} \int_{\mathbf{T}} \frac{\chi_\Gamma^+(\tau)\zeta'(\tau)\varphi(\tau)}{\zeta(\tau) - \zeta(\sigma)}\, d\tau - \frac{\chi_\Gamma^-(\sigma)}{\pi i} \int_{\mathbf{T}} \frac{\chi_\Gamma^+(\tau)\varphi(\tau)}{\tau - \sigma}\, d\tau =$$
$$= \big[\chi_\Gamma^+ K(\chi_\Gamma^+\varphi)\big](t) \quad (t \in \Gamma),$$

where $K \in \mathcal{K}(L^p(\mathbf{T}))$. Hence $R_0 \simeq \widehat{R}_0$, where

$$(\widehat{R}_0\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - \widehat{\beta}(t)}\, d\tau \quad \text{for} \quad t \in \Gamma.$$

Thus, we may (and will) assume without loss of generality that the shifts β and δ in the operators R_0 and R_1 given by (14) have derivatives in $SO_{t_0}(\Gamma)$ and $SO_{t_1}(\Gamma)$, respectively (recall that the operators R_0 and R_1 have fixed singularities only at these points).

5. Reduction to Mellin pseudodifferential operators. In what follows we assume that all the coefficients $a_k^\pm, a_{n+k}^\pm \in PSO(\mathbf{T})$ of the operator B given by (8)–(9) admit only finite sets of discontinuities on $\mathbf{T}$, $\alpha', \beta' \in SO(\mathbf{T})$, and t_0 and t_1 are isolated points of SO discontinuities for β' and $\delta' = (\alpha \circ \beta)'$, respectively.

Let $\tau_1 \prec \tau_2 \prec \ldots \prec \tau_{m-1}$ be the finite set of all discontinuities of the matrix functions $\mathcal{A}_\pm \in PSO(\Gamma)$ on the arc $\Gamma^0 := \Gamma \setminus \{t_0, t_1\}$. Consider the arc segments $\gamma_s = [\tau_{s-1}, \tau_s] \subset \Gamma$ for $s = 1, 2, \ldots, m$, where $\tau_0 = t_0$ and $\tau_m = t_1$. Without loss of generality we may assume that $\beta' \in SO_{t_0}(\gamma_1)$ and $(\alpha \circ \beta)' \in SO_{t_1}(\gamma_m)$.

For every $s = 0, 1, \ldots, m$ we introduce the operators

$$\begin{aligned} B_0 &= \mathcal{C}_+^{(0)} P_\Gamma^+ + \mathcal{C}_-^{(0)} P_\Gamma^- + 2^{-1}\mathcal{A}^{(0)} \begin{bmatrix} 0 & -H_0 \\ H_0 & 0 \end{bmatrix}, \\ B_s &= \mathcal{C}_+^{(s)} P_\Gamma^+ + \mathcal{C}_-^{(s)} P_\Gamma^- \quad (s = 1, 2, \ldots, m-1), \\ B_m &= \mathcal{C}_+^{(m)} P_\Gamma^+ + \mathcal{C}_-^{(m)} P_\Gamma^- + 2^{-1}\mathcal{A}^{(m)} \begin{bmatrix} 0 & -H_1 \\ H_1 & 0 \end{bmatrix} \end{aligned} \tag{17}$$

in $\mathcal{B}(L^p_{2n}(\Gamma))$, where

$$H_0 = \begin{bmatrix} R_0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & R_0 \\ 0 & 0 & \ldots & R_0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & R_0 & \ldots & 0 & 0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0 & 0 & \ldots & 0 & R_1 \\ 0 & 0 & \ldots & R_1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & R_1 & \ldots & 0 & 0 \\ R_1 & 0 & \ldots & 0 & 0 \end{bmatrix}. \tag{18}$$

Here the matrix functions $\mathcal{C}_{\pm}^{(s)} \in PSO(\Gamma)$ are continuous on $\Gamma \setminus \{\tau_s\}$, admit PSO discontinuities at the points τ_s, equal I_{2n} on $(\Gamma \setminus \gamma_1) \cup u_{\tau_0}$, $(\Gamma \setminus (\gamma_s \cup \gamma_{s+1})) \cup (u_{\tau_{s-1}} \cup u_{\tau_{s+1}})$ and $(\Gamma \setminus \gamma_m) \cup u_{\tau_m}$, respectively, for $s = 0$, $s = 1, 2, \dots, m-1$ and $s = m$, where $u_{\tau_s} \subset \Gamma$ are some open neighborhoods of points τ_s, and

$$\mathcal{C}_{\pm}^{(s-1)}(t)\mathcal{C}_{\pm}^{(s)}(t) = \mathcal{C}_{\pm}(t) \quad \text{for all} \quad t \in \gamma_s \quad (s = 1, 2, \dots, m). \tag{19}$$

For all $s = 0, 1, \dots, m$ the matrix functions $\mathcal{A}^{(s)} \in PSO(\Gamma) \cap C(\Gamma \setminus \{\tau_s\})$ coincide with $\mathcal{A}$ on u_{τ_s} and equal zero matrix on $\Gamma \setminus \gamma_1$, $\Gamma \setminus (\gamma_s \cup \gamma_{s+1})$ and $\Gamma \setminus \gamma_m$, respectively, for $s = 0$, $s = 1, 2, \dots, m-1$ and $s = m$.

Under these conditions we infer that $B_\Gamma \simeq \prod_{s=0}^{m} B_s$, where the multiples commute to within compact operators. Hence, the operator B_Γ is Fredholm on the space $L^p_{2n}(\Gamma)$ if and only if all the operators B_s $(s = 0, 1, \dots, m)$ are Fredholm on this space, and

$$\operatorname{Ind} B_\Gamma = \sum_{s=0}^{m} \operatorname{Ind} B_s\,. \tag{20}$$

In its turn, we may consider the operators B_s on the spaces $L^p(\gamma_1)$, $L^p(\gamma_s \cup \cup \gamma_{s+1})$ and $L^p(\gamma_m)$ instead of $L^p(\Gamma)$ for $s = 0$, $s = 1, 2, \dots, m-1$ and $s = m$, respectively.

For every $s = 1, 2, \dots, m$ we introduce the diffeomorphisms

$$\begin{aligned} \eta_s &: [0,1] \to \gamma_s, \quad x \mapsto \exp\big\{i[\arg \tau_{s-1} + \theta_s(x)(\arg \tau_s - \arg \tau_{s-1})]\big\}, \\ \widetilde{\eta}_s &: [0,1] \to \gamma_s, \quad x \mapsto \exp\big\{i[\arg \tau_s - \theta_s(x)(\arg \tau_s - \arg \tau_{s-1})]\big\}, \end{aligned} \tag{21}$$

where θ_s is a diffeomorphism of $\mathbf{I} := [0,1]$ onto itself such that

$$\theta_s(0) = 0, \quad \theta_s(1) = 1, \quad \theta_s'(0) = (\arg \tau_s - \arg \tau_{s-1})^{-1} > 0. \tag{22}$$

Such θ_s exists. Indeed, let $c := (\arg \tau_s - \arg \tau_{s-1})^{-1}$ and take $\theta_s(x) = = cx + (1-c)x^\mu$, where $\mu > 1$ if $c \le 1$ and $1 < \mu < c/(c-1)$ if $c > 1$. Then (22) holds and $\theta_s'(x) = c + \mu(1-c)x^{\mu-1} > 0$ for $x \in \mathbf{I}$. Thus, $\theta_s : \mathbf{I} \to \mathbf{I}$ is an orientation-preserving diffeomorphism, and

$$\eta_s'(0) = i\tau_{s-1}, \quad \widetilde{\eta}_s'(0) = -i\tau_s. \tag{23}$$

Further, for $s = 0$ and $s = m$ we introduce the isomorphisms

$$\begin{aligned} \Upsilon_0 &: L^p_{2n}(\gamma_1) \to L^p_{2n}(\mathbf{I}), \quad f \mapsto f \circ \eta_1, \\ \Upsilon_m &: L^p_{2n}(\gamma_m) \to L^p_{2n}(\mathbf{I}), \quad f \mapsto f \circ \widetilde{\eta}_m, \end{aligned}$$

where η_1 and $\widetilde{\eta}_m$ are given by (21). Take the diffeomorphism

$$\widetilde{\eta}_0 : [0,1] \to \beta(\gamma_1), \quad x \mapsto \exp\big\{i\big[\arg\tau_0 - \theta_0(x)\big(\arg\tau_0 - \arg\beta(\tau_1)\big)\big]\big\},$$

where θ_0 is a diffeomorphism of $\mathbf{I}$ onto itself such that

$$\theta_0(0) = 0, \quad \theta_0(1) = 1, \quad \theta_0'(0) = \big(\arg\tau_0 - \arg\beta(\tau_1)\big)^{-1}.$$

Then the map $\eta : [-1,1] \to \beta(\gamma_1) \cup \gamma_1$ given by

$$\eta(x) = \begin{cases} \widetilde{\eta}_0(-x) & \text{if } \ x \in [-1,0], \\ \eta_1(x) & \text{if } \ x \in [0,1], \end{cases}$$

is a diffeomorphism. Consequently, for $t \in \mathbf{I}$ we infer that

$$\begin{aligned}\big(\Upsilon_0 R_0 \Upsilon_0^{-1}\varphi\big)(t) &= \frac{1}{\pi i}\int_{\mathbf{I}} \frac{\eta_1'(\tau)\,\varphi(\tau)}{\eta_1(\tau) - (\beta\circ\eta_1)(t)}\,d\tau = \\ &= \frac{1}{\pi i}\int_{\mathbf{I}} \frac{\eta'(\tau)\,\varphi(\tau)}{\eta(\tau) - \eta[-(\widetilde{\eta}_0^{-1}\circ\beta\circ\eta_1)(t)]}\,d\tau = \\ &= \frac{1}{\pi i}\int_{\mathbf{I}} \frac{\varphi(\tau)}{\tau + \widetilde{\beta}(t)}\,d\tau + (K\varphi)(t),\end{aligned}$$

where K is a compact operator and $\widetilde{\beta} = \widetilde{\eta}_0^{-1}\circ\beta\circ\eta_1$ is an orientation-preserving map of $\mathbf{I}$ onto itself. Thus, $\Upsilon_0 R_0 \Upsilon_0^{-1} \simeq \widetilde{R}_0$, where

$$\big(\widetilde{R}_0\varphi\big)(t) = \frac{1}{\pi i}\int_{\mathbf{I}} \frac{\varphi(\tau)}{\tau + \widetilde{\beta}(t)}\,d\tau \quad \text{for } \ t \in \mathbf{I}, \tag{24}$$

and we may assume without loss of generality that $\widetilde{\beta}'(1) = 0$.

Analogously, take the diffeomorphism $\eta_{m+1} : [0,1] \to \delta(\gamma_m)$ given by

$$\eta_{m+1}(x) = \exp\big\{i\big[\arg\tau_m + \theta_{m+1}(x)\big(\arg\delta(\tau_{m-1}) - \arg\tau_m\big)\big]\big\},$$

where θ_{m+1} is a diffeomorphism of $\mathbf{I}$ onto itself such that

$$\theta_{m+1}(0) = 0, \quad \theta_{m+1}(1) = 1, \quad \theta_{m+1}'(0) = \big(\arg\delta(\tau_{m-1}) - \arg\tau_m\big)^{-1}.$$

Then the map $\widehat{\eta} : [-1,1] \to \delta(\gamma_m) \cup \gamma_m$ given by

$$\widehat{\eta}(x) = \begin{cases} \eta_{m+1}(-x) & \text{if } \ x \in [-1,0], \\ \widetilde{\eta}_m(x) & \text{if } \ x \in [0,1], \end{cases}$$

is a diffeomorphism. Consequently, for $t \in \mathbf{I}$ we deduce that

$$\begin{aligned}\big(\Upsilon_m R_1 \Upsilon_m^{-1}\varphi\big)(t) &= -\frac{1}{\pi i}\int_{\mathbf{I}} \frac{\widetilde{\eta}_m'(\tau)\,\varphi(\tau)}{\widetilde{\eta}_m(\tau) - (\delta\circ\widetilde{\eta}_m)(t)]}\,d\tau = \\ &= -\frac{1}{\pi i}\int_{\mathbf{I}} \frac{\widehat{\eta}'(\tau)\,\varphi(\tau)}{\widehat{\eta}(\tau) - \widehat{\eta}[-(\eta_{m+1}^{-1}\circ\delta\circ\widetilde{\eta}_m)(t)]}\,d\tau = \\ &= -\frac{1}{\pi i}\int_{\mathbf{I}} \frac{\varphi(\tau)}{\tau+\widetilde{\delta}(t)}\,d\tau + (K\varphi)(t),\end{aligned}$$

where K is a compact operator and $\widetilde{\delta} = \eta_{m+1}^{-1}\circ\delta\circ\widetilde{\eta}_m$ is an orientation-preserving map of $\mathbf{I}$ onto itself. Thus, $\Upsilon_m R_1 \Upsilon_m^{-1} \simeq \widetilde{R}_1$, where

$$\big(\widetilde{R}_1\varphi\big)(t) = -\frac{1}{\pi i}\int_{\mathbf{I}} \frac{\varphi(\tau)}{\tau+\widetilde{\delta}(t)}\,d\tau \quad \text{for} \quad t\in\mathbf{I}, \tag{25}$$

where again without loss of generality we may assume that $\widetilde{\delta}'(1) = 0$.

It follows from (17), (18) that

$$\begin{aligned}\Upsilon_0 B_0 \Upsilon_0^{-1} &\simeq \big(\mathcal{C}_+^{(0)}\circ\eta_1\big)P_{\mathbf{I}}^+ + \big(\mathcal{C}_-^{(0)}\circ\eta_1\big)P_{\mathbf{I}}^- + 2^{-1}\big(\mathcal{A}^{(0)}\circ\eta_1\big)\begin{bmatrix}0 & -\widetilde{H}_0\\ \widetilde{H}_0 & 0\end{bmatrix},\\ \Upsilon_m B_m \Upsilon_m^{-1} &\simeq \big(\mathcal{C}_-^{(m)}\circ\widetilde{\eta}_m\big)P_{\mathbf{I}}^+ + \big(\mathcal{C}_+^{(m)}\circ\widetilde{\eta}_m\big)P_{\mathbf{I}}^- + \\ &\quad + 2^{-1}(\mathcal{A}^{(m)}\circ\widetilde{\eta}_m)\begin{bmatrix}0 & -\widetilde{H}_1\\ \widetilde{H}_1 & 0\end{bmatrix},\end{aligned} \tag{26}$$

where

$$\widetilde{H}_0 = \begin{bmatrix}\widetilde{R}_0 & 0 & \dots & 0 & 0\\ 0 & 0 & \dots & 0 & \widetilde{R}_0\\ 0 & 0 & \dots & \widetilde{R}_0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \widetilde{R}_0 & \dots & 0 & 0\end{bmatrix}, \quad \widetilde{H}_1 = \begin{bmatrix}0 & 0 & \dots & 0 & \widetilde{R}_1\\ 0 & 0 & \dots & \widetilde{R}_1 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \widetilde{R}_1 & \dots & 0 & 0\\ \widetilde{R}_1 & 0 & \dots & 0 & 0\end{bmatrix}, \tag{27}$$

and the operators $\widetilde{R}_0, \widetilde{R}_1 \in \mathcal{B}(L^p(\mathbf{I}))$ with fixed singularities at 0 are given by (24) and (25), respectively.

For $s = 1, 2, \dots, m-1$, we introduce the isomorphisms

$$\Upsilon_s:\ L_{2n}^p(\gamma_s\cup\gamma_{s+1}) \to L_{4n}^p(\mathbf{I}), \quad f \mapsto \begin{Bmatrix} f\circ\eta_{s+1}\\ f\circ\widetilde{\eta}_s\end{Bmatrix}.$$

Then for these s we infer from (17) that

$$\Upsilon_s B_s \Upsilon_s^{-1} \simeq \operatorname{diag}\{\mathcal{C}_+^{(s)} \circ \eta_{s+1}, \mathcal{C}_+^{(s)} \circ \widetilde{\eta}_s\} \begin{bmatrix} P_{\mathbf{I}}^+ & 0 \\ 0 & P_{\mathbf{I}}^- \end{bmatrix} + \\ + \operatorname{diag}\{\mathcal{C}_-^{(s)} \circ \eta_{s+1}, \mathcal{C}_-^{(s)} \circ \widetilde{\eta}_s\} \begin{bmatrix} P_{\mathbf{I}}^- & 0 \\ 0 & P_{\mathbf{I}}^+ \end{bmatrix} + \\ + 2^{-1} \operatorname{diag}\{\mathcal{D}^{(s)} \circ \eta_{s+1}, \mathcal{D}^{(s)} \circ \widetilde{\eta}_s\} \begin{bmatrix} 0 & T_s^+ \\ T_s^- & 0 \end{bmatrix},$$

where $\mathcal{D}^{(s)} := \mathcal{C}_+^{(s)} - \mathcal{C}_-^{(s)}$, $T_s^\pm := \operatorname{diag}\{\widetilde{R}_s^\pm\}_{k=0}^{2n-1}$, and the operators $\widetilde{R}_s^\pm \in \mathcal{B}(L^p(\mathbf{I}))$ are given by

$$(\widetilde{R}_s^+ \varphi)(t) = -\frac{1}{\pi i} \int_{\mathbf{I}} \frac{\widetilde{\eta}_s'(\tau)\, \varphi(\tau)}{\widetilde{\eta}_s(\tau) - \eta_{s+1}(t)}\, d\tau,$$
$$(\widetilde{R}_s^- \varphi)(t) = \frac{1}{\pi i} \int_{\mathbf{I}} \frac{\eta_{s+1}'(\tau)\, \varphi(\tau)}{\eta_{s+1}(\tau) - \widetilde{\eta}_s(t)}\, d\tau.$$

In its turn, applying (23), we can easily obtain the relation

$$\widetilde{R}_s^\pm \simeq \mp R_{\mathbf{I}} \quad \text{for all} \quad s = 1, 2, \dots, m-1,$$

where

$$(R_{\mathbf{I}} \varphi)(t) = \frac{1}{\pi i} \int_{\mathbf{I}} \frac{\varphi(\tau)}{\tau + t}\, d\tau \quad \text{for} \quad t \in \mathbf{I}.$$

Thus, for $s = 1, 2, \dots, m-1$,

$$\Upsilon_s B_s \Upsilon_s^{-1} \simeq \operatorname{diag}\{\mathcal{C}_+^{(s)} \circ \eta_{s+1}, \mathcal{C}_-^{(s)} \circ \widetilde{\eta}_s\} P_{\mathbf{I}}^+ + \operatorname{diag}\{\mathcal{C}_-^{(s)} \circ \eta_{s+1}, \mathcal{C}_+^{(s)} \circ \widetilde{\eta}_s\} P_{\mathbf{I}}^- + \\ + 2^{-1} \operatorname{diag}\{\mathcal{D}^{(s)} \circ \eta_{s+1}, \mathcal{D}^{(s)} \circ \widetilde{\eta}_s\} \begin{bmatrix} 0 & -T \\ T & 0 \end{bmatrix}, \tag{28}$$

where $T := \operatorname{diag}\{R_{\mathbf{I}}\}_{k=0}^{2n-1}$.

Taking $\Upsilon_s B_s \Upsilon_s^{-1}$ given for $s = 0, 1, \dots, m$ by (26) and (28), let

$$\widehat{B}_s := \chi_{\mathbf{I}} \Upsilon_s B_s \Upsilon_s^{-1} \chi_{\mathbf{I}} I + (1 - \chi_{\mathbf{I}}) I \in \mathcal{B}(L_{n_s}^p(\mathbf{R}_+)), \tag{29}$$

where $\chi_{\mathbf{I}}$ is the characteristic function of $\mathbf{I}$, $n_0 = n_m = 2n$ and $n_s = 4n$

for $s = 1, 2, \dots, m-1$. Hence $\widehat{B}_s \simeq \widetilde{B}_s$, where

$$\widetilde{B}_0 = \widehat{\mathcal{C}}_+^{(0)} P_{\mathbf{R}_+}^+ + \widehat{\mathcal{C}}_-^{(0)} P_{\mathbf{R}_+}^- + 2^{-1}\widehat{\mathcal{A}}^{(0)} \begin{bmatrix} 0 & -H_\beta \\ H_\beta & 0 \end{bmatrix}, \tag{30}$$

$$\widetilde{B}_s = \operatorname{diag}\{\widehat{\mathcal{C}}_+^{(s)}, \widetilde{\mathcal{C}}_-^{(s)}\} P_{\mathbf{R}_+}^+ + \operatorname{diag}\{\widehat{\mathcal{C}}_-^{(s)}, \widetilde{\mathcal{C}}_+^{(s)}\} P_{\mathbf{R}_+}^- + \\ + 2^{-1}\operatorname{diag}\{\widehat{\mathcal{D}}^{(s)}, \widetilde{\mathcal{D}}^{(s)}\} \begin{bmatrix} 0 & -\widetilde{T} \\ \widetilde{T} & 0 \end{bmatrix} \quad (s = 1, 2, \dots, m-1), \tag{31}$$

$$\widetilde{B}_m = \widetilde{\mathcal{C}}_-^{(m)} P_{\mathbf{R}_+}^+ + \widetilde{\mathcal{C}}_+^{(m)} P_{\mathbf{R}_+}^- + 2^{-1}\widetilde{\mathcal{A}}^{(m)} \begin{bmatrix} 0 & -H_\delta \\ H_\delta & 0 \end{bmatrix}, \tag{32}$$

the operator matrices H_β and H_δ are given by (27) with $\widetilde{R}_0$ and $\widetilde{R}_1$ replaced by R_β and R_δ, respectively,

$$\widetilde{T} := \operatorname{diag}\{R\}_{k=0}^{2n-1}, \quad (R\varphi)(t) = \frac{1}{\pi i}\int_{\mathbf{R}_+} \frac{\varphi(\tau)}{\tau + t}\, d\tau \quad \text{for } t \in \mathbf{R}_+,$$

$$(R_\beta\varphi)(t) = \frac{1}{\pi i}\int_{\mathbf{R}_+} \frac{\varphi(\tau)}{\tau + \widehat{\beta}(t)}\, d\tau, \quad (R_\delta\varphi)(t) = -\frac{1}{\pi i}\int_{\mathbf{R}_+} \frac{\varphi(\tau)}{\tau + \widehat{\delta}(t)}\, d\tau,$$

$$\widehat{\beta}(t) = \begin{cases} \widetilde{\beta}(t) & \text{if } t \in \mathbf{I}, \\ 1 & \text{if } t \in \mathbf{R}_+ \setminus \mathbf{I}, \end{cases} \qquad \widehat{\delta}(t) = \begin{cases} \widetilde{\delta}(t) & \text{if } t \in \mathbf{I}, \\ 1 & \text{if } t \in \mathbf{R}_+ \setminus \mathbf{I}. \end{cases}$$

The matrix coefficients in (30) — (32) are defined as follows.

$$\begin{aligned} \widehat{\mathcal{A}}^{(0)}(t) &= \big(\mathcal{A}^{(0)} \circ \eta_1\big)(t) \quad \text{if } t \in \mathbf{I}, \quad \widehat{\mathcal{A}}^{(0)}(t) = 0 \quad \text{if } t \in \mathbf{R}_+ \setminus \mathbf{I}, \\ \widetilde{\mathcal{A}}^{(m)}(t) &= \big(\mathcal{A}^{(m)} \circ \widetilde{\eta}_m\big)(t) \quad \text{if } t \in \mathbf{I}, \quad \widetilde{\mathcal{A}}^{(m)}(t) = 0 \quad \text{if } t \in \mathbf{R}_+ \setminus \mathbf{I}. \end{aligned} \tag{33}$$

If $s = 0, 1, \dots, m-1$, then

$$\begin{aligned} \widehat{\mathcal{C}}_\pm^{(s)}(t) &= \big(\mathcal{C}_\pm^{(s)} \circ \eta_{s+1}\big)(t) \quad \text{if } t \in \mathbf{I}, \ \widehat{\mathcal{C}}_\pm^{(s)}(t) = I_{2n} \ \text{if } t \in \mathbf{R}_+ \setminus \mathbf{I}, \\ \widehat{\mathcal{D}}^{(s)}(t) &= \big([\mathcal{C}_+^{(s)} - \mathcal{C}_-^{(s)}] \circ \eta_{s+1}\big)(t) \ \text{if } t \in \mathbf{I}, \ \widehat{\mathcal{A}}^{(s)}(t) = 0_{2n} \ \text{if } t \in \mathbf{R}_+ \setminus \mathbf{I}; \end{aligned} \tag{34}$$

and if $s = 1, 2, \dots, m$, then

$$\begin{aligned} \widetilde{\mathcal{C}}_\pm^{(s)}(t) &= \big(\mathcal{C}_\pm^{(s)} \circ \widetilde{\eta}_s\big)(t) \quad \text{if } t \in \mathbf{I}, \ \widetilde{\mathcal{C}}_\pm^{(s)}(t) = I_{2n} \ \text{if } t \in \mathbf{R}_+ \setminus \mathbf{I}, \\ \widetilde{\mathcal{D}}^{(s)}(t) &= \big([\mathcal{C}_+^{(s)} - \mathcal{C}_-^{(s)}] \circ \widetilde{\eta}_s\big)(t) \ \text{if } t \in \mathbf{I}, \ \widetilde{\mathcal{A}}^{(s)}(t) = 0_{2n} \ \text{if } t \in \mathbf{R}_+ \setminus \mathbf{I}; \end{aligned} \tag{35}$$

where the entries of the matrix functions $\widehat{\mathcal{C}}_\pm^{(s)}$, $\widetilde{\mathcal{C}}_\pm^{(s)}$, $\widehat{\mathcal{D}}^{(s)}$, $\widetilde{\mathcal{D}}^{(s)}$, $\widehat{\mathcal{A}}^{(0)}$, $\widetilde{\mathcal{A}}^{(m)}$ belong to $SO_0(\mathbf{R}_+)$.

Thus, applying (20), (29) and the relations $\widehat{B}_s \simeq \widetilde{B}_s$, where the operators $\widetilde{B}_s$ are given by (30) — (32), we arrive at the following assertion.

Lemma 8. *Let the matrix coefficients $\mathcal{C}_\pm, \mathcal{A} \in PSO(\Gamma)$ of the operator* (16) *admit discontinuities only at the points $\tau_0, \tau_1, \dots, \tau_m \in \Gamma$, $\alpha', \beta' \in SO(\mathbf{T})$, and let t_0 and t_1 be isolated discontinuity points for the derivatives β' and δ', respectively. Then the operator B_Γ given by* (16) *is Fredholm on the space $L^p_{2n}(\Gamma)$ if and only if the operators $\widetilde{B}_0$ and $\widetilde{B}_m$ are Fredholm on the space $L^p_{2n}(\mathbf{R}_+)$ and the operators $\widetilde{B}_s$ for all $s = 1, 2, \dots, m-1$ are Fredholm on the space $L^p_{4n}(\mathbf{R}_+)$. In that case*

$$\operatorname{Ind} B_\Gamma = \sum_{s=0}^{m} \operatorname{Ind} \widetilde{B}_s. \tag{36}$$

Clearly, the functions ω_β and ω_δ given by

$$\omega_\beta(t) := \ln\big(\widehat{\beta}(t)/t\big), \quad \omega_\delta(t) := \ln\big(\widehat{\delta}(t)/t\big) \tag{37}$$

belong to $SO_0(\mathbf{R}_+)$ along with $\widehat{\beta}'$ and $\widehat{\delta}'$ (see, e.g. [13, Lemma 2.2]).

Consider the isometric isomorphism

$$\Phi : L^p(\mathbf{R}_+) \to L^p(\mathbf{R}_+, d\mu), \quad (\Phi f)(t) := t^{1/p} f(t) \quad (t \in \mathbf{R}_+).$$

Then (see, e.g. [13, Theorem 4.3])

$$\Phi P^{\pm}_{\mathbf{R}_+} \Phi^{-1} = \operatorname{Op}(\mathfrak{p}_\pm), \quad \Phi R \Phi^{-1} = \operatorname{Op}(\mathfrak{r}) \tag{38}$$

and, by the proof of [13, Lemma 8.3],

$$\Phi \widetilde{R}_\beta \Phi^{-1} = \operatorname{Op}(\mathfrak{b}), \quad \Phi \widetilde{R}_\delta \Phi^{-1} = \operatorname{Op}(\mathfrak{d}), \tag{39}$$

where the functions $\mathfrak{p}_\pm, \mathfrak{r}, \mathfrak{b}, \mathfrak{d} \in C_b(\mathbf{R}_+, V(\mathbf{R}))$ are defined for $(t, x) \in \mathbf{R}_+ \times \mathbf{R}$ by

$$\begin{gathered} \mathfrak{p}_\pm(t, x) = \mathcal{P}_\pm(x), \quad \mathfrak{r}(t, x) = r_p(x), \\ \mathcal{P}_\pm(x) := \big[1 \pm \coth(\pi x + \pi i/p)\big]/2, \quad r_p(x) = 1/\sinh(\pi x + \pi i/p), \\ \mathfrak{b}(t, x) = e^{i\omega_\beta(t)(x+i/p)} r_p(x), \quad \mathfrak{d}(t, x) = e^{i\omega_\delta(t)(x+i/p)} r_p(x), \end{gathered}$$

and the functions ω_β, $\omega_\delta \in SO_0(\mathbf{R}_+)$ are given by (37). Obviously, $\mathfrak{p}_\pm, \mathfrak{r} \in \widetilde{\mathcal{E}}(\mathbf{R}_+, V(\mathbf{R}))$. By analogy with [13, Lemmas 7.3, 7.4] one can prove that $\mathfrak{b}, \mathfrak{d} \in \widetilde{\mathcal{E}}(\mathbf{R}_+, V(\mathbf{R}))$ as well.

Thus, for all $s = 0, 1, \dots, m$ we infer from (38) and (39) that

$$\Phi \widetilde{B}_s \Phi^{-1} = \mathrm{Op}(\mathfrak{B}_s), \tag{40}$$

where $\mathrm{Op}(\mathfrak{B}_0), \mathrm{Op}(\mathfrak{B}_m) \in \mathcal{B}(L^p_{2n}(\mathbf{R}_+, d\mu))$, $\mathrm{Op}(\mathfrak{B}_s) \in \mathcal{B}(L^p_{4n}(\mathbf{R}_+, d\mu))$ for all $s = 1, 2, \dots, m-1$, and the symbols of these operators are given for $(t, x) \in \mathbf{R}_+ \times \overline{\mathbf{R}}$ by

$$\mathfrak{B}_0(t,x) = \widehat{\mathcal{C}}^{(0)}_+(t)\mathcal{P}_+(x) + \widehat{\mathcal{C}}^{(0)}_-(t)\mathcal{P}_-(x) + 2^{-1}\widehat{\mathcal{A}}^{(0)}(t)\begin{bmatrix} 0 & -\mathcal{H}_\beta(t,x) \\ \mathcal{H}_\beta(t,x) & 0 \end{bmatrix},$$

$$\mathfrak{B}_s(t,x) = \mathrm{diag}\big\{\widehat{\mathcal{C}}^{(s)}_+(t),\, \widetilde{\mathcal{C}}^{(s)}_-(t)\big\}\mathcal{P}_+(x) + \mathrm{diag}\big\{\widehat{\mathcal{C}}^{(s)}_-(t),\, \widetilde{\mathcal{C}}^{(s)}_+(t)\big\}\mathcal{P}_-(x) +$$
$$+2^{-1}\begin{bmatrix} 0 & -\widehat{\mathcal{D}}^{(s)}(t)r_p(x) \\ \widetilde{\mathcal{D}}^{(s)}(t)r_p(x) & 0 \end{bmatrix} \quad (s = 1, 2, \dots, m-1), \tag{41}$$

$$\mathfrak{B}_m(t,x) = \widetilde{\mathcal{C}}^{(m)}_-(t)\mathcal{P}_+(x) + \widetilde{\mathcal{C}}^{(m)}_+(t)\mathcal{P}_-(x) + 2^{-1}\widetilde{\mathcal{A}}^{(m)}(t)\begin{bmatrix} 0 & -\mathcal{H}_\delta(t,x) \\ \mathcal{H}_\delta(t,x) & 0 \end{bmatrix},$$

where

$$\mathcal{H}_\beta(t,x) = \begin{bmatrix} \mathfrak{b}(t,x) & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \mathfrak{b}(t,x) \\ 0 & 0 & \dots & \mathfrak{b}(t,x) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mathfrak{b}(t,x) & \dots & 0 & 0 \end{bmatrix},$$

$$\mathcal{H}_\delta(t,x) = -\begin{bmatrix} 0 & 0 & \dots & 0 & \mathfrak{d}(t,x) \\ 0 & 0 & \dots & \mathfrak{d}(t,x) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mathfrak{d}(t,x) & \dots & 0 & 0 \\ \mathfrak{d}(t,x) & 0 & \dots & 0 & 0 \end{bmatrix},$$

6. A Fredholm criterion for the operator *B*. Consider the following matrix functions:

$$\mathcal{H}_{t_0}(\xi^+,x) = \begin{bmatrix} \mathfrak{b}(\xi^+,x) & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \mathfrak{b}(\xi^+,x) \\ 0 & 0 & \dots & \mathfrak{b}(\xi^+,x) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mathfrak{b}(\xi^+,x) & \dots & 0 & 0 \end{bmatrix}$$

for all $(\xi, x) \in M_{t_0}(SO(\mathbf{T})) \times \overline{\mathbf{R}}$, and

$$\mathcal{H}_{t_1}(\xi^-, x) = -\begin{bmatrix} 0 & 0 & \dots & 0 & \mathfrak{d}(\xi^-, x) \\ 0 & 0 & \dots & \mathfrak{d}(\xi^-, x) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mathfrak{d}(\xi^-, x) & \dots & 0 & 0 \\ \mathfrak{d}(\xi^-, x) & 0 & \dots & 0 & 0 \end{bmatrix}$$

for all $(\xi, x) \in M_{t_1}(SO(\mathbf{T})) \times \overline{\mathbf{R}}$, where

$$\mathfrak{b}(\xi^+, x) = |\beta'(\xi^+)|^{ix-1/p} r_p(x) \quad \text{for} \quad (\xi, x) \in M_{t_0}(SO(\Gamma)) \times \overline{\mathbf{R}},$$
$$\mathfrak{d}(\xi^-, x) = |\delta'(\xi^-)|^{ix-1/p} r_p(x) \quad \text{for} \quad (\xi, x) \in M_{t_1}(SO(\Gamma)) \times \overline{\mathbf{R}}.$$

By (16), Lemma 8 and (40), the operator B is Fredholm if and only if so are the operators $\mathrm{Op}(\mathfrak{B}_s)$ $(s = 0, 1, \dots, m)$. Applying the matrix version of Theorem 4 to the Mellin pseudodifferential operators $\mathrm{Op}(\mathfrak{B}_s)$ given by (40) with matrix symbols (41) that have entries in $\widetilde{\mathcal{E}}(\mathbf{R}_+, V(\mathbf{R}))$, we obtain the following.

Theorem 9. *Let the operator B be given by* (8), (9)*, where the coefficients $a_k^\pm, a_{n+k}^\pm \in PSO(\mathbf{T})$ $(k = 0, 1, \dots, n-1)$ admit only a finite set Y of discontinuities on $\mathbf{T}$, $Y \cap \Gamma = \{\tau_0, \tau_1, \dots, \tau_m\}$, $\alpha', \beta' \in SO(\mathbf{T})$, and let $t_0 = \tau_0$ and $t_1 = \tau_m$ be the isolated points of discontinuities for $\beta' \in SO(\mathbf{T})$ and $\delta' \in SO(\mathbf{T})$, respectively. Then the operator B is Fredholm on the space $L^p(\mathbf{T})$ if and only if the following four conditions are fulfilled:*

(i) *the functions* $\det \mathcal{C}_\pm$ *are separated from zero on* Γ*;*

(ii) *for every $t \in (t_0, t_1)$ and every $(\xi, x) \in M_t(SO(\mathbf{T})) \times \overline{\mathbf{R}}$, the next $4n \times 4n$ matrix is invertible:*

$$\mathcal{B}_t(\xi, x) := \mathrm{diag}\big\{\mathcal{C}_+(\xi^+), \mathcal{C}_-(\xi^-)\big\}\mathcal{P}_+(x) + \mathrm{diag}\big\{\mathcal{C}_-(\xi^+), \mathcal{C}_+(\xi^-)\big\}\mathcal{P}_-(x) + \\ + 2^{-1}\begin{bmatrix} 0 & -\big(\mathcal{C}_+(\xi^+) - \mathcal{C}_-(\xi^+)\big) r_p(x) \\ \big(\mathcal{C}_+(\xi^-) - \mathcal{C}_-(\xi^-)\big) r_p(x) & 0 \end{bmatrix};$$

(iii) *for every $(\xi, x) \in M_{t_0}(SO(\mathbf{T})) \times \overline{\mathbf{R}}$, the $2n \times 2n$ matrix*

$$\mathcal{B}_{t_0}(\xi, x) := \mathcal{C}_+(\xi^+)\mathcal{P}_+(x) + \mathcal{C}_-(\xi^+)\mathcal{P}_-(x) + \\ + 2^{-1}\mathcal{A}(\xi^+)\begin{bmatrix} 0 & -\mathcal{H}_{t_0}(\xi^+, x) \\ \mathcal{H}_{t_0}(\xi^+, x) & 0 \end{bmatrix}$$

is invertible;

(iv) *for every* $(\xi, x) \in M_{t_1}(SO(\mathbf{T})) \times \overline{\mathbf{R}}$, *the* $2n \times 2n$ *matrix*

$$\mathcal{B}_{t_1}(\xi, x) := \mathcal{C}_-(\xi^-)\mathcal{P}_+(x) + \mathcal{C}_+(\xi^-)\mathcal{P}_-(x) + \\ + 2^{-1}\mathcal{A}(\xi^-)\begin{bmatrix} 0 & -\mathcal{H}_{t_1}(\xi^-, x) \\ \mathcal{H}_{t_1}(\xi^-, x) & 0 \end{bmatrix}$$

is invertible.

Proof. By definition, $\det \widehat{\mathcal{C}}_\pm^{(s)}(t) = 1$ $(s = 0, 1, \ldots, m-1)$ and $\det \widetilde{\mathcal{C}}_\pm^{(s)}(t) = 1$ $(s = 1, 2, \ldots, m)$ for all $t \in \mathbf{R}_+ \setminus \mathbf{I}$. Hence $\det \mathfrak{B}_s(\xi^-, x) = 1$ for all $s = 0, 1, \ldots, m$ and all $(\xi, x) \in M_\infty(SO_0(\mathbf{R}_+)) \times \overline{\mathbf{R}}$. Then we infer from Theorem 4 that the operator $\mathrm{Op}(\mathfrak{B}_0)$ is Fredholm on the space $L^p_{2n}(\mathbf{R}_+, d\mu)$ if and only if

$$\det \widehat{\mathcal{C}}_\pm^{(0)}(t) \neq 0 \quad \text{for all} \ \ t \in (0, 1], \tag{42}$$

$$\det \mathfrak{B}_0(\xi^+, x) \neq 0 \quad \text{for all} \ \ (\xi, x) \in M_0(SO_0(\mathbf{R}_+)) \times \overline{\mathbf{R}}. \tag{43}$$

Analogously, for every $s = 1, 2, \ldots, m-1$, the operator $\mathrm{Op}(\mathfrak{B}_s)$ is Fredholm on the space $L^p_{4n}(\mathbf{R}_+, d\mu)$ if and only if

$$\det \widehat{\mathcal{C}}_\pm^{(s)}(t) \neq 0, \ \ \det \widetilde{\mathcal{C}}_\pm^{(s)}(t) \neq 0 \quad \text{for all} \ \ t \in (0, 1], \tag{44}$$

$$\det \mathfrak{B}_s(\xi^+, x) \neq 0 \quad \text{for all} \ \ (\xi, x) \in M_0(SO_0(\mathbf{R}_+)) \times \overline{\mathbf{R}}\,. \tag{45}$$

Finally, the operator $\mathrm{Op}(\mathfrak{B}_m)$ is Fredholm on the space $L^p_{2n}(\mathbf{R}_+, d\mu)$ if and only if

$$\det \widetilde{\mathcal{C}}_\pm^{(m)}(t) \neq 0 \quad \text{for all} \ \ t \in (0, 1], \tag{46}$$

$$\det \mathfrak{B}_m(\xi^+, x) \neq 0 \quad \text{for all} \ \ (\xi, x) \in M_0(SO_0(\mathbf{R}_+)) \times \overline{\mathbf{R}}\,. \tag{47}$$

Making use of definitions (33)–(35) of the matrix functions $\widehat{\mathcal{A}}^{(0)}$, $\widetilde{\mathcal{A}}^{(m)}$, $\widehat{\mathcal{C}}_\pm^{(s)}$, $\widetilde{\mathcal{C}}_\pm^{(s)}$, $\widehat{\mathcal{D}}^{(s)}$, $\widetilde{\mathcal{D}}^{(s)}$ and the equalities (19), we deduce that assertion (i) is equivalent to the fulfilment of all conditions (42), (44) and (46), if assertions (43), (45) and (47) hold. On the other hand, conditions (43), (45) and (47) are equivalent to assertions (ii) — (iv), because $M_0(SO_0(\mathbf{R}_+)) = M_{\tau_s}(SO(\mathbf{T}))$ and hence the matrices $\mathfrak{B}_s(\xi, x)$ for $(\xi, x) \in M_0(SO_0(\mathbf{R}_+)) \times \overline{\mathbf{R}}$ coincide with matrices $\mathcal{B}_{\tau_s}(\xi, x)$ for $(\xi, x) \in M_{\tau_s}(SO(\mathbf{T})) \times \overline{\mathbf{R}}$. Finally, if $t \in \Gamma \setminus \{\tau_0, \tau_1, \ldots, \tau_m\}$ and $(\xi, x) \in M_t(SO(\mathbf{T})) \times \overline{\mathbf{R}}$, then $\det \mathcal{B}_t(\xi, x) = C_+(t)C_-(t)$, and therefore the invertibility of the matrix $\mathcal{B}_t(\xi, x)$ is equivalent to the invertibility of both matrices $C_\pm(t)$, which completes the proof.

Remark. *Applying results of* [13, 14]*, one can prove that Theorem* 9 *remains valid for arbitrary coefficients* $a_k^\pm \in PSO(\mathbf{T})$ $(k = 0, 1, \ldots, 2n-1)$. *The arguments presented at the end of Section* 4 *also say that Theorem* 9 *is true for arbitrary* $\alpha', \delta' \in SO(\mathbf{T})$.

7. An index formula for the operator B. Given $s = 0, 1, \ldots, m$ and $0 < \varepsilon_0 < \varepsilon_1 < 1$, let $l_0 = [\varepsilon_0, \varepsilon_1]$, $l_s = [\zeta_s(\varepsilon_{n_s}), \zeta_s(\varepsilon_{n'_s})]$, where $\zeta_0(t) = t$, $\zeta_s(t) = (\widetilde{\eta}_s^{-1} \circ \eta_s \circ \widetilde{\eta}_{s-1}^{-1} \circ \eta_{s-1} \circ \ldots \circ \widetilde{\eta}_1^{-1} \circ \eta_1)(t)$ for $s = 1, 2, \ldots, m$ and $t \in [0, 1]$, $n_s = [1 - (-1)^s]/2$ and $n'_s = [1 + (-1)^s]/2$.

Theorem 10. *If all the conditions of Theorem* 9 *are fulfilled, then the index of the Fredholm operator B acting on the space $L^p(\mathbf{T})$ is calculated by the formula*

$$\begin{aligned} \operatorname{Ind} B = \lim_{\varepsilon_0 \to 0,\ \varepsilon_1 \to 1} \frac{1}{2\pi} \Bigg(&- \sum_{s=0}^{m} \Big\{ \arg\det \mathfrak{B}_s\big(\zeta_s(\varepsilon_{n_s}), x\big) \Big\}_{x \in \overline{\mathbf{R}}} + \\ &+ \sum_{s=1}^{m} \Big\{ \arg\det \mathcal{C}_-[\eta_s(t)] \Big\}_{t \in l_{s-1}} - \sum_{s=1}^{m} \Big\{ \arg\det \mathcal{C}_+[\eta_s(t)] \Big\}_{t \in l_{s-1}} \Bigg). \end{aligned} \tag{48}$$

Proof. Since $\operatorname{Ind} B = \operatorname{Ind} B_\Gamma$ by Corollary 7 and since $\operatorname{Ind} \widetilde{B}_s = \operatorname{Ind} \operatorname{Op}(\mathfrak{B}_s)$ for every $s = 0, 1, \ldots, m$ due to (40), we infer from (36) that

$$\operatorname{Ind} B = \operatorname{Ind} B_\Gamma = \sum_{s=0}^{m} \operatorname{Ind} \operatorname{Op}(\mathfrak{B}_s). \tag{49}$$

Because $\det \widehat{\mathcal{C}}_\pm^{(s)}(t) = 1$ $(s = 0, 1, \ldots, m-1)$ and $\det \widetilde{\mathcal{C}}_\pm^{(s)}(t) = 1$ $(s = 1, 2, \ldots, m)$ for all $t \in [1, \infty)$, and therefore $\det \mathfrak{B}_s(t, x) = 1$ for all $s = 0, 1, \ldots, m$ and all $(t, x) \in [1, \infty) \times \overline{\mathbf{R}}$, we deduce from Theorem 4 that the indices of the Mellin pseudodifferential operators $\operatorname{Op}(\mathfrak{B}_s)$ on the space $L^p(\mathbf{R}_+, d\mu)$ for $s = 0, 1, \ldots, m$ are calculated by the formula:

$$\operatorname{Ind} \operatorname{Op}(\mathfrak{B}_s) = \lim_{\varepsilon_0 \to 0,\ \varepsilon_1 \to 1} \frac{1}{2\pi} \big\{ \arg\det \mathfrak{B}_s(t, x) \big\}_{(t,x) \in \partial(l_s \times \overline{\mathbf{R}})} \tag{50}$$

where $\big\{ \arg\det \mathfrak{B}_s(t, x) \big\}_{(t,x) \in \partial(l_s \times \overline{\mathbf{R}})}$ denotes the increment of the function $\arg\det \mathfrak{B}_s(t, x)$ when the point (t, x) traces the boundary $\partial(l_s \times \overline{\mathbf{R}})$ of $l_s \times \overline{\mathbf{R}}$ counter-clockwise. It follows from conditions (ii) — (iv) of Theorem 9 that in (50) the functions $\det \mathfrak{B}_s(t, \cdot)$ given by (41) are separated from zero for all $s = 0, 1, \ldots, m$ and all $t \in (0, \varepsilon]$, where $\varepsilon > 0$ is sufficiently small.

Consequently, we infer from (41) in view of (33) — (35) that

$$\operatorname{Ind}\operatorname{Op}(\mathfrak{B}_0) = \lim_{\varepsilon_0\to 0,\ \varepsilon_1\to 1} \frac{1}{2\pi}\Big[\big\{\arg\det(C_-^{(0)}\circ\eta_1)(t)\big\}_{t\in[\varepsilon_0,\varepsilon_1]} - \\ - \big\{\arg\det(C_+^{(0)}\circ\eta_1)(t)\big\}_{t\in[\varepsilon_0,\varepsilon_1]} - \big\{\arg\det\mathfrak{B}_0(\varepsilon_0,x)\big\}_{x\in\overline{\mathbf{R}}}\Big], \qquad (51)$$

$$\operatorname{Ind}\operatorname{Op}(\mathfrak{B}_s) = \lim_{\varepsilon_0\to 0,\ \varepsilon_1\to 1} \frac{1}{2\pi}\Big[\big\{\arg\det(C_-^{(s)}\circ\eta_{s+1})(t)\big\}_{t\in l_s} + \\ + \big\{\arg\det(C_+^{(s)}\circ\widetilde{\eta}_s)(t)\big\}_{t\in l_s} - \big\{\arg\det(C_+^{(s)}\circ\eta_{s+1})(t)\big\}_{t\in l_s} - \\ - \big\{\arg\det(C_-^{(s)}\circ\widetilde{\eta}_s)(t)\big\}_{t\in l_s} - \big\{\arg\det\mathfrak{B}_s(\zeta_s(\varepsilon_{n_s}),x)\big\}_{x\in\overline{\mathbf{R}}}\Big] \qquad (52)$$

for $s = 1, 2, \ldots, m-1$, and

$$\operatorname{Ind}\operatorname{Op}(\mathfrak{B}_m) = \lim_{\varepsilon_0\to 0,\ \varepsilon_1\to 1} \frac{1}{2\pi}\Big[\big\{\arg\det(C_+^{(m)}\circ\widetilde{\eta}_m)(t)\big\}_{t\in l_m} - \\ - \big\{\arg\det(C_-^{(m)}\circ\widetilde{\eta}_m)(t)\big\}_{t\in l_m} - \big\{\arg\det\mathfrak{B}_m(\zeta_m(\varepsilon_{n_m}),x)\big\}_{x\in\overline{\mathbf{R}}}\Big]. \qquad (53)$$

Further, by (19), $\mathcal{C}_\pm^{(s-1)}(t)\mathcal{C}_\pm^{(s)}(t) = \mathcal{C}_\pm(t)$ for $t \in \gamma_s$ for all $s = = 1, 2, \ldots, m$, and therefore

$$\big\{\arg\det\mathcal{C}_\pm^{(s-1)}[\eta_s(t)]\big\}_{t\in l_{s-1}} - \big\{\arg\det\mathcal{C}_\pm^{(s)}[\widetilde{\eta}_s(t)]\big\}_{t\in l_s} = \\ = \big\{\arg\det\mathcal{C}_\pm[\eta_s(t)]\big\}_{t\in l_{s-1}}. \qquad (54)$$

Substituting (51) — (53) into (49) and applying (54), we obtain (48), which completes the proof.

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References

[1] Böttcher A., Silbermann B. *Analysis of Toeplitz operators*. 2nd edn. — Berlin: Springer-Verlag, 2006. — xiii, 665 p.

[2] Bashkarev P. G., Karlovich Yu. I., Nechaev A. P. *On the theory of singular integral operators with a finite group of shifts* // Soviet Math. Dokl. — 1974. — **15**, № 6. — P. 1553—1556.

[3] Litvinchuk G. S. *Boundary value problemas and singular integral equations with shift.* — Moscow: Nauka, 1977. — 448 p. (in Russian).

[4] Karapetiants N., Samko S. *Equations with involutive operators.* — Boston: Birkhäuser, 2001. — xxii, 427 p.

[5] Karlovich Yu. I. *Singular integral operators with a shift of contour of integration into the domain and their applications.* — PhD thesis, Odessa, 1974. — 111 p. (in Russian).

[6] Karlovich Yu. I. *An algebra of pseudodifferential operators with slowly oscillating symbols* // Proc. London Math. Soc. — 2006. — **92**. — P. 713—761.

[7] Karlovich Yu. I. *Pseudodifferential operators with compound slowly oscillating symbols* // The Extended Field of Operator Theory. — Operator Theory: Advances and Applications. — Basel: Birkhäuser, 2007, **171**. — P. 189—224.

[8] Karlovich Yu. I. *An algebra of shift-invariant singular integral operators with slowly oscillating data and its application to operators with a Carleman shift* // Analysis, Partial Differential Equations and Applications. The Vladimir Maz'ya Anniversary Volume. — Operator Theory: Advances and Applications. — Basel: Birkhäuser, 2009, **193**. — P. 81—95.

[9] Bastos M. A., Fernandes C. A., Karlovich Yu. I. *C^*-algebras of integral operators with piecewise slowly oscillating coefficients and shifts acting freely* // Integr. Equ. Oper. Theory. — 2006. — **55**. — P. 19—67.

[10] Sarason D. *Functions of vanishing mean oscillation* // Trans. Amer. Math. Soc. — 1975. — **207**. — P. 391—405.

[11] Hartman P. *On completely continuous Hankel matrices* // Proc. Amer. Math. Soc. — 1958. —- **9**. — P. 862—866.

[12] Muhly P. S., Xia J. *Calderón-Zygmund operators, local mean oscillation and certain automorphisms of the Toeplitz algebra* // Amer. J. Math. — 1995. — **117**. — P. 1157—1201.

[13] Karlovich A. Yu., Karlovich Yu. I., Lebre A. B. *Sufficient conditions for Fredholmness of singular integral operators with shifts and slowly oscillating data* // Integr. Equ. Oper. Theory, — 2011. — **70**. — P. 451—483.

[14] Böttcher A., Roch S., Silbermann B., Spitkovsky I. M. *A Gohberg–Krupnik–Sarason symbol calculus for algebras of Toeplitz, Hankel, Cauchy, and Carleman operators* // Topics in Operator Theory. Ernst D. Hellinger Memorial Volume. — Operator Theory: Advances and Applications. — Basel: Birkhäuser, 1990, **48**. — P. 189—234.